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SOME LIMIT THEOREMS FOR THE CRITICAL GALTON – WATSON BRANCHING PROCESSES

ДЕЯКІ ГРАНИЧНІ ТЕОРЕМИ ДЛЯ КРИТИЧНИХ РОЗГАЛУЖЕНИХ ПРОЦЕСІВ ГАЛЬТОНА – ВАТСОНА

We consider critical Galton – Watson processes starting from a random number of particles and determine the effect of the mean value of the initial state on the asymptotic state of the process. For processes starting from a large number of particles and satisfying the condition (S), we prove the limit theorem similar to the result of W. Feller. We also prove the theorem under the condition $W(n) > 0$ for critical processes satisfying the conditions (S) and (M).

Розглянуто критичні процеси Гальтона – Ватсона, починаючи з випадкової кількості частинок, та визначено вплив середнього значення початкового стану на асимптотичний стан процесу. Для процесів, що починаються з великої кількості частинок і задовольняють умову (S), доведено граничну теорему, подібну до результату В. Феллера. Також доведено теорему за умови, що $W(n) > 0$ для критичних процесів, які задовольняють умови (S) і (M).

1. Introduction. Suppose that $\{\xi(k,j),\ k,j \in \mathbb{N}\}$ be a sequence of independent identically distributed random variables taking nonnegative integer values. Let the random variable $\xi(1,1)$ have the distribution

$$p_k = P(\xi(1,1) = k), \quad k = 0, 1, \ldots,$$

with the generating function

$$F(s) := Es^{\xi(1,1)} = \sum_{k=0}^{\infty} p_k s^k, \quad 0 \le s \le 1,$$

and $p_0 + p_1 \neq 1$. Consider the process $W(k),\ k \geq 0$, defined by the following recurrent relation:

$$W(0) = \eta, \quad W(n) = \sum_{j=1}^{W(n-1)} \xi(n,j), \quad n \in \mathbb{N}, \tag{1.1}$$

where η is a random variable that takes positive integer values and independent on the sequence of random variables $\{\xi(k,j),\ k,j \in \mathbb{N}\}$.

We call the process $\{W(k),\ k \geq 0\}$ the Galton – Watson process starting with a random number of particles η. It is well-known [1], that the asymptotic state of the process $\{W(k),\ k \geq 0\}$ depends on the mean value of the random variable $\xi(1,1)$ and it is divided into the classes as follows. It is clear that $F'(1) = E\xi(1,1)$. The process (1.1) is called subcritical, critical and supercritical if $F'(1) < 1,\ F'(1) = 1$ and $F'(1) > 1$, respectively.

In this paper, we consider only critical processes.

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We denote the Galton – Watson process generated by the ith particle in the initial state by $W_i(n)$, $n = 0, 1, \ldots$. Obviously, $W_i(n),\ n = 0, 1, \ldots,\ i \geq 1$, form independent and identically distributed Galton – Watson branching processes. It is known [1] that $W(n)$ can be represented as

$$W(n) = \sum_{i=1}^{\eta} W_i(n), \quad n \in \mathbb{N}. \tag{1.2}$$

Independence of random variables η and $\xi(i,j),\ i \geq 1,\ j \geq 1$, implies independence of $W_i(n)$ and the random variable η. Denote by $P(n)$ the probability of degeneration of the process $\{W(k),\ k \geq 0\}$ at the nth step, i.e., $P(n) = P(W(n) = 0)$. We denote by $R(n)$ the probability of continuation of the process $W_1(n)$ at the nth step, i.e., $R(n) = P(W_1(n) > 0)$. In what follows, we need the following designations:

$$Q(n) = 1 - P(n), \quad h(s) := Es^{\eta}, \quad H_n(s) := Es^{W(n)}, \quad A = h'(1), \quad \sigma^2 = F''(1),$$

$F_0(s) = s,\ F_1(s) = F(s),\ F_n(s) = F(F_{n-1}(s))$ is the nth iteration of $F(s)$.

Further, the sign $a_n \sim b_n$ indicates that $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 1$.

