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# GENERAL LOCAL COHOMOLOGY MODULES IN VIEW OF LOW POINTS AND HIGH POINTS

# ЗАГАЛЬНІ ЛОКАЛЬНІ КОГОМОЛОГІЧНІ МОДУЛІ З ТОЧКИ ЗОРУ НИЗЬКИХ І ВИСОКИХ ТОЧОК

Let $R$ be a commutative Noetherian ring, let $\Phi$ be a system of ideals of $R$, let $M$ be a finitely generated $R$-module, and let $t$ be a nonnegative integer. We first show that a general local cohomology module $H^i_\Phi(M)$ is a finitely generated $R$-module for all $i < t$ if and only if $\mathrm{Ass}_R(H^i_\Phi(M))$ is a finite set and $H^i_{\Phi_\mathfrak{p}}(M_\mathfrak{p})$ is a finitely generated $R_\mathfrak{p}$-module for all $i < t$ and all $\mathfrak{p} \in \mathrm{Spec}(R)$. Then, as a consequence, we prove that if $(R, \mathfrak{m})$ is a complete local ring, $\Phi$ is countable, and $n \in \mathbb{N}_0$ is such that $\big(\mathrm{Ass}_R\big(H_\Phi^{h^n_\Phi(M)}(M)\big)\big)_{\geq n}$ is a finite set, then $f^n_\Phi(M) = h^n_\Phi(M)$. In addition, we show that the properties of vanishing and finiteness of general local cohomology modules are equivalent on high points over an arbitrary Noetherian (not necessary local) ring. For each covariant $R$-linear functor $T$ from $\mathrm{Mod}(R)$ into itself, which has the global vanishing property on $\mathrm{Mod}(R)$ and for an arbitrary Serre subcategory $\mathcal{S}$ and $t \in \mathbb{N}$, we prove that $\mathcal{R}^iT(R) \in \mathcal{S}$ for all $i \geq t$ if and only if $\mathcal{R}^iT(M) \in \mathcal{S}$ for any finitely generated $R$-module $M$ and all $i \geq t$. Then we obtain some results on general local cohomology modules.

Нехай $R$ — комутативне нетерове кільце, $\Phi$ — система ідеалів для $R$, $M$ — скінченнопороджений $R$-модуль, а $t$ — невід'ємне ціле число. Спочатку показано, що загальний локальний когомологічний модуль $H^i_\Phi(M)$ є скінченнопородженим $R$-модулем для всіх $i < t$ тоді й лише тоді, коли $\mathrm{Ass}_R(H^i_\Phi(M))$ є скінченною множиною, а $H^i_{\Phi_\mathfrak{p}}(M_\mathfrak{p})$ — скінченнопородженим $R_\mathfrak{p}$-модулем для всіх $i < t$ і всіх $\mathfrak{p} \in \mathrm{Spec}(R)$. Далі, як наслідок, доведено, що якщо $(R, \mathfrak{m})$ є повним локальним кільцем, $\Phi$ — зліченним, а $n \in \mathbb{N}_0$ — таким, що $\big(\mathrm{Ass}_R\big(H_\Phi^{h^n_\Phi(M)}(M)\big)\big)_{\geq n}$ є скінченною множиною, то $f^n_\Phi(M) = h^n_\Phi(M)$. Крім того, показано, що властивості спадання і скінченності загальних локальних когомологічних модулів еквівалентні у високих точках над довільним нетеровим (необов'язково локальним) кільцем. Для кожного коваріантного $R$-лінійного функтора $T$ з $\mathrm{Mod}(R)$ в себе, який має глобальну властивість спадання на $\mathrm{Mod}(R)$, і для довільної підкатегорії Серра $\mathcal{S}$ і $t \in \mathbb{N}$ доведено, що $\mathcal{R}^iT(R) \in \mathcal{S}$ для всіх $i \geq t$ тоді й лише тоді, коли $\mathcal{R}^iT(M) \in \mathcal{S}$ для будь-якого скінченнопородженого $R$-модуля $M$ і всіх $i \geq t$. Отримано деякі результати щодо загальних локальних когомологічних модулів.

**1. Introduction.** Throughout this article, $R$ denotes a commutative Noetherian ring with non-zero identity and $M$ denotes an $R$-module. We use $\mathbb{N}_0$ to denote the set of nonnegative integers and $\mathrm{Mod}(R)$ to denote the category of all $R$-modules and $R$-homomorphisms. Also, we use $\mathrm{V}(\mathfrak{a})$ to denote the variety of an ideal $\mathfrak{a}$ of $R$ and $\mathrm{Max}(R)$ to denote the set of maximal ideals of $R$. For any subset $Y$ of $\mathrm{Spec}(R)$, we set $(Y)_{\geq n} := \big\{\mathfrak{p} \in Y \,|\, \dim R/\mathfrak{p} \geq n\big\}$. Let $\Phi$ be a nonempty set of ideals of $R$. We recall that $\Phi$ is a system of ideals of $R$ if, whenever $\mathfrak{a}$, $\mathfrak{b} \in \Phi$, then there is an ideal $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$.

For an $R$-module $M$ and an ideal $\mathfrak{a}$ of $R$, the ith local cohomology module of $M$ with respect to $\mathfrak{a}$ is defined as

$$H^i_\mathfrak{a}(M) \cong \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}^i_R\big(R/\mathfrak{a}^n, M\big).$$

As a generalization of these modules, for a system of ideals $\Phi$ of $R$, Bijanzadeh in [7] defined submodule $\Gamma_\Phi(M)$ of $M$ as follows:

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$$\Gamma_\Phi(M) = \{x \in M |\ \mathfrak{a}x = 0 \quad \text{for some} \quad \mathfrak{a} \in \Phi\}.$$

Then $\Gamma_\Phi(-)$ is a covariant, $R$-linear and left exact functor from $\mathrm{Mod}(R)$ to itself. The author in [7] denoted the functor $\Gamma_\Phi(-)$ by $\mathrm{L}_\Phi(-)$ and called the *general local cohomology functor with respect to* $\Phi$. For each $i \geq 0$, the $i$th right derived functor of $\Gamma_\Phi(-)$ is denoted by $H^i_\Phi(-)$. For an ideal $\mathfrak{a}$ of $R$, if $\Phi = \{\mathfrak{a}^i | i > 0\}$, then the functor $H^i_\Phi(-)$ coincides with the ordinary local cohomology functor $H^i_\mathfrak{a}(-)$. From now on, we refer to $H^i_\Phi(M)$ as the general local cohomology module. Some introductory properties of the functors $\Gamma_\Phi(-)$ and $H^i_\Phi(-)$ that will be used throughout this article are collected in Proposition 2.1. For more information on the ordinary local cohomology and its generalization on system of ideals, the reader is referred to [6, 7, 9].

It is well-known that one of the most important theorems in local cohomology is Faltings' theorem [12, Satz (1)], which is another formulation of the finiteness dimension $f_\mathfrak{a}(M)$ of $M$ relative to $\mathfrak{a}$, as follows:

$$\begin{aligned} f_\mathfrak{a}(M) &= \inf\{i \geq 0 |\ H^i_\mathfrak{a}(M) \text{ is not finitely generated}\} \\ &= \inf\{f_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}) |\ \mathfrak{p} \in \mathrm{Spec}(R)\}. \end{aligned}$$

This theorem is known *Local-global principal for finiteness dimension of local cohomology modules* [9, 9.6.2]. This motivated Bahmanpour et al. in [5] to define the *n*th finiteness dimension $f^n_\mathfrak{a}(M)$ of $M$ relative to $\mathfrak{a}$ by

$$f^n_\mathfrak{a}(M) := \inf\{f_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}) |\ \mathfrak{p} \in \mathrm{Supp}_R(M/\mathfrak{a}M) \text{ and } \dim\ R/\mathfrak{p} \geq n\}$$

for any nonnegative integer $n$, that is, a generalization of the finiteness dimension $f_\mathfrak{a}(M)$. Then Asadollahi et al. in [3] introduced the class of in dimension $< n$ modules. They showed that on a complete local ring $R$, for any finitely generated $R$-module $M$ and ideal $\mathfrak{a}$ of $R$

$$f^n_\mathfrak{a}(M) = \inf\{i \geq 0 | H^i_\mathfrak{a}(M) \text{ is not in dimension} < n\},$$

which is the most important result of [3].

