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d-GAUSSIAN FIBONACCI, d-GAUSSIAN LUCAS POLYNOMIALS AND THEIR MATRIX REPRESENTATIONS

d-ГАУССОВІ ПОЛІНОМИ ФІБОНАЧЧІ, d-ГАУССОВІ ПОЛІНОМИ ЛУКАСА ТА ЇХНІ МАТРИЧНІ ЗОБРАЖЕННЯ

We define d-Gaussian Fibonacci polynomials and d-Gaussian Lucas polynomials. We present the matrix representations of these polynomials. By using the Riordan method, we obtain the factorizations of the Pascal matrix including the polynomials. In addition, we define the infinite d-Gaussian Fibonacci polynomial matrix and the d-Gaussian Lucas polynomial matrix and give their inverses.

Визначено d-гауссові поліноми Фібоначчі та d-гауссові поліноми Лукаса. Наведено матричні зображення цих поліномів. Використовуючи метод Ріордана, отримано факторизації матриці Паскаля, що включають поліноми. Крім того, визначено нескінченну матрицю d-гауссових поліномів Фібоначчі та матрицю d-гауссових поліномів Лукаса і наведено їхні обернені матриці.

1. Introduction. Fibonacci numbers, which emerged with the solution of the famous rabbit problem, have been made many generalizations until today and still find application in many scientific fields [6]. One of the most well-known number sequences is also the Lucas numbers [9]. Many generalizations of number sequences were then described and studied [1, 10 – 12, 15]. We know that the Fibonacci numbers F_n are defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

with $F_0 = 0$ and $F_1 = 1$ [6]. Similarly, the Lucas numbers L_n are defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2,$$

with $L_0 = 2$ and $L_1 = 1$ [6].

Definition 1.1. *Let $p_i(x)$ be a real coefficient for $i = 1, \ldots, d+1$. Then d-Fibonacci polynomials are defined by*

$$F_{n+1}(x) = p_1(x)F_n(x) + p_2(x)F_{n-1}(x) + \ldots + p_{d+1}(x)F_{n-d}(x)$$

with $F_n(x) = 0$ for $n \leq 0$ and $F_1(x) = 1$ [13].

Fibonacci numbers are of great importance in the study of many fields such as mathematics, physics, biology, statistics, etc. Falcon et al. [3] presented a general Fibonacci sequence.

In [7], Nalli and Haukkanen defined $h(x)$, Fibonacci and Lucas polynomials. Gaussian Fibonacci and Gaussian Lucas numbers were studied in [5]. Özkan et al. introduced Gaussian Fibonacci polynomials and Gaussian Lucas polynomials and presented some properties for these polynomials in [8].

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Shapiro et al. described Riordan matrices and the Riordan group as a set of matrices $M = (m_{ij})$, $i, j \geq 0$, whose elements are complex numbers [14].

The Riordan group was examined by Shapiro et al. as a set of infinite lower-triangular integer matrices where each matrix is defined by pair of formal power series $g(z) = \sum_{n=0}^{\infty} g_n z^n$ and $f(z) = \sum_{n=0}^{\infty} f_n z^n$ with $g_0 \neq 0$ and $f_1 \neq 1$ [14]. An infinite lower triangular matrix $D = [d_{n,k}]_{n,k\geq 0}$ is called a Riordan array, if its ith column generating function is $g(x)(f(x))^i$ for $i \geq 0$ where the first column is indexed by 0. Generally, we assume $d_{0,0} = g_0 = 1$ [14]. Sadaoui et al. have recently been studying introduced d-Fibonacci and d-Lucas polynomials [13].

In this paper, we give new generalizations of Gaussian Fibonacci and Gaussian Lucas polynomials. We find the matrix representations for these polynomials. Using the Riordan method, we obtain the factorizations of the Pascal matrix including these polynomials. Also, we present d-Gaussian Fibonacci polynomials matrix and d-Gaussian Lucas polynomials matrix and their inverses.

2. Generalization of Gaussian Fibonacci and Gaussian Lucas polynomials. ***2.1. Generalization of Gaussian Fibonacci polynomials.***

Definition 2.1.1. *d-Gaussian Fibonacci polynomials $GF_n(x)$ are defined by*

$$GF_{n+1}(x) = p_1(x)GF_n(x) + p_2(x)GF_{n-1}(x) + \ldots + p_{d+1}(x)GF_{n-d}(x) \tag{2.1}$$

with $GF_n(x) = 0$ for $n \leq 0$ and $GF_1(x) = p_1(x) + i$.

Let us give a few terms of d-Gaussian Fibonacci polynomials as follows:

$$GF_0(x) = 0, \qquad GF_1(x) = p_1(x) + i, \qquad GF_2(x) = {p_1}^2(x) + ip_1(x)$$

and

$$GF_3(x) = {p_1}^3(x) + i{p_1}^2(x) + p_1(x)p_2(x) + ip_2(x).$$

From Eq. (2.1), the characteristic equation of d-Gaussian Fibonacci polynomials is given by

$$r^{d+1} - p_1(x)r^d - p_2(x)r^{d-1} - \ldots - p_{d+1}(x) = 0.$$

The roots of this equation are $\{\alpha_1(x), \alpha_2(x), \ldots, \alpha_{d+1}(x)\}$. Thus, we can give the generating function for these polynomials as follows.

Theorem 2.1.1. *The generating function of $GF_n(x)$ is given as follows*:

$$G(x, r) = \sum_{n=0}^{\infty} GF_n(x)r^n = \frac{r(p_1(x) + i)}{(1 - p_1(x)r - p_2(x)r^2 - \ldots - p_{d+1}(x)r^{d+1})}.$$

Proof. We have

$$G(x, r) = GF_0(x) + GF_1(x)r + \ldots + GF_n(x)r^n + \ldots . \tag{2.2}$$

Let us multiply Eq. (2.2) by $p_1(x)r, p_2(x)r^2, \ldots, p_{d+1}(x)r^{d+1}$, respectively. So, the following equations are obtained:

$$G(x, r) = GF_0(x) + GF_1(x)r + \ldots + GF_n(x)r^n + \ldots,$$

$$p_1(x)rG(x,r) = p_1(x)rGF_0(x) + p_1(x)r^2GF_1(x) + \ldots,$$

$$p_2(x)r^2G(x,r) = p_2(x)r^2GF_0(x) + p_2(x)r^3GF_1(x) + \ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$p_{d+1}(x)r^{d+1}G(x,r) = p_{d+1}(x)r^{d+1}GF_0(x) + p_{d+1}(x)r^{d+2}GF_1(x) + \ldots .$$

If we take the necessary calculations are made, we obtain the equations

$$G(x,r)\Big(1 - p_1(x)r - p_2(x)r^2 - \ldots - p_{d+1}(x)r^{d+1}\Big) = GF_0(x) + GF_1(x)r - p_1(x)rGF_0(x),$$

$$G(x,r) = \frac{r(p_1(x)+i)}{1 - p_1(x)r - p_2(x)r^2 - \ldots - p_{d+1}(x)r^{d+1}}.$$

In this case, the desired formula is obtained.