The case when the process $\{W(k),\ k \geq 0\}$ starts with one particle $(\eta = 1)$ has been studied by many authors. So, in 1938, A. N. Kolmogorov [2] obtained the following famous result for the probability of continuation $R(n)$ of the critical Galton – Watson process:

$$R(n) \sim \frac{2}{\sigma^2 n}. \tag{1.3}$$

In 1947, A. M. Yaglom [3] studied the conditional distribution of the variable $W(n)$ given $W(n) > 0$ and obtained the following result:

$$\lim_{n\to\infty} P\left(\frac{2}{\sigma^2 n} W(n) \leq y / W(n) > 0\right) = 1 - e^{-y}, \quad y > 0, \tag{1.4}$$

where it was required $F'''(1) < \infty$. The given results (1.3), (1.4) were later obtained by Spitzer, Kesten, Ney [4] under the condition $F''(1) < \infty$. In [5], V. M. Zolotarev obtained similar results for branching processes with continuous parameters.

In 1968, Slack [6] considered the case of

$$F(s) = s + (1-s)^{1+\alpha} L(1-s), \quad \alpha \in (0, 1], \tag{S}$$

where $L(x)$ is a slowly varying function on a neighborhood of zero, and obtained the following:

$$(1 - F_n(0))^{\alpha} L(1 - F_n(0)) \sim \frac{1}{\alpha n}, \tag{1.5}$$

$$\lim_{n\to\infty} E(\exp\{-\lambda(1 - F_n(0)) W(n)\} / W(n) > 0) = 1 - \lambda(1 + \lambda^{\alpha})^{-1/\alpha}, \quad \lambda > 0. \tag{1.6}$$

This result implies the result by Yaglom (1.4) if $\alpha \equiv 1$ and $F''(1) < \infty$. It should be noted that in the case considered by Slack, the equality $F''(1) = \infty$ can be satisfied.

In [8], K. V. Mitov, G. K. Mitov, N. M. Yanev considered the critical case $(F'(1) = 1)$ when the second factorial moment was finite: $F''(1) = \sigma^2 < \infty$ and the generating function of the number of particles in the initial state was satisfied the condition

$$h(s) = 1 - (1-s)^{\theta} L_0\left(\frac{1}{1-s}\right), \quad \theta \in (0,1). \tag{M}$$

Here, $L_0(x)$ is a slowly varying function at infinity, and obtained the following results:

$$P(W(n) > 0) = 1 - h(F_n(0)) \sim \left(\sigma^2 n\right)^{-\theta} L_0(n), \tag{1.7}$$

$$\lim_{n\to\infty} E(\exp\{-\lambda(1 - F_n(0))W(n)\}/W(n) > 0) = 1 - \lambda^{\theta}(1+\lambda)^{-\theta}, \quad \lambda > 0. \tag{1.8}$$

With the help of Tauber's theorem, it is not difficult to see that condition (M) implies that the average number of particles in the initial state is infinitely. But it follows from (1.7) that in this case, too, the critical Galton – Watson process will degenerate with probability 1.

In 2007, S. V. Nagaev and V. Wachtel [9] considered the case of $\alpha = 0$ in condition (S), i.e.,

$$F(s) = s + (1-s) L_0\left(\frac{1}{1-s}\right) \tag{1.9}$$

and obtained the following results:

$$\lim_{n\to\infty} P\big(H\big(R^{-1}(n)\big) V(W(n)) < x/W(n) > 0\big) = 1 - e^{-x}, \quad x > 0.$$

Here,

$$H(x) = x\big(F\big(1 - x^{-1}\big) - 1 + x^{-1}\big), \quad x \geq 1,$$

and

$$V(y) = \int\limits_0^{1-1/y} \frac{ds}{F(s) - s} = \int\limits_1^{y} \frac{dx}{xH(x)}, \quad y \geq 1.$$

Thus, the analog of the Yaglom theorem is set for all critical processes that satisfy the condition (S) in the case of $\alpha \in [0,1]$. It should be noted that in the case of $\alpha = 0$ not the distribution of the process itself, but the distribution of the process obtained after substitution converges to an exponential distribution.