In this paper, we investigate some properties of local cohomology modules generalized for an arbitrary system of ideals on points which we name *low points* and *high points*. For this purpose, we divide this article in two sections as follows:

In Section 2, we study general local cohomology modules on low points. By low points, we mean study of some properties of $H^i_\Phi$ for $0 \leq i \leq r-1$, $r \in \mathbb{N}$. One of our main results of this section is a generalization of the main theorem of [3] on a system of ideals. To achieve this goal, we first generalize Faltings' theorem for an arbitrary system of ideals as follows (Theorem 2.2).

**Theorem 1.1.** *Let* $\Phi$ *be a system of ideals of* $R$, $M$ *be a finite* $R$*-module, and* $t \in \mathbb{N}$. *Then the following conditions are equivalent*:

(i) $H^i_\Phi(M)$ *is finitely generated module for all* $i < t$;

(ii) $\mathrm{Ass}_R(H^i_\Phi(M))$ *is a finite set and* $H^i_{\Phi_\mathfrak{p}}(M_\mathfrak{p})$ *is a finitely generated* $R_\mathfrak{p}$*-module for all* $i < t$ *and all* $\mathfrak{p} \in \mathrm{Spec}(R)$, *where* $\Phi_\mathfrak{p} := \{\mathfrak{a}R_\mathfrak{p} |\ \mathfrak{a} \in \Phi\}$.

Using Theorem 1.1, we deduce the following result. The procedure of the proof is notable (Theorem 2.3).

**Theorem 1.2.** *Let* $(R, \mathfrak{m})$ *be a complete local ring,* $\Phi$ *be a countable system of ideals of* $R$, *and* $M$ *be a finitely generated* $R$*-module. Assume that there exists* $n \in \mathbb{N}_0$ *such that the set* $\big(\mathrm{Ass}\,(H_\Phi^{h_\Phi^n(M)}(M))\big)_{\geq n}$ *is finite. Then* $f_\Phi^n(M) = h_\Phi^n(M)$.

In Section 3, we study general local cohomology modules on high points. By high points, we mean study of some properties of $H_\Phi^i$ for $s \leq i \leq \dim M$, $s \in \mathbb{N}$. As the first main result of this section, we present a relationship between the vanishing and finiteness of generalized local cohomology modules on an arbitrary Noetherian ring at high points (Theorem 3.1). This is proved by Yoshida in [17, Proposition 3.1] for the ordinary local cohomology modules on local rings.

**Theorem 1.3.** *Let* $R$ *be an arbitrary Noetherian ring*, $M$ *be an* $R$*-module with finite dimension*, *and* $t \in \mathbb{N}$. *The following conditions are equivalent*:

(i) $H_\Phi^i(M) = 0$ *for all* $i \geq t$;

(ii) $H_\Phi^i(M)$ *is finitely generated for all* $i \geq t$;

(iii) *there exists an ideal* $\mathfrak{a}$ *in* $\Phi$ *such that* $\mathfrak{a}H_\Phi^i(M) = 0$ *for all* $i \geq t$.

Consequently, we show that minimaxness and Artinianness of general local cohomology modules are equivalent on high points as follows (Proposition 3.1). Recall that an $R$-module $M$ is ZD-module if for any submodule $N$ of $M$ the set of zero divisors of $M/N$ is a union of finitely many prime ideals in $\mathrm{Ass}_R(M/N)$ (see [10, 11]).

**Proposition 1.1.** *Let* $M$ *be a finite dimensional* $ZD$*-module and* $t \in \mathbb{N}$. *Then* $H_\Phi^i(M)$ *is Artinian* $R$*-module for all* $i \geq t$ *if and only if* $H_\Phi^i(M)$ *is minimax for all* $i \geq t$.

As the other main result of this section, we prove the following theorem (Theorem 3.2).

**Theorem 1.4.** *Let* $T\colon \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(R)$ *be a covariant*, $R$*-linear functor satisfies the global vanishing property on* $\mathrm{Mod}(R)$. *Let* $\mathcal{S}$ *be an arbitrary Serre subcategory and* $t \in \mathbb{N}$. *Then* $\mathcal{R}^iT(R) \in S$ *for all* $i \geq t$ *if and only if* $\mathcal{R}^iT(M) \in \mathcal{S}$ *for all finitely generated* $R$*-module* $M$ *and all* $i \geq t$.

Finally, as applications of above theorem, we obtain some results on general local cohomology modules and their vanishing of their tensor products on high points (Corollaries 3.2, 3.3, 3.4 and Proposition 3.2).

**2. General local cohomology on low points.** As it is mentioned in the introduction, in this section, we generalize some of main results of the ordinary local cohomology to the general local cohomology on low points. To this end, we need some preliminaries.

**Definition 2.1.** *An* $R$*-module* $M$ *is called a* $\Phi$*-torsion module whenever* $\Gamma_\Phi(M) = M$ *and is called a* $\Phi$*-torsion-free module whenever* $\Gamma_\Phi(M) = 0$.

Some properties of $\Gamma_\Phi(-)$ and $H_\Phi^i(-)$ are as follows.

**Proposition 2.1.** *Let* $M$ *be an* $R$*-module. Then*:

(i) $M/\Gamma_\Phi(M)$ *is* $\Phi$*-torsion-free.*

(ii) *If* $M$ *is an injective* $R$*-module*, *then* $\Gamma_\Phi(M)$ *is an injective* $R$*-module.*

(iii) *If* $M$ *is a* $\Phi$*-torsion* $R$*-module*, *then there exists an injective resolution of* $M$ *in which each term is a* $\Phi$*-torsion* $R$*-module.*

(iv) *If* $M$ *is a* $\Phi$*-torsion* $R$*-module*, *then* $H_\Phi^i(M) = 0$ *for all* $i > 0$. *Specially*, $H_\Phi^i(\Gamma_\Phi(M)) = 0$ *for all* $i > 0$.

(v) $H_\Phi^i(M) \cong H_\Phi^i(M/\Gamma_\Phi(M))$ *for all* $i > 0$.

(vi) $H_\Phi^i(M) \cong \varinjlim_{\mathfrak{a}\in\Phi} H_\mathfrak{a}^i(M)$ *for all* $i \geq 0$.

(vii) *For any multiplicatively closed subset $S$ of $R$, $S^{-1}(H^i_\Phi(M)) \cong H_{S^{-1}\Phi}(S^{-1}M)$ for all $i \geq 0$, in which $S^{-1}\Phi = \{S^{-1}\mathfrak{a} \mid \mathfrak{a} \in \Phi\}$.*

(viii) *For all $i \in \mathbb{N}_0$, the local cohomology functor $H^i_\Phi$ commutes with direct limits. More precisely, let $\{M_\alpha\}_{\alpha\in\Lambda}$ be a direct system of $R$-modules over the directed partially ordered set $(\Lambda, \leq)$. Then $H^i_\Phi(\varinjlim_{\alpha\in\Lambda} M_\alpha) \cong \varinjlim_{\alpha\in\Lambda} H^i_\Phi(M_\alpha)$ for all $i \geq 0$.*

***Proof.*** For (i) – (v) see [13, Lemma 2.4] and for (vi) – (viii) see [7, Lemma 2.1] and [6, Propositions 2.4, 2.6].

For the next proposition, let $F, G\colon \operatorname{Mod}(R) \longrightarrow \operatorname{Mod}(R)$ be two left exact covariant functors. For all $n \geq 1$, we denote the $n$th right derived functor of $F, G$ and their composite $FG$ by $F^n, G^n$ and $(FG)^n$, respectively. Also assume that $\mathcal{S}$ is a Serre subcategory of $\operatorname{Mod}(R)$. Recall that a subcategory $\mathcal{S}$ of $\operatorname{Mod}(R)$ is called a Serre subcategory, if it is closed under taking submodules, quotients and extensions. In other words, for any short exact sequence $0 \to N \to M \to L \to 0$ of $R$-modules and $R$-homomorphisms, $M \in \mathcal{S}$ if and only if $N, L \in \mathcal{S}$.

The following proposition plays a main role on this section which is useful to achieve some of the results of this section.