The theorem is proved.

Let us now derive the classical Binet formula for sequences of numbers and their polynomials. The Binet formula of $GF_n(x)$ has the following form:

$$GF_n(x) = \sum_{i=1}^{d+1} A_i(x)[\alpha_i(x)]^n.$$

Let us write the following equations for some values of n:

$$GF_0(x) = \sum_{i=1}^{d+1} A_i(x),$$

$$GF_1(x) = \sum_{i=1}^{d+1} A_i(x)\alpha_i(x),$$

$$GF_2(x) = \sum_{i=1}^{d+1} A_i(x)[\alpha_i(x)]^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$GF_n(x) = \sum_{i=1}^{d+1} A_i(x)[\alpha_i(x)]^n.$$

If we multiply both sides of the last equations by 1, r, $r^2, \ldots, r^n$, respectively, we obtain

$$GF_0(x) = \sum_{i=1}^{d+1} A_i(x),$$

$$rGF_1(x) = \sum_{i=1}^{d+1} A_i(x)\alpha_i(x)r,$$

$$r^2 GF_2(x) = \sum_{i=1}^{d+1} A_i(x)[\alpha_i(x)]^2 r^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$r^n GF_n(x) = \sum_{i=1}^{d+1} A_i(x)[\alpha_i(x)]^n r^n.$$

So, we have

$$\sum_{n=0}^{\infty} GF_n(x) r^n = \sum_{i=1}^{d+1} A_i(x)(1 + \alpha_i(x) r + [\alpha_i(x)]^2 r^2 + \ldots) = \sum_{i=1}^{d+1} \left(\frac{A_i(x)}{1 - \alpha_i(x) r} \right).$$

From Theorem 2.1.1, we obtain

$$\frac{r(p_1(x) + i)}{1 - p_1(x) r - p_2(x) r^2 - \ldots - p_{d+1}(x) r^{d+1}} = \sum_{i=1}^{d+1} \left(\frac{A_i(x)}{1 - \alpha_i(x) r} \right).$$

More precisely, the coefficients allow us to give the explicit form of d-Gaussian Fibonacci polynomials.

Theorem 2.1.2. *For $n \geq 0$, the following equality is true*:

$$GF_n(x) = (p_1(x) + i) \sum_{\substack{n_1, n_2, \ldots, n_{d+1} \\ 1 + n_1 + 2n_2 + \ldots + (d+1) n_{d+1} = n}} \binom{n_1 + n_2 + \ldots + n_{d+1}}{n_1, n_2, \ldots, n_{d+1}} {p_1}^{n_1}(x) {p_2}^{n_2}(x) \ldots {p_{d+1}}^{n_{d+1}}(x) r^n.$$

Proof. Let us use the generating function to prove the theorem:

$$\sum_{n=0}^{\infty} GF_{n+1}(x) r^n = \frac{p_1(x) + i}{1 - p_1(x) r - p_2(x) r^2 - \ldots - p_{d+1}(x) r^{d+1}}$$

$$= (p_1(x) + i) \sum_{n=0}^{\infty} \left(p_1(x) r + p_2(x) r^2 + \ldots + p_{d+1}(x) r^{d+1} \right)^n$$

$$= (p_1(x) + i) \sum_{n=0}^{\infty} \left[\sum_{n_1 + n_2 + \ldots + n_{d+1} = n}^{\infty} \binom{n}{n_1, n_2, \ldots, n_{d+1}} \right.$$

$$\left. \times\, {p_1}^{n_1}(x) {p_2}^{n_2}(x) \ldots {p_{d+1}}^{n_{d+1}}(x) \right] r^{n_1 + 2n_2 + \ldots + (d+1) n_{d+1}}$$

$$= (p_1(x) + i) \sum_{n=0}^{\infty} \left[\sum_{\substack{n_1, n_2, \ldots, n_{d+1} \\ n_1 + 2n_2 + \ldots + (d+1) n_{d+1} = n}} \binom{n_1 + n_2 + \ldots + n_{d+1}}{n_1, n_2, \ldots, n_{d+1}} \right.$$

$$\left. \times\, {p_1}^{n_1}(x){p_2}^{n_2}(x)\ldots {p_{d+1}}^{n_{d+1}}(x)\right] r^n.$$

The theorem is proved.

***Corollary* 2.1.1.** *Let $SGF_n(x)$ be sum of the d-Gaussian Fibonacci polynomials. Then we have*

$$SGF_n(x)=\sum_{n=0}^{\infty} GF_n(x)=\frac{p_1(x)+i}{1-p_1(x)-p_2(x)-\ldots-p_{d+1}(x)}.$$

Proof. We get the following equation:

$$SGF_n(x)=GF_0(x)+GF_1(x)+\ldots+GF_n(x)+\ldots .$$

If we multiply the last equation by $p_1(x), p_2(x),\ldots,p_{d+1}(x)$, respectively, then we obtain

$$p_1(x)\; SGF_n(x)=p_1(x)GF_1(x)+p_1(x)GF_2(x)+\ldots+p_1(x)GF_n(x)+\ldots,$$

$$p_2(x)SGF_n(x)=p_2(x)GF_1(x)+p_2(x)GF_2(x)+\ldots+p_2(x)GF_n(x)+\ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$p_{d+1}(x)SGF_n(x)=p_{d+1}(x)GF_1(x)+p_{d+1}(x)GF_2(x)+\ldots+p_{d+1}(x)GF_n(x)+\ldots .$$

If the necessary operations are done, we get

$$SGF_n(x)(1-1-p_1(x)-p_2(x)-\ldots-p_{d+1}(x))=p_1(x)+i.$$

Thus, we have

$$SGF_n(x)=\sum_{n=0}^{\infty} GF_n(x)=\frac{p_1(x)+i}{1-p_1(x)-p_2(x)-\ldots-p_{d+1}(x)}.$$