All of the above results were obtained for distributions under the condition $W(n) > 0$.

In 1951, W. Feller [7] studied the critical Galton – Watson process starting with a large number of particles and satisfying the condition $F''(1) = \sigma^2 < \infty$, i.e., he considered the case when for process (1.1), the equality $W(0) = \frac{1}{2} n\sigma^2 x + o(n)$ holds, where the parameter is x, and received the following result without the condition $W(n) > 0$:

$$\lim_{n\to\infty} E\left(\exp\left[-\frac{2\lambda W(n)}{\sigma^2 n}\right] / W(0) = \left[\frac{1}{2} n\sigma^2 x + o(n)\right]\right) = \exp\left[-\frac{\lambda x}{1+\lambda}\right], \quad \lambda > 0, \quad x > 0.$$

In this paper, we consider critical Galton – Watson processes starting from a random number of particles and determine the effect of the mean value of the initial state on the asymptotic state of the process. We prove the limit theorem that generalizes W. Feller's result for processes starting from a large number of particles and satisfying the condition (*S*). We prove the limit theorem for critical processes $W(n)$ satisfied the conditions (*S*) and (*M*) under the condition $W(n) > 0$.

2. Main results. Suppose that a critical Galton – Watson process is given, defined by relation (1.1). The following theorem shows the influence of the average number of particles in the initial state on the asymptotic's of the survival probability of the process.

Theorem 2.1. *If the condition* (S) *is satisfied and* $h''(1) < \infty$, *then*

$$Q^{\alpha}(n)L(1 - F_n(0)) \sim \frac{A^{\alpha}}{\alpha n}$$

as $n \to \infty$.

Theorem 2.2. *If the condition* (S) *holds, then*

$$\lim_{n\to\infty} E(\exp\{-\lambda(1 - H_n(0))W(n)\}/W(n) > 0) = 1 - A\lambda(1 + (A\lambda)^{\alpha})^{-1/\alpha}, \quad \lambda > 0.$$

In the case of $\eta = 1$, the equality $A = 1$ holds, and in this case Theorem 2.2 turns of the Slack theorem.

The following theorem determines the asymptotic distribution of the critical Galton – Watson process, which initially has average many particles and the law of particle multiplication satisfies the condition (S).

Theorem 2.3. *If the condition* (S) *is satisfied and, for the initial state* $W(0)$, *the condition* $W(0) = \left[bn^{1/\alpha}L^{1/\alpha}(1 - F_n(0))\right]$ *is valid, then*

$$E\Big(\exp\big\{ -\lambda(1 - F_n(0))W(n)\big\}/W(0) = \Big[bn^{1/\alpha}L^{1/\alpha}(1 - F_n(0))\Big]\Big)$$
$$\to \exp\Big\{-\lambda b(\alpha(1 + \lambda^{\alpha}))^{-1/\alpha}\Big\}$$

as $n \to \infty$.

Theorem 2.4. *If the conditions* (M) *and* (S) *are satisfied, then*

$$\lim_{n\to\infty} E(\exp\{-\lambda(1 - F_n(0))W(n)\}/W(n) > 0) = 1 - \lambda^{\theta}(1 + \lambda^{\alpha})^{-\theta/\alpha}, \quad \lambda > 0.$$

In the case of $F''(1) < \infty$, Theorem 2.4 implies the result by Mitov, Mitov, and Yanev. If we set formal $\theta = 1$, $\alpha = 1$ in the last Laplace substitution, we get the Laplace substitution $(1 + \lambda)^{-1}$ of the exponential distribution.