**Proposition 2.2.** *Let $M$ be an $R$-module, $\mathfrak{a}$ be an ideal of $R$, and $\mathcal{S}$ be a Serre subcategory of $\operatorname{Mod}(R)$. Let $G\colon \operatorname{Mod}(R) \longrightarrow \operatorname{Mod}(R)$ be a left exact covariant functor such that $\left(0 \underset{N}{:} \mathfrak{a}\right) = \left(0 \underset{G(N)}{:} \mathfrak{a}\right)$ for all $R$-modules $N$. Suppose that $G(E)$ is an injective $R$-module for all injective $R$-modules $E$. For $t \in \mathbb{N}$, consider natural homomorphism*

$$\psi\colon \operatorname{Ext}^t_R(R/\mathfrak{a}, M) \longrightarrow \operatorname{Hom}_R\left(R/\mathfrak{a}, G^t(M)\right).$$

*Then we have the following*:

(i) *If $\operatorname{Ext}^{t-j}_R\left(R/\mathfrak{a}, G^j(M)\right) \in \mathcal{S}$ for all $j < t$, then $\operatorname{Ker}\psi \in \mathcal{S}$.*

(ii) *If $\operatorname{Ext}^{t+1-j}_R\left(R/\mathfrak{a}, G^j(M)\right) \in \mathcal{S}$ for all $j < t$, then $\operatorname{Coker}\psi \in \mathcal{S}$.*

(iii) *If $\operatorname{Ext}^{n-j}_R\left(R/\mathfrak{a}, G^j(M)\right) \in \mathcal{S}$ for $t = n, n+1$ and for all $j < t$, then $\operatorname{Ker}\psi$ and $\operatorname{Coker}\psi$ both belong to $\mathcal{S}$. Thus, $\operatorname{Ext}^t_R(R/\mathfrak{a}, M) \in \mathcal{S}$ if and only if $\operatorname{Hom}_R\left(R/\mathfrak{a}, G^t(M)\right) \in \mathcal{S}$.*

***Proof.*** Let $F(-) = \operatorname{Hom}_R(R/\mathfrak{a}, -)$. Then for any $R$-module $M$, we have $FG(M) = F(M)$. Now, the assertion follows from [1, Proposition 3.1].

In [15, Definition 2.1], the author introduces FSF modules and some of their properties and applications. An $R$-module $M$ is an FSF module if there is a finitely generated submodule $N$ of $M$ such that the quotient module $M/N$ has finite support.

Also, the authors in [3], introduce the class of $R$-modules in dimension $< n$ for any nonnegative integer $n$. An $R$-module $M$ is called in dimension $< n$ if there is a finitely generated submodule $N$ of $M$ such that $\dim\operatorname{Supp}_R(M/N) < n$.

Note that when $R$ is a Noetherian ring and $M$ is an FSF module, since, for any finitely generated submodule $N$ of $M$, $\dim\operatorname{Supp}_R(M/N) \leq 1$, then $M$ is in dimension $< 2$. Therefore, we shall consider the class of in dimension $< n$ modules as a generalization of the class of FSF modules. Also, obviously, the class of in dimension $< n$ modules and FSF modules both are Serre subcategories of $\operatorname{Mod}(R)$ (see [15, Proposition 2.2]).

As some applications of Proposition 2.2, we have the following corollaries, which are generalizations of the main results of [8] and [15]. For all, we use [4, Lemma 2.1].

***Corollary* 2.1** (see [15, Proposition 3.1]). *Let $M$ be an FSF module. Let $t \in \mathbb{N}_0$ be such that $H^i_\Phi(M)$ is FSF for all $i < t$. Then $\operatorname{Hom}_R\big(R/\mathfrak{a}, H^t_\Phi(M)\big)$ is FSF for all $\mathfrak{a} \in \Phi$. Consequently, $\operatorname{Ass}_R\big(H^t_\Phi(M)\big) \cap V(\mathfrak{a})$ is finite.*

***Corollary* 2.2.** *Let $n, t \in \mathbb{N}_0$ be such that the $R$-modules $M$ and $H^i_\Phi(M)$ are in dimension $< n$ for all $i < t$. Then $\operatorname{Hom}_R\big(R/\mathfrak{a}, H^t_\Phi(M)\big)$ is in dimension $< n$ for all $\mathfrak{a} \in \Phi$. Consequently, $(\operatorname{Ass}_R\big(H^t_\Phi(M)\big) \cap V(\mathfrak{a}))_{\geq n}$ is finite.*

***Corollary* 2.3** (see [8, Proposition 2.1]). *Let $M$ be a finitely generated $R$-module. Let $t \in \mathbb{N}_0$ be such that $H^i_\Phi(M)$ is finitely generated for all $i < t$. Then $\operatorname{Hom}_R\big(R/\mathfrak{a}, H^t_\Phi(M)\big)$ is finitely generated for all $\mathfrak{a} \in \Phi$. Consequently, $\operatorname{Ass}_R\big(H^t_\Phi(M)\big) \cap V(\mathfrak{a})$ is finite.*

As another application of Proposition 2.2, we get the following proposition which shows that $H^i_\mathfrak{a}(M)$ can be considered as a submodule of $H^i_\Phi(M)$ on low points $i$ for all $\mathfrak{a} \in \Phi$.

**Proposition 2.3.** *Let $M$ be an $R$-module and $t \in \mathbb{N}_0$ be such that $H^i_\Phi(M) = 0$ for all $i < t$. Then, for all $\mathfrak{a} \in \Phi$ and all $n \in \mathbb{N}$, the following hold*:

(i) $\operatorname{Ext}^t_R\big(R/\mathfrak{a}^n, M\big) \cong \operatorname{Hom}_R\big(R/\mathfrak{a}^n, H^t_\Phi(M)\big) \cong \operatorname{Hom}_R\big(R/\mathfrak{a}^n, H^t_\mathfrak{a}(M)\big)$.

(ii) $H^t_\mathfrak{a}(M) \subseteq H^t_\Phi(M)$ *and hence* $H^i_\mathfrak{a}(M) \subseteq H^i_\Phi(M)$ *for all* $i \leq t$.

(iii) $\operatorname{Ass}_R(H^t_\mathfrak{a}(M)) = \operatorname{Ass}_R\big(H^t_\Phi(M)\big) \cap V(\mathfrak{a})$.

(iv) *If $\mathfrak{a} \neq 0$ and $M$ is a* ZD-*module, then $\mathfrak{a}$ contains a regular $M$-sequence of length $t$.*

***Proof.*** (i) Using notations of Proposition 2.2 for $G_1(-) = \Gamma_\mathfrak{a}(-)$, $G_2(-) = \Gamma_\Phi(-)$, $\mathcal{S} = \{0\}$, and the ideal $\mathfrak{a}^n$ of $R$, then the homomorphism $\psi$ is an isomorphism. Therefore, the assertion follows.

(ii) By part (i), for all $n \geq 1$, $\operatorname{Ext}^i_R\big(R/\mathfrak{a}^n, M\big) \cong \operatorname{Hom}_R\big(R/\mathfrak{a}^n, H^t_\Phi(M)\big)$. Therefore easily we conclude that $H^t_\mathfrak{a}(M) \cong \Gamma_\mathfrak{a}\big(H^t_\Phi(M)\big) \subseteq H^t_\Phi(M)$.

(iii) It follows immediately from part (i).

(iv) First note that $H^i_\mathfrak{a}(M) = 0$ for all $i < t$. Now, by induction on $t$, we construct a regular $M$-sequence of length $t$. Let $t = 1$, then $\Gamma_\mathfrak{a}(M) = 0$ and, since $M$ is ZD-module by the Prime Avoidance Theorem, there exists $x_1 \in \mathfrak{a}$ which is a non-zero divisor on $M$. Now, the assertion follows easily from the inductive hypothesis and the exact sequence $H^i_\mathfrak{a}(M) \longrightarrow H^i_\mathfrak{a}(M/x_1M) \longrightarrow H^{i+1}_\mathfrak{a}(M)$ for all $i \leq t$.

The vanishing of $H^i_\Phi(M)$ and $H^i_\mathfrak{a}(M)$ (where $\mathfrak{a} \in \Phi$ and $i \in \mathbb{N}_0$) effects on each other. This is a consequence of Proposition 2.3 as follows.

***Corollary* 2.4.** *Let $M$ be an $R$-module and $t \in \mathbb{N}_0$. Then*:

(i) $H^i_\mathfrak{a}(M) = 0$ *for all* $i < t$ *and all* $\mathfrak{a} \in \Phi$ *if and only if* $H^i_\Phi(M) = 0$ *for all* $i < t$.