Definition 2.1.2. *The d-Gaussian Fibonacci polynomials matrix G_d is given by*

$$G_d=(p_1(x)+i)Q_d$$

$$=\begin{bmatrix} (p_1(x)+i)p_1(x) & (p_1(x)+i)p_2(x) & \ldots & (p_1(x)+i)p_{d+1}(x)\\ p_1(x)+i & 0 & & \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & 0 & p_1(x)+i & 0 \end{bmatrix}$$

$$=\begin{bmatrix} (p_1(x))^2+ip_1(x) & p_2(x)p_1(x)+ip_2(x) & \ldots & p_{d+1}(x)p_1(x)+ip_{d+1}(x)\\ p_1(x)+i & 0 & & 0\\ 0 & \ddots & & 0\\ & & \ddots & \\ 0 & 0 & p_1(x)+i & 0 \end{bmatrix}, \qquad (2.3)$$

where

$$Q_d = \begin{bmatrix} p_1(x) & p_2(x) & \dots & p_{d+1}(x) \\ 1 & 0 & & \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & 0 & 1 & 0 \end{bmatrix} [13]$$

and

$$\det G_d = (p_1(x) + i)^{d+1}(-1)^d p_{d+1}(x).$$

Now, we can give matrix representation for $GF_n(x)$ in the next theorem.

Theorem 2.1.3. *The matrix representation for $GF_n(x)$ has the following form*:

$$G_d^n = \begin{bmatrix} GF_{n+1}(x) & p_2(x)GF_n(x) + \ldots + p_{d+1}(x)GF_{n-d+1}(x) & \dots & p_{d+1}(x)GF_n(x) \\ GF_n(x) & p_2(x)GF_{n-1}(x) + \ldots + p_{d+1}(x)GF_{n-d}(x) & \dots & p_{d+1}(x)GF_{n-1}(x) \\ \vdots & \vdots & \dots & \vdots \\ GF_{n-d+1}(x) & p_2(x)GF_{n-d}(x) + \ldots + p_{d+1}(x)GF_{n-2d+1}(x) & \dots & p_{d+1}(x)GF_{n-d}(x) \end{bmatrix}, \tag{2.4}$$

where $G_d^n = G_d^{n-1}Q_d$.

Proof. To prove the theorem, let us use mathematical induction on n.

If we take $n = 1$ in Eq. (2.4), we get the following matrix:

$$G_d = \begin{bmatrix} GF_2(x) & p_2(x)GF_1(x) & p_3(x)GF_1(x) & \dots & p_{d+1}(x)GF_1(x) \\ GF_1(x) & 0 & 0 & & 0 \\ 0 & \begin{matrix} p_2(x)GF_{n-2}(x) + \ldots \\ + p_{d+1}(x)GF_{n-d-1}(x) \end{matrix} & 0 & & 0 \\ 0 & 0 & \begin{matrix} p_3(x)GF_{n-3}(x) + \ldots \\ + p_{d+1}(x)GF_{n-d-1}(x) \end{matrix} & & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \ddots & 0 \end{bmatrix}. \tag{2.5}$$

From the recurrence relation of $GF_n(x)$, it will be seen that the matrices in (2.3) and (2.5) are equal. Now, assume the equation (2.4) satisfy for n. So, we have

$$G_d^n = \begin{bmatrix} GF_{n+1}(x) & p_2(x)GF_n(x) + \ldots + p_{d+1}(x)GF_{n-d+1}(x) & \dots & p_{d+1}(x)GF_n(x) \\ GF_n(x) & p_2(x)GF_{n-1}(x) + \ldots + p_{d+1}(x)GF_{n-d}(x) & \dots & p_{d+1}(x)GF_{n-1}(x) \\ \vdots & \vdots & \dots & \vdots \\ GF_{n-d+1}(x) & p_2(x)GF_{n-d}(x) + \ldots + p_{d+1}(x)GF_{n-2d+1}(x) & \dots & p_{d+1}(x)GF_{n-d}(x) \end{bmatrix}.$$

Let show that it is true for $n + 1$. Then we obtain

$$G_d^{n+1} = G_d^n Q_d \begin{bmatrix} GF_{n+1}(x) & \begin{matrix} p_2(x)GF_n(x) + \ldots \\ +p_{d+1}(x)GF_{n-d+1}(x) \end{matrix} & \ldots & p_{d+1}(x)GF_n(x) \\ GF_n(x) & \begin{matrix} p_2(x)GF_{n-1}(x) + \ldots \\ + p_{d+1}(x)GF_{n-d}(x) \end{matrix} & \ldots & p_{d+1}(x)GF_{n-1}(x) \\ \vdots & \vdots & \ldots & \vdots \\ GF_{n-d+1}(x) & \begin{matrix} p_2(x)GF_{n-d}(x) + \ldots \\ + p_{d+1}\ (x)GF_{n-2d+1}(x) \end{matrix} & \ldots & p_{d+1}(x)GF_{n-d}(x) \end{bmatrix}$$

$$\times \begin{bmatrix} p_1(x) & p_2(x) & \ldots & p_{d+1}(x) \\ 1 & 0 & & \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} GF_{n+2}(x) & \begin{matrix} p_2(x)GF_{n+2}(x) + \ldots \\ + p_{d+1}(x)GF_{n-d+2}(x) \end{matrix} & \ldots & p_{d+1}(x)GF_{n+1}(x) \\ GF_{n+1}(x) & \begin{matrix} p_2(x)GF_{n+1}(x) + \ldots \\ + p_{d+1}(x)GF_{n-d+1}(x) \end{matrix} & \ldots & p_{d+1}(x)GF_n(x) \\ \vdots & \vdots & \vdots & \vdots \\ GF_{n-d+2}(x) & \begin{matrix} p_2(x)GF_{n-d+2}(x) + \ldots \\ + p_{d+1}(x)GF_{n-2d+2}(x) \end{matrix} & \ldots & p_{d+1}(x)GF_{n-d+1}(x) \end{bmatrix}.$$

The theorem is proved.

***Corollary* 2.1.2.** *For $n, m \geq 0$, the following equality is provided*:

$$(p_1(x) + i)GF_{n+m+1}(x) = GF_{n+1}(x)GF_{m+1}(x) + p_2(x)(GF_n(x)GF_m(x)) + \ldots$$
$$+ p_{d+1}(x)(GF_{n-d+1}(x)GF_m(x) + \ldots$$
$$+ GF_n(x)GF_{m-d+1}(x)).$$

Proof. From the product of matrices G_d^n and G_d^m, we get

$$G_d^n G_d^m = G_d^{n+m}.$$

The result is the first row and column of matrix G_d^{n+m}.