Theorem 2.5. *If the conditions* (M) *and* (1.9) *are satisfied, then*

$$\lim_{n\to\infty} P\big(H\big(R^{-1}(n)\big)V(W(n)) < x/W(n) > 0\big) = 1 - e^{-\theta x}, \quad x > 0.$$

3. Proof of main results.

***Proof of Theorem* 2.1.** It is not difficult to see that

$$H_n(s) = h(F_n(s)), \quad 0 \le s \le 1. \tag{3.1}$$

It is clear that according to (3.1)

$$Q(n) = 1 - H_n(0) = 1 - h(F_n(0)). \tag{3.2}$$

Since $h''(1) < \infty$, according to the Taylor formula,

$$h(s) = h(1) + h'(1)(s - 1) + \frac{h''(\theta_s)}{2}(s - 1)^2 = 1 + A(s - 1) + \frac{h''(\theta_s)}{2}(s - 1)^2, \tag{3.3}$$

where θ_s is such that $s \le \theta_s \le 1$. Since h is a generating function, it and its derivatives increase monotonically. Therefore,

$$h''(\theta_s) \le h''(1) < \infty. \tag{3.4}$$

Now, replacing s in (3.3) with $F_n(0)$ and taking into account (3.2), we obtain

$$Q(n) = A(1 - F_n(0)) - \frac{h''(\theta_s)}{2}(1 - F_n(0))^2,$$

what implies

$$Q^\alpha(n)L(1 - F_n(0)) = A^\alpha(1 - F_n(0))^\alpha L(1 - F_n(0))\left[1 - \frac{h''(\theta_s)}{2A}(1 - F_n(0))\right]^\alpha. \tag{3.5}$$

Now taking into account that $F_n(0) \to 1$ as $n \to \infty$ and the relations

$$(1 - x)^\alpha \approx 1 - \alpha x, \quad x \to 0,$$

(3.5), and the result (1.5), we get the following relation:

$$Q^\alpha(n)L(1 - F_n(0)) = \frac{A^\alpha}{\alpha n}(1 + o(1)).$$

Theorem 2.1 is proved.

***Proof of Theorem* 2.2.** It is clear that according to the total probability formula, we have

$$\begin{aligned} E[\exp\{-\lambda(1 - H_n(0))W(n)\}] &= E(\exp\{-\lambda(1 - H_n(0))W(n)\}I(W(n) = 0)) \\ &\quad + E(\exp\{-\lambda(1 - H_n(0))W(n)\}I(W(n) > 0)) \\ &= P(W(n) = 0) + P(W(n) > 0)E(\exp\{-\lambda(1 - H_n(0))W(n)\}/W(n) > 0), \end{aligned} \tag{3.6}$$

what implies

$$\begin{aligned} &E(\exp\{-\lambda(1 - H_n(0))W(n)\}/W(n) > 0) \\ &\quad = \frac{1}{1 - P(W(n) = 0)}\{E(\exp\{-\lambda(1 - H_n(0))W(n)\}) - P(W(n) = 0)\}. \end{aligned} \tag{3.7}$$

The asymptotic's of $P(W(n) = 0)$ in the last relation is known according to Theorem 2.1. Now we determine the asymptotic's of $E(\exp\{-\lambda(1 - H_n(0))W(n)\})$. Taking into account the fact that variables $W_i(n)$ are independent, identically distributed, and independent of the random variable η, and also relation (1.2), we obtain the following:

$$\begin{aligned} E(\exp\{-\lambda(1 - H_n(0))W(n)\}) &= E\left(\exp\left\{-\lambda(1 - H_n(0))\sum_{i=1}^{\eta} W_i(n)\right\}\right) \\ &= E\left[E\left(\exp\left\{-\lambda(1 - H_n(0))\sum_{i=1}^{\eta} W_i(n)\right\}\right)/\eta\right] \end{aligned}$$

$$= E\prod_{i=1}^{\eta} E(\exp\{-\lambda(1-H_n(0))W_i(n)\}) = E(F_n(\exp\{-\lambda(1-H_n(0))\}))^{\eta}. \tag{3.8}$$

According to the total probability formula, we have

$$F_n(\exp\{-\lambda(1-H_n(0))\}) = E(\exp\{-\lambda(1-H_n(0))W_1(n)\})$$
$$= P(W_1(n)=0) + P(W_1(n)>0)E(\exp\{-\lambda(1-H_n(0))W_1(n)\}/W_1(n)>0)$$
$$= F_n(0) + (1-F_n(0))E(\exp\{-\lambda(1-H_n(0))W_1(n)\}/W_1(n)>0). \tag{3.9}$$