(ii) *If* $H^i_\Phi(M) = 0$ *for all* $i < t$, *then*

$$\operatorname{Ass}_R\big(H^t_\Phi(M)\big) = \bigcup_{\mathfrak{a}\in\Phi} \operatorname{Ass}_R(H^t_\mathfrak{a}(M)) \quad \textit{and} \quad \operatorname{Supp}_R H^t_\Phi(M) = \bigcup_{\mathfrak{a}\in\Phi} \operatorname{Supp}_R(H^t_\mathfrak{a}(M)).$$

***Proof.*** Each both parts are easily derived from Propositions 2.3 and 2.1. Note that we always have $\bigcup_{i<t} \operatorname{Supp}_R(H^i_\Phi(M)) = \bigcup_{\substack{i<t\\ \mathfrak{a}\in\Phi}} \operatorname{Supp}_R(H^i_\mathfrak{a}(M))$ for all $t \in \mathbb{N}_0$.

One of our main results of this section is to achieve the generalization of Faltings' theorem for systems of ideals (Theorem 2.2). In order to prove it, we prove Theorem 2.1 which needs the following lemma.

**Lemma 2.1.** *Let $\mathcal{S}$ be a Serre subcategory of $\operatorname{Mod}(R)$, $\mathfrak{a}$ be an ideal of $R$, and $M$ be an $R$-module. Then $\mathfrak{a}M \in \mathcal{S}$ if and only if $M/\Big(0 \underset{M}{:} \mathfrak{a}\Big) \in \mathcal{S}$.*

***Proof.*** Let $\mathfrak{a}M \in \mathcal{S}$ and $\mathfrak{a} = \sum_{i=1}^{n} Ra_i$. Define $f\colon M \to (\mathfrak{a}M)^n$ by $f(m) = (a_i m)_{i=1}^n$ for all $m \in M$. Since $\operatorname{Ker} f = \left(0 \underset{M}{:} \mathfrak{a}\right)$, so that the $R$-module $M/(0 \underset{M}{:} \mathfrak{a})$ is isomorphic to a submodule of $(\mathfrak{a}M)^n$. Therefore $M/\left(0 \underset{M}{:} \mathfrak{a}\right) \in \mathcal{S}$, as $(\mathfrak{a}M)^n \in \mathcal{S}$. For inverse, let $M/\left(0 \underset{M}{:} \mathfrak{a}\right) \in \mathcal{S}$. Define the homomorphism $g\colon M^n \to aM$ by $g((m_i)_{i=1}^n) = \sum_{i=1}^{n} a_i m_i$ (for all $(m_i)_{i=1}^n \in M^n$). Then $g$ is surjective and since $\left(0 \underset{M}{:} \mathfrak{a}\right)^n \subseteq \operatorname{Ker} g$, so $\mathfrak{a}M$ is a homomorphic image of $\left(M/(0 \underset{M}{:} \mathfrak{a})\right)^n$. Now, the assertion follows from $\left(M/\left(0 \underset{M}{:} \mathfrak{a}\right)\right)^n \in \mathcal{S}$.

**Theorem 2.1.** *Let $M$ be a finitely generated $R$-module and $t \in \mathbb{N}_0$. The following statements are equivalent*:

(i) $H^i_\Phi(M)$ *is finitely generated module for all $i < t$.*

(ii) *There exists an ideal $\mathfrak{a} \in \Phi$ such that $\mathfrak{a}H^i_\Phi(M) = 0$ for all $i < t$.*

(iii) *There exists an ideal $\mathfrak{a} \in \Phi$ such that $\mathfrak{a}H^i_\Phi(M)$ is finitely generated module for all $i < t$.*

***Proof.*** (i)→(ii)→(iii) are trivial.

(iii)→(i) Assume that $\mathfrak{a} \in \Phi$ is such that $\mathfrak{a}H^i_\Phi(M)$ is finitely generated module for all $i < t$. We show by induction on $t$, that $H^i_\Phi(M)$ is finitely generated module for all $i < t$. For $t = 0, 1$ there is nothing to prove. Suppose that $t > 1$ and the assertion is settled for all $i \leq t-2$. We show that $H^{t-1}_\Phi(M)$ is finitely generated module. Since, for each $i < t-1$, $H^i_\Phi(M)$ is finitely generated module, the $R$-module $\left(0 \underset{H^{t-1}_\Phi(M)}{:} \mathfrak{a}\right)$ is also finitely generated by Corollary 2.3. Now, consider the exact sequence

$$0 \to \left(0 \underset{H^{t-1}_\Phi(M)}{:} \mathfrak{a}\right) \to H^{t-1}_\Phi(M) \to H^{t-1}_\Phi(M)/\left(0 \underset{H^{t-1}_\Phi(M)}{:} \mathfrak{a}\right) \to 0,$$

and since $\mathfrak{a}H^{t-1}_\Phi(M)$ is finitely generated module, the assertion follows by Lemma 2.1 for the category of finitely generated $R$-modules.

**Definition 2.2.** *Let $\Phi$ be a system of ideals of $R$, $N$ be an $R$-module, and $\mathcal{S}$ an arbitrary Serre subcategory of* $\operatorname{Mod}(R)$. *We define the $\mathcal{S}$-dimension $f_{(\Phi,\mathcal{S})}(M)$ of $M$ relative to $\Phi$ and $\mathcal{S}$ by*

$$f_{(\Phi,\mathcal{S})}(N) := \inf\left\{i \geq 0 |\, H^i_\Phi(N) \notin \mathcal{S}\right\}$$

*with the usual convention that the infimum of the empty set of integers is interpreted as $\infty$.*

***Remark* 2.1.** In Definition 2.2, when $\mathcal{S}$ is the class of finitely generated $R$-modules, we use $f_\Phi(N)$ instead of $f_{(\Phi,\mathcal{S})}(N)$. That is, the generalization of the finiteness dimension of $N$ relative to an ideal, whenever $\Phi$ is the powers of that ideal. Also, for $\mathcal{S} = \{0\}$, by Corollary 2.4,

$$\begin{aligned} f_{(\Phi,\{0\})}(N) &= \inf\left\{i \geq 0 \mid H^i_\Phi(N) \neq 0\right\} \\ &= \inf\left\{\operatorname{grade}\,(\mathfrak{a}, N) \mid \mathfrak{a} \in \Phi\right\}. \end{aligned}$$

We denote $f_{(\Phi,\{0\})}(N)$ by $\operatorname{grade}(\Phi, N)$. Note that if $\mathfrak{a}N \neq N$ for some $\mathfrak{a} \in \Phi$ (or $H^i_\mathfrak{a}(N) \neq 0$ for some $\mathfrak{a} \in \Phi$ and some $i \in \mathbb{N}$), then $\operatorname{grade}(\Phi, N)$ is a nonnegative integer. For finite $R$-modules $M$ and $N$, the author in [7, Definition 5.3] defined $\Phi$-grade of $M$ w.r.t. $N$ as follows:

$$\Phi - \operatorname{grade}_N M = \inf_{\mathfrak{a} \in \Phi}\{\operatorname{grade}_N(M/\mathfrak{a}M)\}.$$

In the case of $M = R$, these two definitions are coincide. Moreover, we have

$$\operatorname{grade}(\Phi, N) \leq f_\Phi(N) = \inf\big\{i \geq 0 \mid H^i_\Phi(N) \text{ is not finitely generated}\big\}.$$

In addition, if $t := \operatorname{grade}(\Phi, N) < f_\Phi(N)$, then $H^t_\Phi(N) = H^t_\mathfrak{a}(N)$ for some $\mathfrak{a} \in \Phi$, by Proposition 2.3 and Theorem 2.1. More precisely, if $H^{\operatorname{grade}(\Phi,N)}_\Phi(N)$ is finitely generated module, then $H^{\operatorname{grade}(\Phi,N)}_\Phi(N) = H^{\operatorname{grade}(\mathfrak{a},N)}_\mathfrak{a}(N)$ for some $\mathfrak{a} \in \Phi$.

In this stage, we are in position to present the generalization of Faltings' theorem for an arbitrary system of ideals (see [12, Satz (1)] or [9, Theorem 9.6.1]).