Lemma 2.1.1. *For $n \geq 1$ the following equality is true*:

$$GF_n = (p_1(x) + i)F_n.$$

Proof. This result can be easily proved by induction on n.

2.2. Generalization of Gaussian Lucas polynomials.

Definition 2.2.1. *d-Gaussian Lucas polynomials are defined by*

$$GL_{n+1}(x) = p_1(x)GL_n(x) + p_2(x)GL_{n-1}(x) + \ldots + p_{d+1}(x)GL_{n-d}(x) \tag{2.6}$$

with $GL_n(x) = 0$ for $n \leq 0$ and $GL_1(x) = p_1(x) + 2i$.

We give a few terms of d-Gaussian Lucas polynomials as follows:

$$GL_0(x) = 0, \qquad GL_1(x) = p_1(x) + 2i, \qquad GL_2(x) = {p_1}^2(x) + 2ip_1(x),$$

and

$$GL_3(x) = {p_1}^3(x) + 2i{p_1}^2(x) + p_1(x)p_2(x) + 2ip_2(x).$$

Theorem 2.2.1. *The generating function of $GL_n(x)$ has the form*

$$G(x,r) = \sum_{n=0}^{\infty} GL_n(x)r^n = \frac{r(p_1(x)+2i)}{1 - p_1(x)r - p_2(x)r^2 - \ldots - p_{d+1}(x)r^{d+1}}.$$

Proof. It is like that of Theorem 2.1.1.

Let us now derive the Binet formula for d-Gaussian Lucas polynomials. The Binet formula of $GL_n(x)$ has the following form:

$$GL_n(x) = \sum_{i=1}^{d+1} B_i(x)[\alpha_i(x)]^n.$$

If operations similar to Subsection 2.1 are carried out, we have the following equations:

$$\frac{r(p_1(x)+2i)}{1 - p_1(x)r - p_2(x)r^2 - \ldots - p_{d+1}(x)r^{d+1}} = \sum_{i=1}^{d+1}\left(\frac{B_i(x)}{1-\alpha_i(x)r}\right).$$

More precisely, the coefficients allow us to give the explicit form of d-Gaussian Lucas polynomials.

Theorem 2.2.2. *For $n \geq 0$, the following equality is true*:

$$GL_n(x) = (p_1(x)+2i) \times \sum_{\substack{n_1,n_2,\ldots,\, n_{d+1} \\ 1+n_1+2n_2+\ldots+(d+1)n_{d+1}=n}} \binom{n_1+n_2+\ldots+n_{d+1}}{n_1,n_2,\ldots,\, n_{d+1}} {p_1}^{n_1}(x){p_2}^{n_2}(x)\ldots{p_{d+1}}^{n_{d+1}}(x)r^n.$$

Proof. It is like that of Theorem 2.1.2.

***Corollary* 2.2.1.** *Let $SGL_n(x)$ be sum of the d-Gaussian Lucas polynomials. Then we have*

$$SGL_n(x) = \sum_{n=0}^{\infty} GL_n(x) = \frac{p_1(x)+2i}{1 - p_1(x) - p_2(x) - \ldots - p_{d+1}(x)}.$$

Proof. It is like that of Theorem 2.1.3.

Now, we give the d-Gaussian Lucas polynomials matrix L_d for $GL_n(x)$.

Definition 2.2.2. *The d-Gaussian Lucas polynomials matrix L_d is given by*

$$L_d = (p_1(x) + 2i)Q_d$$

$$= \begin{bmatrix} (p_1(x))^2 + 2ip_1(x) & p_2(x)p_1(x) + 2ip_2(x) & \dots & p_{d+1}(x)p_1(x) + 2ip_{d+1}(x) \\ p_1(x) + 2i & 0 & & 0 \\ 0 & \ddots & & 0 \\ & & \ddots & \\ 0 & 0 & p_1(x) + 2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (p_1(x))^2 + 2ip_1(x) & p_2(x)p_1(x) + 2ip_2(x) & \dots & p_{d+1}(x)p_1(x) + 2ip_{d+1}(x) \\ p_1(x) + 2i & 0 & & 0 \\ 0 & \ddots & & 0 \\ & & \ddots & \\ 0 & 0 & p_1(x) + 2i & 0 \end{bmatrix}.$$

Theorem 2.2.3. *The matrix representation for $GL_n(x)$ has the following form*:

$$L_d^n = \begin{bmatrix} GL_{n+1}(x) & p_2(x)GL_n(x) + \ldots + p_{d+1}(x)GL_{n-d+1}(x) & \dots & p_{d+1}(x)GL_n(x) \\ GL_n(x) & p_2(x)GL_{n-1}(x) + \ldots + p_{d+1}(x)GL_{n-d}(x) & \dots & p_{d+1}(x)GL_{n-1}(x) \\ \vdots & \vdots & \vdots & \vdots \\ GL_{n-d+1}(x) & p_2(x)GL_{n-d}(x) + \ldots + p_{d+1}(x)GL_{n-2d+1}(x) & \dots & p_{d+1}(x)GL_{n-d}(x) \end{bmatrix},$$

where $L_d^n = L_d^{n-1}Q_d$.

Proof. It is easily demonstrated by induction on n.

***Corollary* 2.2.2.** *For $n, m \geq 0$, the following equality is provided*:

$$\begin{aligned}(p_1(x) + 2i)GL_{n+m+1}(x) = GL_{n+1}(x)GL_{m+1}(x) + p_2(x)(GL_n(x)GL_m(x)) \\ + p_3(x)(GL_{n-1}(x)GL_m(x) + GL_n(x)GL_{m-1}(x)) + \ldots \\ + p_{d+1}(x)(GL_{n-d+1}(x)GL_m(x) + \ldots \\ + GL_n(x)GL_{m-d+1}(x)).\end{aligned}$$

Proof. From the product of matrices L_d^n and L_d^m, we have

$$L_d^{n+m} = L_d^n L_d^m.$$

The result is the first row and column of matrix L_d^{n+m}.

Lemma 2.2.1. *For $n \geq 1$, the following equality is true*:

$$2GF_n(x) - GL_n(x) = p_1(x)F_n(x).$$

Proof. It can be easily proved by induction on n.