Now, applying Theorem 2.1 and the result (1.5), we get

$$\left[\frac{1-H_n(0)}{1-F_n(0)}\right]^{\alpha} \sim \frac{Q^{\alpha}(n)L(1-F_n(0))}{(1-F_n(0))^{\alpha}L(1-F_n(0))} \sim \frac{\dfrac{A^{\alpha}}{\alpha n}(1+o(1))}{\dfrac{1}{\alpha n}(1+o(1))} \sim A^{\alpha}(1+o(1)). \tag{3.10}$$

It is well-known that

$$\left|e^{-x}-e^{-y}\right| \le |x-y|, \quad x\ge 0, \quad y\ge 0.$$

Taking into account the inequality, the relation

$$(1-F_n(0))E(W_1(n)/W_1(n)>0) = 1$$

valid for the critical process, and (3.10), we have

$$|E(\exp\{-\lambda(1-H_n(0))W_1(n)\}/W_1(n)>0)$$
$$- E(\exp\{-\lambda A(1-F_n(0))W_1(n)\}/W_1(n)>0)|$$
$$\le \left|\frac{1-H_n(0)}{1-F_n(0)} - A\right|(1-F_n(0))E(W_1(n)/W_1(n)>0) = \left|\frac{1-H_n(0)}{1-F_n(0)} - A\right| \to 0. \tag{3.11}$$

Now by virtue of (3.11), the results (1.5) and (1.6), we obtain from equality (3.9) the following:

$$F_n(\exp\{-\lambda(1-H_n(0))\}) \sim 1 - \frac{1}{(L(1-F_n(0))\alpha n)^{1/\alpha}}$$
$$+ \frac{1}{(L(1-F_n(0))\alpha n)^{1/\alpha}}\left(1 - A\lambda(1+(A\lambda)^{\alpha})^{-1/\alpha}\right)$$
$$= 1 - \frac{A\lambda(1+(A\lambda)^{\alpha})^{-1/\alpha}}{(L(1-F_n(0))\alpha n)^{1/\alpha}}. \tag{3.12}$$

From (3.8) and (3.12), we get

$$E(\exp\{-\lambda(1-H_n(0))W(n)\}) = E\left(1 - \frac{A\lambda(1+(A\lambda)^{\alpha})^{-1/\alpha}}{(L(1-F_n(0))\alpha n)^{1/\alpha}}\right)^{\eta}. \tag{3.13}$$

Next, according to the asymptotic relations,

$$\ln(1-x) = -x + o(x), \quad x\to 0,$$

$$e^{-x} = 1 - x + o(x), \quad x \to 0,$$

we have

$$E\left(1 - \frac{A\lambda(1+(A\lambda)^\alpha)^{-1/\alpha}}{(L(1-F_n(0))\alpha n)^{1/\alpha}}\right)^\eta = Ee^{\eta \ln\left(1 - \frac{A\lambda(1+(A\lambda)^\alpha)^{-1/\alpha}}{(L(1-F_n(0))\alpha n)^{1/\alpha}}\right)} = Ee^{-\frac{\eta A\lambda(1+(A\lambda)^\alpha)^{-1/\alpha}}{(L(1-F_n(0))\alpha n)^{1/\alpha}}}$$

$$= E\left(1 - \frac{\eta A\lambda(1+(A\lambda)^\alpha)^{-1/\alpha}}{(L(1-F_n(0))\alpha n)^{1/\alpha}}\right) \sim 1 - \frac{A^2\lambda(1+(A\lambda)^\alpha)^{-1/\alpha}}{(L(1-F_n(0))\alpha n)^{1/\alpha}}. \tag{3.14}$$