**Theorem 2.2.** *Let $\Phi$ be a system of ideals of $R$, $M$ be a finitely generated $R$-module, and $t \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) *$H^i_\Phi(M)$ is finitely generated module for all $i < t$;*

(ii) *$\operatorname{Ass}_R(H^i_\Phi(M))$ is a finite set and $H^i_{\Phi_\mathfrak{p}}(M_\mathfrak{p})$ is a finitely generated $R_\mathfrak{p}$-module for all $i < t$ and all $\mathfrak{p} \in \operatorname{Spec}(R)$, where $\Phi_\mathfrak{p} := \{\mathfrak{a}R_\mathfrak{p} \mid \mathfrak{a} \in \Phi\}$.*

***Proof.*** (i) $\to$ (ii) is trivial by Proposition 2.1.

(ii) $\to$ (i) We argue by induction on $t$. For $t \leq 1$ there is nothing to prove. Now, suppose that $t \geq 2$ and the assertion is settled for $i \leq t-2$. We show that $H^{t-1}_\Phi(M)$ is also finite module. According to the assumption $X := \displaystyle\bigcup_{i=1}^{t-1} \operatorname{Ass}_R(H^i_\Phi(M))$ is a finite set. Let $X = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Also, by assumption $(H^i_\Phi(M))_{\mathfrak{p}_j}$ is finitely generated $R_{\mathfrak{p}_j}$-module for all $i < t$ and all $\mathfrak{p}_j \in X$. Let $1 \leq j \leq n$. By Theorem 2.1, there exists an ideal $\mathfrak{a}_j R_{\mathfrak{p}_j} \in \Phi_{\mathfrak{p}_j}$ such that $(\mathfrak{a}_j H^i_\Phi(M))_{\mathfrak{p}_j} = (\mathfrak{a}_j R_{\mathfrak{p}_j}) H^i_{\Phi_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) = 0$ for all $i < t-1$. There exists an ideal $\mathfrak{a} \in \Phi$ such that $\mathfrak{a} \subseteq \prod_{j=1}^n \mathfrak{a}_j$, as $\Phi$ is a system of ideals. Hence, $\big(\mathfrak{a} H^i_\Phi(M)\big)_{\mathfrak{p}_j} = 0$ for all $i < t$ and all $1 \leq j \leq n$. Now, since $\operatorname{Ass}_R(\mathfrak{a} H^i_\Phi(M)) \subseteq \operatorname{Ass}_R(H^i_\Phi(M))$ for all $i < t$, we have $\mathfrak{a} H^i_\Phi(M) = 0$ for all $i < t$. Now, the assertion follows from Theorem 2.1.

***Corollary* 2.5** (see [9, Theorem 9.6.1])**.** *Let $\mathfrak{a}$ be an ideal of $R$, $M$ be a finitely generated $R$-module, and $t \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) *$H^i_\mathfrak{a}(M)$ is finitely generated module for all $i < t$;*

(ii) *$H^i_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p})$ is a finitely generated $R_\mathfrak{p}$-module for all $i < t$ and all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

***Proof.*** Considering $\Phi = \{\mathfrak{a}^n \mid n \geq 1\}$, the assertion easily follows from Theorem 2.2 and Corollary 2.3.

***Remark* 2.2.** Let $M$ be a finitely generated $R$-module and $\mathfrak{a}$ be an ideal of $R$. According to [9, Theorems 9.6.1, 9.6.2, Proposition 9.1.2], we can get several equalities for $f_\mathfrak{a}(M)$ as follows:

$$\begin{aligned} f_\mathfrak{a}(M) &= \inf\big\{i \in \mathbb{N}_0 \mid H^i_\mathfrak{a}(M) \text{ is not finitely generated module}\big\} \\ &= \inf\Big\{i \in \mathbb{N}_0 \mid \mathfrak{a} \not\subseteq \sqrt{(0 :_R H^i_\mathfrak{a}(M))}\Big\} \\ &= \inf\big\{i \in \mathbb{N}_0 \mid \mathfrak{a}^n H^i_\mathfrak{a}(M) \neq 0 \text{ for all } n \in \mathbb{N}\big\} \\ &= \inf\big\{f_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\big\} \\ &= \inf\big\{f_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M) \text{ and } \dim R/\mathfrak{p} \geq 0\big\}. \end{aligned}$$

The equality

$$f_{\mathfrak{a}}(M) = \inf\left\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M) \text{ and } \dim R/\mathfrak{p} \geq 0\right\}$$

motivated the authors in [5] to define the *n*th finiteness dimension of $M$ relative to $\mathfrak{a}$ as follows:

$$f_{\mathfrak{a}}^n(M) := \inf\left\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M) \text{ and } \dim R/\mathfrak{p} \geq n\right\}.$$

Then the authors in [3, Theorem 2.5] showed that if $(R, \mathfrak{m})$ is a complete local ring, $\mathfrak{a}$ is an ideal of $R$ and $M$ is a finitely generated $R$-module, then $f_{\mathfrak{a}}^n(M) = h_{\mathfrak{a}}^n(M)$ for all $n \in \mathbb{N}_0$, where $h_{\mathfrak{a}}^n(M) := \inf\left\{i \in \mathbb{N}_0 | H_{\mathfrak{a}}^i(M) \text{ is not in dimension } < n\right\}$ (see [3, Definition 2.4]). Following, in Theorem 2.3, we extend this result for an arbitrary system of ideals which is the last and most important theorem of this section. For this purpose, we need to provide the following definitions, which are generalizations of $f_a^n(M)$ and $h_{\mathfrak{a}}^n(M)$.

**Definition 2.3.** *Let $M$ be an $R$-module, $\Phi$ be a system of ideals of $R$ and $n$ be a nonnegative integer. We define*

$$h_{\Phi}^n(M) = \inf\left\{i \in \mathbb{N}_0 | \, H_{\Phi}^i(M) \textit{ is not in dimension } < n\right\}$$

*and*

$$f_{\Phi}^n(M) := \inf\left\{f_{\Phi_{\mathfrak{p}}}(M_{\mathrm{p}}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M) \textit{ and } \dim R/\mathfrak{p} \geq n\right\}.$$

**Theorem 2.3.** *Let $(R, \mathfrak{m})$ be a complete local ring, $\Phi$ be a countable system of ideals of $R$, and $M$ be a finitely generated $R$-module. Let $n \in \mathbb{N}_0$ be such that the set $\left(\operatorname{Ass}(H_{\Phi}^{h_{\phi}^n(M)}(M))\right)_{\geq n}$ is finite. Then $f_{\Phi}^n(M) = h_{\Phi}^n(M)$.*

***Proof.*** Put $t := h_{\Phi}^n(M)$. By definition, for all $i < t$, there is a finitely generated submodule $N$ of $H_{\Phi}^i(M)$ such that $\dim \operatorname{Supp}_R(H_{\Phi}^i(M)/N) < n$. Thus, for all $\mathfrak{p} \in (\operatorname{Spec}(R))_{\geq n}$, $(H_{\Phi}^i(M)/N)_{\mathfrak{p}} = 0$ and so $H_{\Phi_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$ is a finitely generated $R_{\mathfrak{p}}$-module. Therefore $t \leq f_{\Phi}^n(M)$. Now, we show that $t = f_{\Phi}^n(M)$. Assume the opposite $t < f_{\Phi}^n(M)$ and look for a contradiction. To this end, first we claim that $\mathcal{X}_{\mathfrak{a}} := \left(\operatorname{Ass}_R\left(H_{\Phi}^t(M)\Big/\Big(0 \underset{H_{\Phi}^t(M)}{:} \mathfrak{a}\Big)\right)\right)_{\geq n}$ is a finite set for all $\mathfrak{a} \in \Phi$. To achieve this, suppose contrary to our claim that there is a countable infinite subset $\{\mathfrak{p}_k\}_{k=1}^{\infty}$ of $\mathcal{X}_{\mathfrak{a}}$ for some $\mathfrak{a} \in \Phi$. Let $S := R \setminus \bigcup_{k=1}^{\infty} \mathfrak{p}_k$ and we show that $H_{S^{-1}\Phi}^t(S^{-1}M)$ is finitely generated as $S^{-1}R$-module. For this purpose, by virtue of Theorem 2.2, it is enough to show that $\operatorname{Ass}_{S^{-1}R}(H_{S^{-1}\Phi}^j(S^{-1}M))$ is a finite set and the $R_{\mathfrak{p}}$-module $\left(H_{S^{-1}\Phi}^j(S^{-1}M)\right)_{S^{-1}\mathfrak{p}}$ is finitely generated for all $j \leq t$ and all prime ideals $\mathfrak{p}$ with $S \cap \mathfrak{p} = \varnothing$. First, note that it is easy to see that, for all $j < t$, $\left(\operatorname{Ass}_R(H_{\Phi}^j(M))\right)_{\geq n}$ is finite, since $H_{\Phi}^j(M)$ is in dimension $< n$. Hence, by assumption $\left(\operatorname{Ass}_R(H_{\Phi}^j(M))\right)_{\geq n}$ is finite for all $j \leq t$. On the other hand, since $\mathfrak{p} \subseteq \bigcup_{k=1}^{\infty} \mathfrak{p}_k$, there exists $k \geq 1$ such that $\mathfrak{p} \subseteq \mathfrak{p}_k$, by [14, Lemma 3.2]. Thus, $\dim R/\mathfrak{p} \geq n$. Therefore the set $\operatorname{Ass}_{S^{-1}R}\left(H_{S^{-1}\Phi}^j(S^{-1}M)\right)$ is finite for all $j \leq t$. Now, as $j < f_{\Phi}^n(M)$, it follows that $\left(H_{S^{-1}\Phi}^j(S^{-1}M)\right)_{S^{-1}\mathfrak{p}} \cong \left(H_{\Phi}^j(M)\right)_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module and so $H_{S^{-1}\Phi}^t(S^{-1}M)$ is a Noetherian $S^{-1}R$-module. Therefore $\operatorname{Ass}_{S^{-1}R}\left(S^{-1}\Big(H_{\Phi}^t(M)\Big/\Big(0 \underset{H_{\Phi}^t(M)}{:} \mathfrak{a}\Big)\Big)\right)$