Lemma 2.2.2. *For* $n \geq 1$, *the following equality is true*:

$$(p_1(x) + 2i)F_n(x) = GL_n(x). \tag{2.7}$$

Proof. Let us prove the result by induction on n. For $n = 1$, since $F_1(x) = 1$, we obtain

$$(p_1(x) + 2i)F_1(x) = p_1(x) + 2i = GL_1(x).$$

Now let us assume that (2.7) is true for n. Thus, we get

$$(p_1(x) + 2i)F_n(x) = GL_n(x).$$

We must show that (2.7) is true for $n + 1$:

$$\begin{aligned}(p_1(x) + 2i)F_{n+1}(x) &= (p_1(x) + 2i)(p_1(x)F_n(x) + p_2(x)F_{n-1}(x) + \ldots + p_{d+1}(x)F_{n-d}(x)) \\ &= p_1(x)(p_1(x) + 2i)F_n(x) + p_2(x)(p_1(x) + 2i)F_{n-1}(x) + \ldots \\ &\qquad + p_{d+1}(x)(p_1(x) + 2i)F_{n-d}(x) \\ &= p_1(x)GL_n(x) + p_2(x)GL_{n-1}(x) + \ldots + p_{d+1}(x)GL_n(x) \\ &= GL_{n+1}(x).\end{aligned}$$

The lemma is proved.

Proposition 2.2.1. *For* $n \geq 1$, *the* d*-Gaussian Lucas polynomials satisfy the following recurrence relation*:

$$GL_n(x) = 2GF_n(x) - F_{n+1} + p_2(x)GF_{n-1}(x) + \ldots + p_{d+1}(x)GF_{n-d}(x),$$

$$GL_n(x) = (p_1(x) + 2i)(p_1(x)F_n(x) + p_2(x)F_{n-1}(x) + \ldots + p_{d+1}(x)F_{n-d}(x)).$$

Proof is easily seen from Lemmas 2.2.1 and 2.2.2.

3. The infinite *d*-Gaussian Fibonacci and the infinite *d*-Gaussian Lucas polynomials matrix.

3.1. The infinite d-Gaussian Fibonacci polynomial matrix. Now, we can define a matrix called d-Gaussian Fibonacci polynomials matrix as in [13].

Definition 3.1.1. *The infinite* d*-Gaussian Fibonacci polynomials matrix is denoted by*

$$G\mathcal{F}(x) = \left[G\mathcal{F}_{p_1,p_2,\ldots,p_{d+1},i,j}(x)\right]$$

and defined as follows:

$$G\mathcal{F}(x) = \begin{bmatrix} p_1(x) + i & 0 & 0 & \ldots \\ (p_1(x))^2 + ip_1(x) & p_1(x) + i & 0 & \ldots \\ \begin{matrix}(p_1(x))^3 + i(p_1(x))^2 \\ + p_1(x)p_2(x) + ip_2(x)\end{matrix} & (p_1(x))^2 + ip_1(x) & p_1(x) + i & \ldots \\ t_1(x) & t_2(x) & (p_1(x))^2 + ip_1(x) & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= (g_{G\mathcal{F}(x)}(r), f_{G\mathcal{F}(x)}(r)),$$

where $t_1(x) = {p_1}^4(x) + i{p_1}^3(x) + 2{p_1}^2(x)p_2(x) + 2ip_1(x)p_2(x) + {p_1}^2(x) + ip_1(x)$ *and* $t_2(x) = {p_1}^3(x) + i{p_1}^2(x) + p_1(x)p_2(x) + ip_2(x)$.

We can write the d-Gaussian Fibonacci polynomial matrix as follows:

$$G\mathcal{F}(x) = \begin{bmatrix} GF_1(x) & 0 & 0 & \dots \\ GF_2(x) & GF_1(x) & 0 & \dots \\ GF_3(x) & GF_2(x) & GF_1(x) & \dots \\ GF_4(x) & \vdots & GF_2(x) & \dots \\ \vdots & \vdots & \vdots & \dots \end{bmatrix}.$$

Notice that the matrix $G\mathcal{F}(x)$ is a Riordan matrix.

Theorem 3.1.1. *The first column of matrix $G\mathcal{F}(x)$ has the form*

$$\big(p_1(x) + i,\ {p_1}^2(x) + ip_1(x), {p_1}^3(x) + i{p_1}^2(x) + p_1(x)p_2(x) + ip_2(x), \dots\big)^{\boldsymbol{T}}.$$

From the definition of Riordan array, the generator function of the first column is as follows:

$$g_{G\mathcal{F}(x)}(r) = \sum_{n=0}^{\infty} G\mathcal{F}_{p_1,p_2,\dots,p_{d+1},i,j}(x)r^n = \frac{p_1(x)+i}{1 - p_1(x)r - p_2(x)r^2 - \dots - p_{d+1}(x)r^{d+1}}.$$

Proof. Let us write generating functions of the first column of $G\mathcal{F}(x)$ matrix as follows:

$$\begin{aligned} &(p_1(x) + i) + \big({p_1}^2(x) + ip_1(x)\big)r \\ &\qquad + \big({p_1}^3(x) + i{p_1}^2(x) + p_1(x)p_2(x) + ip_2(x)\big)r^2 + \dots \\ &= GF_1(x) + GF_2(x)r + GF_3(x)r^2 + \dots . \end{aligned}$$

From the generator function of $GF_n(x)$, we have

$$\begin{aligned} G(x,r) &= GF_0(x) + GF_1(x)r + GF_2(x)r^2 + \dots + GF_n(x)r^n + \dots \\ &= \frac{r(p_1(x)+i)}{1 - p_1(x)r - p_2(x)r^2 - \dots - p_{d+1}(x)r^{d+1}}. \end{aligned}$$

So, we can write the following equations:

$$\begin{aligned} &r\big(GF_1(x) + GF_2(x)r + \dots + GF_n(x)r^{n-1} + \dots\big) \\ &\qquad = \frac{r(p_1(x)+i)}{1 - p_1(x)r - p_2(x)r^2 - \dots - p_{d+1}(x)r^{d+1}}, \end{aligned}$$

$$GF_1(x) + GF_2(x)r + GF_3(x)r^2 + \dots = \frac{p_1(x)+i}{1 - p_1(x)r - p_2(x)r^2 - \dots - p_{d+1}(x)r^{d+1}}.$$

Thus, the desired expression is obtained. So,

$$g_{G\mathcal{F}(x)}(r) = \sum_{n=0}^{\infty} G\mathcal{F}_{p_1,p_2,\dots,p_{d+1},i,j}(x)r^n = \frac{p_1(x)+i}{1 - p_1(x)r - p_2(x)r^2 - \dots - p_{d+1}(x)r^{d+1}}.$$

The theorem is proved.