Applying (3.14) and Theorem 2.1, we get

$$E(\exp\{-\lambda(1-H_n(0))W(n)\}\,/W(n)>0)$$

$$\sim \frac{1}{\dfrac{A}{(L(1-F_n(0))\alpha n)^{1/\alpha}}}\left\{1 - \frac{A^2\lambda(1+(A\lambda)^\alpha)^{-1/\alpha}}{(L(1-F_n(0))\alpha n)^{1/\alpha}} - 1 + \frac{A}{(L(1-F_n(0))\alpha n)^{1/\alpha}}\right\}$$

$$= \frac{1}{\dfrac{A}{(L(1-F_n(0))\alpha n)^{1/\alpha}}}\left\{\frac{A}{(L(1-F_n(0))\alpha n)^{1/\alpha}}\Big(1 - A\lambda(1+(A\lambda)^\alpha)^{-1/\alpha}\Big)\right\}$$

$$= 1 - A\lambda(1+(A\lambda)^\alpha)^{-1/\alpha}$$

as $n \to \infty$.

Theorem 2.2 is proved.

Proof of Theorem **2.3.** Because of independence and identically distribution of variables $W_i(n)$, taking into account (1.2), we obtain

$$E(\exp\{-\lambda(1-F_n(0))W(n)\}) = E\left(\exp\left\{-\lambda(1-F_n(0))\sum_{i=1}^{\eta} W_i(n)\right\}\right)$$

$$= \prod_{i=1}^{\eta} E(\exp\{-\lambda(1-F_n(0))W_i(n)\})$$

$$= (E(\exp\{-\lambda(1-F_n(0))W_1(n)\}))^{\left[bn^{1/\alpha}L^{1/\alpha}(1-F_n(0))\right]}. \tag{3.15}$$

Now we determine the asymptotic behavior of $E(\exp\{-\lambda(1-F_n(0))W_1(n)\})$ as $n \to \infty$. By virtue of the total probability formula, we have the following:

$$E(\exp\{-\lambda(1-F_n(0))W_1(n)\})$$

$$= P(W_1(n)=0) + P(W_1(n)>0)E(\exp\{-\lambda(1-F_n(0))W_1(n)\}/W_1(n)>0)$$

$$= F_n(0) + (1-F_n(0))E(\exp\{-\lambda(1-F_n(0))W_1(n)\}/W_1(n)>0).$$

If we use the results (1.5) and (1.6) in the last equation, we get

$$E(\exp\{-\lambda(1-F_n(0))W_1(n)\}) \approx 1 - \frac{1}{(L(1-F_n(0))\alpha n)^{1/\alpha}}$$

$$+\frac{1}{(L(1-F_n(0))\alpha n)^{1/\alpha}}\Big(1-\lambda(1+\lambda^\alpha)^{-1/\alpha}\Big)=1-\frac{\lambda(1+\lambda^\alpha)^{-1/\alpha}}{(L(1-F_n(0))\alpha n)^{1/\alpha}}. \quad (3.16)$$

We have from (3.15) and (3.16) the equality

$$E(\exp\{-\lambda(1-F_n(0))W(n)\})=\left(1-\frac{\lambda(1+\lambda^\alpha)^{-1/\alpha}}{(L(1-F_n(0))\alpha n)^{1/\alpha}}\right)^{\left[bn^{1/\alpha}L^{1/\alpha}(1-F_n(0))\right]}.$$

If we pass to the limit in the last equation as $n\to\infty$, we obtain the statement of Theorem 2.3.

***Proof of Theorem* 2.4.** We have the following:

$$Es^{W(n)}=E\Big(s^{W(n)},W(n)=0\Big)+E\Big(s^{W(n)},W(n)>0\Big)$$
$$=P(W(n)=0)+E\Big(s^{W(n)}/W(n)>0\Big)P(W(n)>0),$$

what implies, according to notations

$$h(F_n(s))=h(F_n(0))+(1-h(F_n(0)))E\Big(s^{W(n)}/W(n)>0\Big).$$

Thus,

$$E\Big(s^{W(n)}/W(n)>0\Big)=\frac{h(F_n(s))-h(F_n(0))}{1-h(F_n(0))}=1-\frac{1-h(F_n(s))}{1-h(F_n(0))}. \quad (3.17)$$