is finite. But $S^{-1}\mathfrak{p}_k \in \operatorname{Ass}_{S^{-1}R}\left(S^{-1}\left(H^t_\Phi(M)\Big/\left(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\right)\right)\right)$ for all $k \geq 1$, which is a contradiction. Consequently, the set $\mathcal{X}_\mathfrak{a}$ is finite for all $\mathfrak{a} \in \Phi$. Now, let $\mathbb{A} = \bigcup_{\mathfrak{a}\in\Phi} \mathcal{X}_\mathfrak{a}$. It is clear that $\mathbb{A}$ is a countable set. Let $S := R \setminus \bigcup_{\mathfrak{p}\in\mathbb{A}} \mathfrak{p}$. It is easy to see that for all $\mathfrak{a} \in \Phi$, the set

$$\operatorname{Ass}_{S^{-1}R}\left(S^{-1}\left(H^t_\Phi(M)\Big/\left(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\right)\right)\right)$$

is finite. Thus, for all $\mathfrak{a} \in \Phi$, the set $\operatorname{Supp}\left(S^{-1}\left(H^t_\Phi(M)\Big/\left(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\right)\right)\right)$ is a closed subset of $\operatorname{Spec}(R)$ in the Zariski topology. To complete the proof of theorem, note that for an arbitrary $\mathfrak{c} \in \Phi$, we have a chain of the form $\mathfrak{c} \supseteq \mathfrak{d} \supseteq \ldots$ in $\Phi$, which induces the following descending chain:

$$\operatorname{Supp}\left(S^{-1}\left(H^t_\Phi(M)\Big/(0 \underset{H^t_\Phi(M)}{:} \mathfrak{c})\right)\right) \supseteq \operatorname{Supp}\left(S^{-1}\left(H^t_\Phi(M)\Big/\left(0 \underset{H^t_\Phi(M)}{:} \mathfrak{d}\right)\right)\right) \supseteq \ldots, \tag{2.1}$$

that is eventually stationary. Let $\mathfrak{b}$ denote its eventual stationary value.

Now, set $E_\mathfrak{b} := \operatorname{Supp}\left(S^{-1}\left(H^t_\Phi(M)\Big/(0 \underset{H^t_\Phi(M)}{:} \mathfrak{b})\right)\right)$ and let $\Sigma$ be the set of such ideals $\mathfrak{b} \in \Phi$. We claim that there exists an ideal $\mathfrak{a} \in \Sigma$ such that $E_\mathfrak{a}$ is the stationary value of all the chains of the form (2.1), in which $\mathfrak{a}$ appears in them. Let us call these ideals of $\Sigma$ as *favorite ideals*. Suppose opposite and no member of $\Sigma$ is favorite ideal. In other words, for each member of $\Sigma$, there are at least two chains of the form (2.1) with different stationary values. Note that since $\Phi$ is nonempty, so $\Sigma$ is nonempty and there exists an ideal $\mathfrak{a}$ in $\Sigma$. Assume that

$$\mathfrak{c} \supseteq \mathfrak{d} \supseteq \ldots \supseteq \mathfrak{a} \supseteq \mathfrak{u} \supseteq \ldots$$

is a chain of elements of $\Phi$ and $E_\mathfrak{a}$ is the eventually stationary value of the following chain:

$$\frac{\operatorname{Supp}\left(S^{-1}\left(H^t_\Phi(M)\right)\right)}{\left(0 \underset{H^t_\Phi(M)}{:} \mathfrak{c}\right)} \supseteq \ldots \supseteq \frac{\operatorname{Supp}\left(S^{-1}\left(H^t_\Phi(M)\right)\right)}{\left(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\right)} = E_\mathfrak{a} = E_\mathfrak{u} = \ldots.$$

According to assumption, there exists another chain including $\mathfrak{a}$,

$$K \supseteq J \supseteq \ldots \supseteq \mathfrak{a} \supseteq \ldots \supseteq I_1 \supseteq \ldots,$$

such that the corresponding chain

$$E_K = \frac{\operatorname{Supp}\left(S^{-1}\left(H^t_\Phi(M)\right)\right)}{\left(0 \underset{H^t_\Phi(M)}{:} K\right)} \supseteq \ldots \supseteq E_\mathfrak{a} \supseteq \ldots \supseteq E_{I_1} = \ldots$$

has a the stationary value other than $E_\mathfrak{a}$, say $E_{I_1}$ (where $E_\mathfrak{a} \neq E_{I_1}$). Proceeding this method for the ideal $I_1$, we can find another ideal $I_2$ of $\Phi$ such that $E_{I_1} \neq E_{I_2}$ (otherwise, $I_1$ would be the favorite ideal). Continuing this method, we get a chain of ideals of $\Sigma$ of the form

$$\mathfrak{a} \supsetneqq I_1 \supsetneqq I_2 \supsetneqq \ldots \supsetneqq I_r \supsetneqq I_{r+1} \supsetneqq \ldots,$$

which induces the following chain:

$$E_{\mathfrak{a}} = \frac{\operatorname{Supp}\Big(S^{-1}\big(H^t_\Phi(M)\big)\Big)}{\Big(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\Big)} \supseteq E_{I_1} = \frac{\operatorname{Supp}\Big(S^{-1}\big(H^t_\Phi(M)\big)\Big)}{\Big(0 \underset{H^t_\Phi(M)}{:} I_1\Big)} \supseteq \ldots .$$

Again, this chain is eventually stationary, say $I_r$ and so $E_{I_r} = E_{I_{r+1}}$ which contradicts with the choice of $I_j$, $j = 1, 2, \ldots$. Therefore the claim is obtained. Thus, there exists $\mathfrak{a} \in \Sigma$ such that, for any ideal $\mathfrak{b} \in \Phi$ with $\mathfrak{b} \subseteq \mathfrak{a}$, we have $E_{\mathfrak{a}} = E_{\mathfrak{b}}$. Now consider a favorite ideal $\mathfrak{a}$ of $\Sigma$. Since $H^i_\Phi(M)$ is in dimension $< n$ for all $i < t$, so by Corollary 2.2, $\Big(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\Big)$ is also in dimension $< n$. Thus, there exists a finitely generated submodule $N$ of $\Big(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\Big)$ such that $\dim \operatorname{Supp}\Big(\Big(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\Big)\Big/N\Big) < n$. Now, we show that

$$\dim \operatorname{Supp}\Big(H^t_\Phi(M)/(0 \underset{H^t_\Phi(M))}{:} \mathfrak{a})\Big) < n.$$

For if there is, $\mathfrak{q} \in \Big(\operatorname{Ass}_R(H^t_\Phi(M)\Big/\Big(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\Big))\Big)_{\geq n}$, then $\mathfrak{q} \in \mathbb{A}$ and so $S \cap \mathfrak{q} = \varnothing$. Thus, $S^{-1}\mathfrak{q} \in E_{\mathfrak{a}}$. On the other hand, $t < f^n_\Phi(M)$ and dim $R/\mathfrak{q} \geq n$ imply that $\big(H^t_\Phi(M)\big)_{\mathfrak{q}}$ is a finitely generated $R_{\mathfrak{q}}$-module. Therefore there exists $\mathfrak{b} \in \Phi$ ( $\mathfrak{b} \subseteq \mathfrak{a}$) such that $(\mathfrak{b}R_{\mathfrak{q}})\big(H^t_\Phi(M)\big)_{\mathfrak{q}} = 0$. This follows that $(H^t_\Phi(M)/(0 \underset{H^t_\Phi(M)}{:} \mathfrak{b}))_{\mathfrak{q}} = 0$, by Lemma 2.1. Hence,

$$\Big(S^{-1}(H^t_\Phi(M)\Big/\Big(0 \underset{H^t_\Phi(M)}{:} \mathfrak{b}\Big))\Big)_{S^{-1}\mathfrak{q}} = 0$$

and so $S^{-1}\mathfrak{q} \notin E_{\mathfrak{b}} = E_{\mathfrak{a}}$, which is a contradiction. Therefore,

$$\dim \operatorname{Supp}\Big(H^t_\Phi(M)\Big/\Big(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\Big)\Big) < n.$$