From the Riordan matrix, we have $f_{G\mathcal{F}(x)}(r) = r$. Then we write $G\mathcal{F}(x)$ as follows:

$$G\mathcal{F}(x) = \big(g_{G\mathcal{F}(x)}(r), f_{G\mathcal{F}(x)}(r)\big) = \left(\frac{p_1(x)+i}{1-p_1(x)r-p_2(x)r^2-\ldots-p_{d+1}(x)r^{d+1}}, r\right).$$

If the Gaussian Fibonacci polynomial matrix $G\mathcal{F}(x)$ is finite, then the matrix is

$$G\mathcal{F}_f(x) = \begin{bmatrix} GF_1(x) & 0 & 0 & \ldots \\ GF_2(x) & GF_1(x) & 0 & \ldots \\ GF_3(x) & GF(x) & GF_1(x) & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ GF_n(x) & GF_{n-1}(x) & \ldots & GF_1(x) \end{bmatrix}$$

and

$$\det G\mathcal{F}_f(x) = |G\mathcal{F}_f(x)| = (GF_1(x))^n.$$

Now, we present two factorizations of Pascal matrix including the d-Gaussian Fibonacci polynomials matrix. We need to find two matrices for these factorizations. Firstly, we define a matrix $C(x) = \dfrac{1}{p_1(x)+i}(c_{i,j}(x))$ as follows:

$$c_{i,j}(x) = \binom{i-1}{j-1} - p_1(x)\binom{i-2}{j-1} - p_2(x)\binom{i-3}{j-1} - \ldots - p_{d+1}(x)\binom{i-d-2}{j-1}.$$

So, we obtain

$$C(x) = \begin{bmatrix} \dfrac{1}{p_1(x)+i} & 1 & 0 & \ldots \\ \dfrac{1-p_1(x)}{p_1(x)+i} & \dfrac{1}{p_1(x)+i} & 0 & \ldots \\ \dfrac{1-p_1(x)-p_2(x)}{p_1(x)+i} & \dfrac{2-p_1(x)}{p_1(x)+i} & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ k_1(x) & k_3(x) & \ldots & \ldots \\ k_2(x) & k_4(x) & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \frac{1}{p_1(x)+i}\begin{bmatrix} 1 & 0 & 0 & \ldots \\ 1-p_1(x) & 1 & 0 & \ldots \\ 1-p_1(x)-p_2(x) & 2-p_1(x) & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ k_1(x) & k_3(x) & \ldots & \ldots \\ k_2(x) & k_4(x) & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{3.1}$$

where

$$k_1(x) = \frac{1-p_1(x)-p_2(x)-\ldots-p_d(x)}{p_1(x)+i},$$

$$k_2(x) = \frac{d-(d-1)p_1(x)-(d-2)p_2(x)-\ldots-p_{d-1}(x)}{p_1(x)+i},$$

$$k_3(x) = \frac{1-p_1(x)-p_2(x)-\ldots-p_{d+1}(x)}{p_1(x)+i},$$

and

$$k_4(x) = \frac{(d+1)-dp_1(x)-(d-1)p_2(x)-\ldots-p_d(x)}{p_1(x)+i}.$$

By using the infinite d-Gaussian Fibonacci matrix and the infinite $C(x)$ matrix as in (3.1), we can present the first factorization of the infinite Pascal matrix with the following theorem.

Theorem 3.1.2. *The factorization of the infinite Pascal matrix is as follows*:

$$P(x) = G\mathcal{F}(x) * C(x).$$

Proof. From the definitions of infinite Pascal matrix and the infinite d-Gaussian Fibonacci polynomials matrix, we have the following Riordan representing:

$$P = \left(\frac{1}{1-r}, \frac{r}{1-r}\right), \qquad G\mathcal{F}(x) = \left(\frac{p_1(x)+i}{1-p_1(x)r-p_2(x)r^2-\ldots-p_{d+1}(x)r^{d+1}}, r\right).$$

Now, we can obtain the Riordan representation of the infinite matrix $C(x) = \big(g_{C(x)}(r), f_{C(x)}(r)\big)$ as follows:

$$C(x) = \begin{bmatrix} \dfrac{1}{p_1(x)+i} & 1 & 0 & \ldots \\ \dfrac{1-p_1(x)}{p_1(x)+i} & \dfrac{1}{p_1(x)+i} & 0 & \ldots \\ \dfrac{1-p_1(x)-p_2(x)}{p_1(x)+i} & \dfrac{2-p_1(x)}{p_1(x)+i} & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ k_1(x) & k_3(x) & \ldots & \ldots \\ k_2(x) & k_4(x) & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where

$$k_1(x) = \frac{1 - p_1(x) - p_2(x) - \ldots - p_d(x)}{p_1(x) + i},$$

$$k_2(x) = \frac{d - (d-1)p_1(x) - (d-2)p_2(x) - \ldots - p_{d-1}(x)}{p_1(x) + i},$$

$$k_3(x) = \frac{1 - p_1(x) - p_2(x) - \ldots - p_{d+1}(x)}{p_1(x) + i},$$

and

$$k_4(x) = \frac{(d+1) - dp_1(x) - (d-1)p_2(x) - \ldots - p_d(x)}{p_1(x) + i}.$$

From the first column of the matrix $C(x)$, we get

$$g_{C(x)}(r) = \frac{1}{p_1(x) + i}\left(\frac{1 - p_1(x)r - p_2(x)r^2 - \ldots - p_{d+1}(x)r^{d+1}}{1 - r}\right).$$

From the rule of the matrix $C(x)$, we have

$$f_{C(x)}(r) = \frac{r}{1 - r}.$$

So,

$$C(x) = \big(g_{C(x)}(r), f_{C(x)}(r)\big).$$

The theorem is proved.

Now, we define a matrix $D(x) = \dfrac{1}{p_1(x) + i}(d_{i,j}(x))$ as follows:

$$d_{i,j}(x) = \binom{i-1}{j-1} - p_1(x)\binom{i-2}{j} - p_2(x)\binom{i-3}{j+1} - \ldots - p_{d+1}(x)\binom{i-d-2}{j+d}.$$

We give the infinite $D(x)$ by

$$D(x) = \frac{1}{p_1(x) + i}\begin{bmatrix} 1 & 0 & 0 & \ldots \\ 1 - p_1(x) & 1 & 0 & \ldots \\ 1 - 2p_1(x) - p_2(x) & 2 - p_1(x) & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ l_1(x) & l_3(x) & \ldots & \ldots \\ l_2(x) & l_4(x) & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{3.2}$$

where

$$l_1(x) = \frac{1 - dp_1(x) - \dfrac{d(d-1)}{2!}p_2(x) - \ldots - p_d(x)}{p_1(x) + i},$$

$$l_2(x) = \frac{d - (d-1)p_1(x) - (d-2)p_2(x) - \ldots - p_{d-1}(x)}{p_1(x) + i},$$

$$l_3(x) = \frac{1 - (d+1)p_1(x) - \dfrac{d(d-1)}{2!}p_2(x) - \ldots - p_d(x)}{p_1(x) + i},$$

and

$$l_4(x) = \frac{(d+1) - dp_1(x) - (d-1)p_2(x) - \ldots - p_d(x)}{p_1(x) + i}.$$

Now, we present another factorization of the Pascal matrix by the following corollary.