In the last relation, if we replace s with $e^{-\lambda(1-F_n(0))}$, where $\lambda>0$, we get

$$E\Big[e^{-\lambda(1-F_n(0))W(n)}/W(n)>0\Big]=1-\frac{1-h\big(F_n\big(e^{-\lambda(1-F_n(0))}\big)\big)}{1-h(F_n(0))}. \quad (3.18)$$

According to the condition (M) set to the function h, we obtain

$$\frac{1-h\big(F_n\big(e^{-\lambda(1-F_n(0))}\big)\big)}{1-h(F_n(0))}=\left(\frac{1-F_n\big(e^{-\lambda(1-F_n(0))}\big)}{1-F_n(0)}\right)^\theta\frac{L_0\Big(\big(1-F_n\big(e^{-\lambda(1-F_n(0))}\big)\big)^{-1}\Big)}{L_0\Big((1-F_n(0))^{-1}\Big)}. \quad (3.19)$$

By virtue of the result (1.6),

$$\frac{1-F_n\big(e^{-\lambda(1-F_n(0))}\big)}{1-F_n(0)}\to\lambda(1+\lambda^\alpha)^{-1/\alpha}. \quad (3.20)$$

It is not difficult to see that

$$0<\frac{\lambda}{(1+\lambda^\alpha)^{1/\alpha}}\le 1,\quad \lambda\ge 0.$$

In this case, according to (3.20), for an arbitrary number $\varepsilon>0$, there exists a number N such that, for any $n\ge N$,

$$\varepsilon<\frac{1-F_n\big(e^{-\lambda(1-F_n(0))}\big)}{1-F_n(0)}\le 1+\varepsilon.$$

In this case, according to Lemma 1 from the paper [6],

$$\frac{L_0\Big(\big(1-F_n\big(e^{-\lambda(1-F_n(0))}\big)\big)^{-1}\Big)}{L_0((1-F_n(0))^{-1})}\to 1$$

as $n\to\infty$. The statement of Theorem 2.4 follows from the last relation, (3.18), (3.19), and (3.20).

***Proof of Theorem* 2.5.** We first prove the following lemma.

Lemma 3.1. *Let ξ_n, $n=1,2,\ldots$, be some sequence of nonnegative random variables, and $V(x)$ a continuous, increasing, slowly varying function. We denote by $G(x)$ the function inverse to $V(x)$. If there exist a continuous function $\varphi(x)$ and a sequence of numbers $a_n>0$ such that $a_n\to\infty$, $n\to\infty$, for all $x>0$,*

$$\lim_{n\to\infty} E\bigg(\exp\bigg\{-\frac{\xi_n}{G(a_n x)}\bigg\}\bigg)=\psi(x), \tag{3.21}$$

then, for all $x>0$,

$$\lim_{n\to\infty} P\big(a_n^{-1}V(\xi_n)<x\big)=\psi(x).$$

If

$$\lim_{n\to\infty} E\bigg(\exp\bigg\{-\frac{\xi_n}{G(a_n x)}\bigg\}/\xi_n>0\bigg)=\varphi(x),$$

then

$$\lim_{n\to\infty} P\big(a_n^{-1}V(\xi_n)<x/\xi_n>0\big)=\varphi(x).$$

Proof. The proof follows the same scheme as the proof of Lemma 1 from [9]. Let $\varepsilon>0$ be an arbitrary fixed number. It is not difficult to see that