Finally, from the exact sequence

$$0 \longrightarrow \Big(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\Big)\Big/N \longrightarrow H^t_\Phi(M)/N \longrightarrow H^t_\Phi(M)\Big/\Big(0 \underset{H^t_\Phi(M)}{:} \mathfrak{a}\Big) \longrightarrow 0,$$

we conclude that $\dim \operatorname{Supp}(H^t_\Phi(M)/N) < n$. That is, the $R$-module $H^t_\Phi(M)$ is in dimension $< n$, which is a contradiction. This contradiction comes from $t < f^n_\Phi(M)$ and, therefore, $t = f^n_\Phi(M)$.

***Corollary* 2.6.** *Let $(R, \mathfrak{m})$ be a complete local ring, $\mathfrak{a}$ be an ideal of $R$, and $M$ be a finitely generated $R$-module. Then, for all $n \in \mathbb{N}_0$, $f^n_{\mathfrak{a}}(M) = h^n_{\mathfrak{a}}(M)$.*

***Proof.*** Apply $\Phi = \big\{\mathfrak{a}^i | i > 0\big\}$ in Theorem 2.3. Note that, by Corollary 2.2, if $t = h^n_{\mathfrak{a}}(M)$, then $\big(\operatorname{Ass}(H^t_{\mathfrak{a}}(M))\big)_{\geq n}$ is finite.

**3. General local cohomology on high points.** In this section, we will study general local cohomology modules on high points and examine vanishing conditions and their tensor products. The following statement is the first important result of this section. The vanishing and finiteness of general local cohomology modules are equivalent on high points. In fact, Theorem 3.1 is a generalization of [17, Proposition 3.1] for a system of ideals over an arbitrary Noetherian (not necessary local) ring.

**Theorem 3.1.** *Let $M$ be a finite dimensional $R$-module and $t \in \mathbb{N}$. Then the following conditions are equivalent*:

(i) $H^i_\Phi(M) = 0$ *for all* $i \geq t$.

(ii) $H^i_\Phi(M)$ *is finitely generated for all* $i \geq t$.

(iii) *There exists* $\mathfrak{a} \in \Phi$ *such that* $\mathfrak{a}H^i_\Phi(M) = 0$ *for all* $i \geq t$ $\Big($*or*, *equivalently*, *there exists* $\mathfrak{b} \in \Phi$ *such that* $\mathfrak{b} \subseteq \sqrt{(0 : H^i_\Phi(M))}$ *for all* $i \geq t\Big)$.

***Proof.*** (i) $\rightarrow$ (ii) $\rightarrow$ (iii) is clear.

(iii) $\rightarrow$ (i) By Proposition 2.1 (vi) and that $M$ can be viewed as the direct limit of its finitely generated submodules, we can assume that $M$ is finitely generated (note that each submodule of $M$ must have dimension not exceeding $\dim M$). We argue by induction on $n := \dim M$. If $n = 0$, then $H^i_\mathfrak{c}(M) = 0$ for all $i \geq t$ and all $\mathfrak{c} \in \Phi$. Thus, $H^i_\Phi(M) = 0$ for all $i \geq t$. Now suppose, inductively, that $n > 0$ and the assertion is settled for every finitely generated $R$-modules of dimension less than $n$. Assume that $\mathfrak{a} \in \Phi$ be such that $\mathfrak{a}H^i_\Phi(M) = 0$ for all $i \geq t$. By Proposition 2.1 (v) and since $M$ is finitely generated module, there exists $x \in \mathfrak{a}$, which is a non-zero divisor on $M$. Now consider the following long exact sequence:

$$H^i_\Phi(M) \xrightarrow{\cdot x} H^i_\Phi(M) \longrightarrow H^i_\Phi(M/xM) \longrightarrow H^{i+1}_\Phi(M). \tag{3.1}$$

Since $\mathfrak{a}H^i_\Phi(M) = 0$ for all $i \geq t$, we get $\mathfrak{a}H^i_\Phi(M/xM) = 0$ for all $i \geq t$, by [9, Lemma 9.1.1]. By inductive hypothesis, $H^i_\Phi(M/xM) = 0$ for all $i \geq t$, as $\dim M/xM < n$. Then by the exact sequence (3.1), $H^i_\Phi(M) = xH^i_\Phi(M)$ for all $i \geq t$. Therefore, by hypothesis, $H^i_\Phi(M) = 0$ for all $i \geq t$.

The following result is an immediate consequence of the above theorem.

***Corollary* 3.1.** *Let $M$ be a finite dimensional $R$-module*, *$t$ be a nonnegative integer such that $H^i_\mathfrak{a}(M)$ is finitely generated for all $i \geq t$ and all $\mathfrak{a} \in \Phi$. Then $H^i_\mathfrak{a}(M) = 0$ for all $i \geq t$ and all $\mathfrak{a} \in \Phi$. Also*, $H^i_\Phi(M) = 0$ *for all* $i \geq t$.

The following proposition is a generalization of [2, Theorem 2.3]. Recall that an $R$-module $M$ is called a minimax module if there is a finitely generated submodule $N$ of $M$ such that $M/N$ is Artinian. Moreover, the class of minimax modules is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of $\mathrm{Mod}(R)$. Minimax modules have been studied by Zink in [18] and Zöschinger in [19, 20].

**Proposition 3.1.** *Let $M$ be a finite dimensional $R$-module and $t \in \mathbb{N}$. Then $H^i_\Phi(M)$ is an Artinian $R$-module for all $i \geq t$ if and only if $H^i_\Phi(M)$ is minimax $R$-module for all $i \geq t$.*

***Proof.*** It is clear that any Artinian $R$-module is minimax. For inverse, first we show that $\mathrm{Supp}_R\left(H^i_\Phi(M)\right) \subseteq \mathrm{Max}(R)$ for all $i \geq t$. For this purpose, let $\mathfrak{p} \in \mathrm{Spec}(R)\backslash\mathrm{Max}(R)$. Then, by assumption, for all $i \geq t$, there is a short exact sequence

$$0 \longrightarrow N \longrightarrow H^i_\Phi(M) \longrightarrow A \longrightarrow 0, \tag{3.2}$$

in which $N$ is Noetherian and $A$ is an Artinian $R$-module. It is easy to see that $(H^i_\Phi(M))_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module for all $i \geq t$. Then, by Theorem 3.1, $(H^i_\Phi(M))_\mathfrak{p} = 0$ for all $i \geq t$. Hence, $\mathrm{Supp}_R(H^i_\Phi(M)) \subseteq \mathrm{Max}(R)$ for all $i \geq t$. Considering the exact sequence (3.2) and since $N$ is Noetherian, we conclude that

$$\mathrm{Ass}_R(N) = \mathrm{Supp}_R(N) = \mathrm{V}(0 \underset{R}{:} N) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\} \subseteq \mathrm{Max}(R).$$

Now, it shows that $N$ is Artinian and so $H^i_\Phi(M)$ is Artinian for all $i \geq t$.

**Definition 3.1.** *Let* $T\colon \operatorname{Mod}(R) \longrightarrow \operatorname{Mod}(R)$ *be a covariant*, $R$*-linear functor*, $\mathcal{S}$ *be an arbitrary Serre subcategory of* $\operatorname{Mod}(R)$, *and* $M$ *be an* $R$*-module. Then we define*

$$U^{(T,\mathcal{S})}(M) = \sup\big\{i \geq 0 |\, \mathcal{R}^iT(M) \notin \mathcal{S}\big\},$$

*where* $\mathcal{R}^iT(-)$ *is the* $i$*th right derived functor of* $T$ *in* $\operatorname{Mod}(R)$. *If the supremum does not exist*, *we put* $U^{(T,\mathcal{S})}(M) = \infty$.