***Corollary* 3.1.1.** *The factorization of the infinite Pascal matrix is as follows*:

$$P(x) = G\mathcal{F}(x) * D(x),$$

where $D(x)$ *is the matrix in* (3.2).

Proof. It is like that of Theorem 3.1.2.

Now, we can find the inverse of d-Gaussian Fibonacci polynomials matrix by using the Riordan representation given matrices as [14].

***Corollary* 3.1.2.** *The inverse of* d*-Gaussian Fibonacci polynomials matrix is given by the following*:

$$G\mathcal{F}^{-1}(x) = \left(\frac{1 - p_1(x)r - p_2(x)r^2 - \ldots - p_{d+1}(x)r^{d+1}}{p_1(x) + i}, r \right).$$

3.2. The infinite d-Gaussian Lucas polynomials matrix. In this subsection, we define a new matrix called d-Gaussian Lucas polynomials matrix.

Definition 3.2.1. *The infinite* d*-Gaussian Lucas polynomials matrix is denoted by*

$$G\mathcal{L}(x) = [G\mathcal{L}_{p_1,p_2,\ldots,p_{d+1},i,j}(x)]$$

and defined as follows:

$$G\mathcal{L}(x) = \begin{bmatrix} p_1(x) + 2i & 0 & 0 & \ldots \\ (p_1(x))^2 + 2ip_1(x) & p_1(x) + 2i & 0 & \ldots \\ \begin{matrix} (p_1(x))^3 + 2i(p_1(x))^2 \\ + p_1(x)p_2(x) + 2ip_2(x) \end{matrix} & (p_1(x))^2 + 2ip_1(x) & p_1(x) + 2i & \ldots \\ t_1(x) & t_2(x) & (p_1(x))^2 + 2ip_1(x) & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= (g_{\mathcal{L}(x)}(r), f_{\mathcal{L}(x)}(r)),$$

where $t_1(x) = {p_1}^4(x) + 2i{p_1}^3(x) + 2{p_1}^2(x)p_2(x) + 4ip_1(x)p_2(x) + p_1(x)p_3(x) + 2ip_3(x)$ *and* $t_2(x) = {p_1}^3(x) + 2i{p_1}^2(x) + p_1(x)p_2(x) + 2ip_2(x)$.

This Gaussian Lucas polynomial matrix can also be written as

$$G\mathcal{L}(x) = \begin{bmatrix} GL_1(x) & 0 & 0 & \dots \\ GL_2(x) & GL_1(x) & 0 & \dots \\ GL_3(x) & GL_2(x) & GL_1(x) & \dots \\ GL_4(x) & \dots & GL_2(x) & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Note that $G\mathcal{L}(x)$ is a Riordan matrix.

Theorem 3.2.1. *The first column of matrix* $G\mathcal{L}(x)$ *is*

$$\big(p_1(x) + 2i,\ {p_1}^2(x) + 2ip_1(x), {p_1}^3(x) + 2i{p_1}^2(x) + p_1(x)p_2(x) + ip_2(x), \dots\big)^{\boldsymbol{T}}.$$

From the definition of Riordan array, the generator function of the first column is as follows:

$$g_{G\mathcal{L}(x)}(r) = \sum_{n=0}^{\infty} G\mathcal{L}_{p_1,p_2,\dots,p_{d+1},i,j}(x) r^n = \frac{p_1(x) + 2i}{1 - p_1(x)r - p_2(x)r^2 - \dots - p_{d+1}(x)r^{d+1}}.$$

Proof. The proof is done analogously to that of Theorem 3.1.1.

So, we get

$$f_{G\mathcal{L}(x)}(r) = r.$$

Then we write $G\mathcal{L}(x)$ as follows:

$$G\mathcal{L}(x) = \big(g_{G\mathcal{L}(x)}(r), f_{G\mathcal{L}(x)}(r)\big).$$

The theorem is proved.

If the Gaussian Lucas polynomials matrix $G\mathcal{L}(x)$ is finite, then the matrix is

$$G\mathcal{L}_f(x) = \begin{bmatrix} GL_1(x) & 0 & 0 & \dots \\ GL_2(x) & GL_1(x) & 0 & \dots \\ GL_3(x) & GL_2(x) & GL_1(x) & \dots \\ \vdots & \vdots & \vdots & \vdots \\ GL_n(x) & GL_{n-1}(x) & \dots & GL_1(x) \end{bmatrix}$$

and

$$\det G\mathcal{L}_f(x) = |G\mathcal{L}_f(x)| = (GL_1(x))^n.$$

Now, we give two factorization of Pascal matrix including the d-Gaussian Lucas polynomials matrix. We need to find two matrices for these factorizations. For that, we define a matrix $C^*(x) = \dfrac{1}{p_1(x) + 2i}(c_{i,j}(x))$ as follows:

$$c_{i,j}(x) = \binom{i-1}{j-1} - p_1(x)\binom{i-2}{j-1} - p_2(x)\binom{i-3}{j-1} - \dots - p_{d+1}(x)\binom{i-d-2}{j-1}.$$

Thus, we obtain

$$C^*(x) = \begin{bmatrix} \frac{1}{p_1(x)+2i} & 0 & 0 & \dots \\ \frac{1-p_1(x)}{p_1(x)+2i} & \frac{1}{p_1(x)+2i} & 0 & \dots \\ \frac{1-p_1(x)-p_2(x)}{p_1(x)+2i} & \frac{2-p_1(x)}{p_1(x)+2i} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ l_1(x) & l_3(x) & \dots & \dots \\ l_2(x) & l_4(x) & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \frac{1}{p_1(x)+2i} \begin{bmatrix} 1 & 0 & 0 & \dots \\ 1-p_1(x) & 1 & 0 & \dots \\ 1-2p_1(x)-p_2(x) & 2-p_1(x) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ l_1(x) & l_3(x) & \dots & \dots \\ l_2(x) & l_4(x) & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{3.3}$$

where

$$l_1(x) = \frac{1-p_1(x)-p_2(x)-\ldots-p_d(x)}{p_1(x)+2i},$$

$$l_2(x) = \frac{d-(d-1)p_1(x)-(d-2)p_2(x)-\ldots-p_{d-1}(x)}{p_1(x)+2i},$$

$$l_3(x) = \frac{1-p_1(x)-p_2(x)-\ldots-p_{d+1}(x)}{p_1(x)+2i},$$

and

$$l_4(x) = \frac{(d+1)-dp_1(x)-(d-1)p_2(x)-\ldots-p_d(x)}{p_1(x)+2i}.$$

By using the infinite d-Gaussian Lucas matrix and the infinite $C^*(x)$ matrix as in (3.3), we introduce the first factorization of the infinite Pascal matrix with the following theorem.