$$\begin{aligned} Ee^{-\frac{\xi_n}{G(a_n x)}} &= Ee^{-\frac{\xi_n}{G(a_n x)}}I(\xi_n<G(a_n(x+\varepsilon)))+Ee^{-\frac{\xi_n}{G(a_n x)}}I(\xi_n\ge G(a_n(x+\varepsilon)))\\ &\le P(\xi_n<G(a_n(x+\varepsilon)))+e^{-\frac{G(a_n(x+\varepsilon))}{G(a_n x)}} \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} Ee^{-\frac{\xi_n}{G(a_n x)}} &= Ee^{-\frac{\xi_n}{G(a_n x)}}I(\xi_n<G(a_n(x-\varepsilon)))+Ee^{-\frac{\xi_n}{G(a_n x)}}I(\xi_n\ge G(a_n(x-\varepsilon)))\\ &\ge Ee^{-\frac{\xi_n}{G(a_n x)}}I(\xi_n<G(a_n(x-\varepsilon)))\ge P(\xi_n<G(a_n(x-\varepsilon)))e^{-\frac{G(a_n(x-\varepsilon))}{G(a_n x)}}. \end{aligned} \tag{3.23}$$

By Theorem 1.11 of [10] we have

$$\lim_{x\to\infty}\frac{G(x)}{G(cx)}=0 \tag{3.24}$$

for every constant $c>1$. Therefore, in (3.22), replacing x by $x-\varepsilon$ and passing to the limit, by virtue of the assumptions made,

$$\varphi(x-\varepsilon)=\lim_{n\to\infty} Ee^{-\frac{\xi_n}{G(a_n x)}}\le\lim_{n\to\infty} P(\xi_n<G(a_n x))$$

$$+\lim_{n\to\infty} e^{-\frac{G(a_n x)}{G(a_n(x-\varepsilon))}} = \lim_{n\to\infty} P(\xi_n < G(a_n x)). \tag{3.25}$$

Now, replacing x by $x+\varepsilon$ in (3.23) and passing to the limit, taking into account (3.24), we have

$$\varphi(x+\varepsilon) = \lim_{n\to\infty} Ee^{-\frac{\xi_n}{G(a_n(x+\varepsilon))}} \geq \lim_{n\to\infty} P(\xi_n < G(a_n x)) \lim_{n\to\infty} e^{-\frac{G(a_n x)}{G(a_n(x+\varepsilon))}}$$

$$= \lim_{n\to\infty} P(\xi_n < G(a_n x)). \tag{3.26}$$

It follows from (3.25) and (3.26) that

$$\varphi(x-\varepsilon) \leq \lim_{n\to\infty} P(\xi_n < G(a_n x)) \leq \varphi(x+\varepsilon).$$

From this, passing to the limit for $\varepsilon \to 0$, taking into account the continuity of $\psi(x)$, the assertion of the lemma follows.

Now we prove the theorem. By virtue of (*M*) and (3.17), we obtain

$$E\Big(s^{W(n)}/W(n) > 0\Big) = 1 - \left[\frac{1-F_n(s)}{1-F_n(0)}\right]^{\theta} \frac{L_0\left(\dfrac{1}{1-F_n(s)}\right)}{L_0\left(\dfrac{1}{1-F_n(0)}\right)}. \tag{3.27}$$

We put $S_n = S_n(x) = \exp\left\{-\dfrac{1}{G(a_n x)}\right\}$, where $G(x)$ is the inverse function to $V(x)$ and $a_n = H\big((1-F_n(0))^{-1}\big)$. Then, as shown in [9],

$$\frac{1-F_n(S_n)}{1-F_n(0)} \to e^{-x}. \tag{3.28}$$

By Lemma 1 of [6], we obtain

$$\frac{L_0\left(\dfrac{1}{1-F_n(s)}\right)}{L_0\left(\dfrac{1}{1-F_n(0)}\right)} \to 1 \tag{3.29}$$

as $n \to \infty$.

Now, substituting in (3.27) instead of s the value $s = S_n$, we obtain, taking into account (3.28), (3.29), that

$$E\left(e^{-\frac{W(n)}{G(a_n x)}}/W(n) > 0\right) \to 1 - e^{-\theta x} \quad \text{as} \quad n \to \infty.$$

Now it follows from the Lemma 3.1 and (3.21) that

$$\lim_{n\to\infty} P\big(H(1-F_n(0))^{-1}V(W(n)) < x/W(n) > 0\big) = 1 - e^{-\theta x}.$$

Theorem 2.5 is proved.

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