Theorem 3.2.2. *The factorization of the infinite Pascal matrix is as follows*:

$$P(x) = G\mathcal{L}(x) * C^*(x).$$

Proof. From the definitions of infinite Pascal matrix and the infinite d-Gaussian Lucas polynomials matrix, we have the following Riordan representing:

$$P = \left(\frac{1}{1-r}, \frac{r}{1-r}\right), \qquad G\mathcal{L}(x) = \left(\frac{p_1(x)+2i}{1-p_1(x)r-p_2(x)r^2-\ldots-p_{d+1}(x)r^{d+1}}, r\right).$$

Now, we can obtain the Riordan representation the infinite matrix $C^*(x)$ as follows:

$$C^*(x) = \big(g_{C^*(x)}(r), f_{C^*(x)}(r)\big),$$

$$C^*(x) = \begin{bmatrix} \dfrac{1}{p_1(x)+2i} & 0 & 0 & \ldots \\ \dfrac{1-p_1(x)}{p_1(x)+2i} & \dfrac{1}{p_1(x)+2i} & 0 & \ldots \\ \dfrac{1-p_1(x)-p_2(x)}{p_1(x)+2i} & \dfrac{2-p_1(x)}{p_1(x)+2i} & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ l_1(x) & l_3(x) & \ldots & \ldots \\ l_2(x) & l_4(x) & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where

$$l_1(x) = \frac{1-p_1(x)-p_2(x)-\ldots-p_d(x)}{p_1(x)+2i},$$

$$l_2(x) = \frac{d-(d-1)p_1(x)-(d-2)p_2(x)-\ldots-p_{d-1}(x)}{p_1(x)+2i},$$

$$l_3(x) = \frac{1-p_1(x)-p_2(x)-\ldots-p_{d+1}(x)}{p_1(x)+2i},$$

and

$$l_4(x) = \frac{(d+1)-dp_1(x)-(d-1)p_2(x)-\ldots-p_d(x)}{p_1(x)+2i}.$$

From the first column of matrix $C^*(x)$, we get

$$g_{C^*(x)}(r) = \frac{1}{p_1(x)+2i}\left(\frac{1-p_1(x)r-p_2(x)r^2-\ldots-p_{d+1}(x)r^{d+1}}{1-r}\right).$$

From the rule of the $C^*(x)$, we write $f_{C^*(x)}(r)$ as follows:

$$f_{C^*(x)}(r) = \frac{r}{1-r}.$$

So,

$$C^*(x) = \big(g_{C^*(x)}(r), f_{C^*(x)}(r)\big).$$

The theorem is proved.

Now, we define a matrix $D^*(x) = \dfrac{1}{p_1(x)+2i}(d_{i,j}(x))$ as follows:

$$d_{i,j}(x) = \binom{i-1}{j-1} - p_1(x)\binom{i-2}{j} - p_2(x)\binom{i-3}{j+1} - \ldots - p_{d+1}(x)\binom{i-d-2}{j+d}.$$

We give the infinite matrix $D^*(x)$ with

$$D^*(x) \begin{bmatrix} \dfrac{1}{p_1(x)+2i} & 0 & 0 & \ldots \\ \dfrac{1-p_1(x)}{p_1(x)+2i} & \dfrac{1}{p_1(x)+2i} & 0 & \ldots \\ \dfrac{1-2p_1(x)-p_2(x)}{p_1(x)+2i} & \dfrac{2-p_1(x)}{p_1(x)+2i} & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ l_1(x) & l_3(x) & \ldots & \ldots \\ l_2(x) & l_4(x) & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \frac{1}{p_1(x)+2i} \begin{bmatrix} 1 & 0 & 0 & \ldots \\ 1-p_1(x) & 1 & 0 & \ldots \\ 1-2p_1(x)-p_2(x) & 2-p_1(x) & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ l_1(x) & l_3(x) & \ldots & \ldots \\ l_2(x) & l_4(x) & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{3.4}$$

where

$$l_1(x) = \frac{1 - dp_1(x) - \dfrac{d(d-1)}{2!}p_2(x) - \ldots - p_d(x)}{p_1(x)+2i},$$

$$l_2(x) = \frac{d - (d-1)p_1(x) - (d-2)p_2(x) - \ldots - p_{d-1}(x)}{p_1(x)+2i},$$

$$l_3(x) = \frac{1 - (d+1)p_1(x) - \dfrac{d(d-1)}{2!}p_2(x) - \ldots - p_d(x)}{p_1(x)+2i},$$

and

$$l_4(x) = \frac{(d+1) - dp_1(x) - (d-1)p_2(x) - \ldots - p_d(x)}{p_1(x)+2i}.$$

***Corollary* 3.2.1.** *The factorization of the infinite Pascal matrix is as follows*:

$$P(x) = G\mathcal{L}(x) * D^*(x).$$

Proof. It is similar to that of Theorem 3.2.2.

Now, we can find the inverse of d-Gaussian Lucas polynomials matrix by using the Riordan representation given matrices as in [14].

***Corollary* 3.2.2.** *The inverse of d-Gaussian Lucas polynomials matrix is given by the following*:

$$GL^{-1}(x) = \left(\frac{1 - p_1(x)r - p_2(x)r^2 - \ldots - p_{d+1}(x)r^{d+1}}{p_1(x) + 2i}, r \right).$$

4. Conclusions. We defined d-Gaussian Fibonacci polynomials and d-Gaussian Lucas polynomials. We gave the matrix representations of d-Gaussian Fibonacci and d-Gaussian Lucas polynomials. Using the Riordan method, we obtained the factorizations of the Pascal matrix including these polynomials. In addition, we defined d-Gaussian Fibonacci polynomials matrix and d-Gaussian Lucas polynomials matrix and gave their inverses.

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