***Example* 3.1.** According to Theorem 3.1, for $T = \Gamma_\Phi(-)$ and Serre subcategories $\mathcal{S}_1 = \{0\}$, $\mathcal{S}_2 = \{f \cdot g\}$ (the category of finitely generated $R$-modules), and for any finite dimensional module $M$, we have

$$U^{(\Gamma_\Phi,\{f\cdot g\})}(M) = U^{(\Gamma_\Phi,\{0\})}(M)$$
$$= \operatorname{cd}(\Phi, M) \leq \sup\big\{U^{(\Gamma_\Phi,\{f\cdot g\})}(M)|\mathfrak{a} \in \Phi\big\},$$

where $\operatorname{cd}(\Phi, M) := \sup\big\{i \geq 0 \mid H^i_\Phi(M) \neq 0\big\}$. Also, by Proposition 3.1, for Serre subcategories $\mathcal{S}_3 = \{$Artinian $R$-modules$\}$, $\mathcal{S}_4 = \{$minimax $R$-modules$\}$, and for any minimax $R$-module $M$, we have $U^{(\Gamma_\Phi,\mathcal{S}_3)}(M) = U^{(\Gamma_\Phi,\mathcal{S}_4)}(M)$. However, For some $R$-modules $M$, some Serre subcategories $\mathcal{S}$, or some functors $T$, it may be happens that $U^{(T,\mathcal{S})}(M) = \infty$. For example, if $M$ has finite length, then $U^{(\Gamma_\Phi,\{f\cdot g\})}(M) = \infty$. Finally, if $(R, \mathfrak{m})$ is a local ring and $M$ is a finite $R$-module, then we obtain

$$U^{(\Gamma_\mathfrak{m},\{f\cdot g\})}(M) = U^{(\Gamma_\mathfrak{m},\{0\})}(M) = \operatorname{cd}(\mathfrak{m}, M) = \dim M.$$

For the last main results of this section, we introduce a homological property that can be used for local cohomology in special cases.

**Definition 3.2.** *Let* $T\colon \operatorname{Mod}(R) \longrightarrow \operatorname{Mod}(R)$ *be a covariant*, $R$*-linear functor and* $M$ *be an* $R$*-module. We say that* $T$ *has the vanishing property on* $M$ *whenever there exists* $n \in \mathbb{N}$ *such that* $\mathcal{R}^iT(M) = 0$ *for all* $i > n$. *Moreover*, *we say that* $T$ *has the global vanishing property on* $\operatorname{Mod}(R)$ *whenever it has the vanishing property on each modules of* $\operatorname{Mod}(R)$.

***Example* 3.2.** Let $\mathfrak{a}$ be an ideal of $R$. It is well-known that for any $R$-module $M$, $H^i_\mathfrak{a}(M) = 0$ for all $i > \operatorname{ara}(\mathfrak{a})$. Therefore $\Gamma_\mathfrak{a}(-)$ has the global vanishing property on $\operatorname{Mod}(R)$. Also, for any arbitrary system of ideals of $R$, say $\Phi$, the functor $\Gamma_\Phi(-)$ has the vanishing property on each finite dimensional $R$-module, by Grothendieck's vanishing theorem. Whenever $R$ has finite dimension itself, then $\Gamma_\Phi(-)$ has the global vanishing property on $\operatorname{Mod}(R)$. Specially, when $R$ is a local ring.

**Theorem 3.2.** *Let* $T\colon \operatorname{Mod}(R) \longrightarrow \operatorname{Mod}(R)$ *be a covariant*, $R$*-linear functor which has the global vanishing property on* $\operatorname{Mod}(R)$. *Also*, *let* $\mathcal{S}$ *be an arbitrary Serre subcategory on* $\operatorname{Mod}(R)$ *and* $t \in \mathbb{N}_0$. *Then* $\mathcal{R}^iT(R) \in \mathcal{S}$ *for all* $i \geq t$ *if and only if* $\mathcal{R}^iT(M) \in \mathcal{S}$ *for all* $i \geq t$ *and any finitely generated* $R$*-module* $M$.

***Proof.*** Let $\mathcal{R}^iT(R) \in \mathcal{S}$ for all $i \geq t$ and $M$ be an arbitrary finitely generated $R$-module. Considering the following exact sequence:

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \to 0,$$

in which $F$ is a finitely generated free $R$-module, by assumption we can choose $n \in \mathbb{N}$ such that $\mathcal{R}^iT(M) = 0 = \mathcal{R}^iT(K)$ for all $i \geq n$. Without loss of generality, we may assume that $n \geq t$. Now, the assertion follows from [16, Theorem 6.43] and descending induction on $i$, $t \leq i \leq n$.

***Corollary* 3.2.** *Let* $T\colon \operatorname{Mod}(R) \longrightarrow \operatorname{Mod}(R)$ *be a covariant*, $R$*-linear functor which commutes with direct limits and has the vanishing property on* $R$. *Let* $\mathcal{S}$ *be an arbitrary Serre subcategory and* $t \in \mathbb{N}_0$. *Then* $\mathcal{R}^i T(R) \in \mathcal{S}$ *for all* $i \geq t$ *if and only if* $\mathcal{R}^i T(M) \in \mathcal{S}$ *for all* $i \geq t$ *and any finitely generated* $R$*-module* $M$.

***Proof.*** Since each $R$-module is the direct limit of its finitely generated submodules, $T$ has the global vanishing property on $\operatorname{Mod}(R)$ and so the assertion follows by Theorem 3.2.

***Corollary* 3.3.** *Let* $R$ *be a finite dimensional ring,* $\mathcal{S}$ *be an arbitrary Serre subcategory and* $t \in \mathbb{N}_0$. *Then* $H^i_\Phi(R) \in \mathcal{S}$ *for all* $i \geq t$ *if and only if* $H^i_\Phi(M) \in \mathcal{S}$ *for all* $i \geq t$ *and any finitely generated* $R$*-module* $M$. *Consequently*,

$$U^{(\Gamma_\Phi, \mathcal{S})}(R) = \sup\big\{U^{(\Gamma_\Phi, \mathcal{S})}(M) \mid M \text{ is finitely generated } R\text{-module}\big\}.$$

***Proof.*** Consider $T := \Gamma_\Phi(-)$. Since $R$ has finite dimension, then $T$ has the global vanishing property on $\operatorname{Mod}(R)$, and the assertion is easily obtained from Theorem 3.2.

***Corollary* 3.4.** *Let* $R$ *be a Noetherian ring* (*not necessary of finite dimension*), $\mathfrak{a}$ *be an ideal of* $R$. *Let* $\mathcal{S}$ *be an arbitrary Serre subcategory and* $t \in \mathbb{N}_0$. *Then* $H^i_{\mathfrak{a}}(R) \in \mathcal{S}$ *for all* $i \geq t$ *if and only if* $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ *for all* $i \geq t$ *and any finitely generated* $R$*-module* $M$.

***Proof.*** Since the arithmetic rank of $\mathfrak{a}$ is finite, therefore $\Gamma_{\mathfrak{a}}(-)$ satisfies in Theorem 3.2.

The following proposition, as the last result of this paper, studies the general local cohomology modules and also the tensor product of them on high points.

**Proposition 3.2.** *Let* $R$ *be a Noetherian ring of finite dimension* $d$ *and* $M$ *be a finitely generated* $R$*-module such that* $U^{(\Gamma_\Phi, f\cdot g)}(M)$ *is an integer* (*or*, *equivalently*, *there exists* $i \in \mathbb{N}$ *such that* $H^i_\Phi(M) \neq 0$). *Then we have the following*:

(i) $U^{(\Gamma_\Phi, \{f\cdot g\})}(M) = \operatorname{cd}(\Phi, M)$.

(ii) *If* $n := \dim M > 0$, *then* $H^n_\Phi(M) \neq 0$ *if and only if* $U^{(\Gamma_\Phi, \{f\cdot g\})}(M) = n$.

(iii) *If* $\mathfrak{a} \in \Phi$ *and* $t := \operatorname{ara}(\mathfrak{a})$, *then* $H^t_{\mathfrak{a}}(M) \neq 0$ *if and only if* $U^{(\Gamma_\Phi, \{f\cdot g\})}(M) = t$.

(iv) $H^i_\Phi(R) \otimes H^j_\Phi(M) = 0$ *for all* $i \in \{U^{(\Gamma_\Phi, \{f\cdot g\})}(M), d\}$ *and all* $j \geq 0$. *Specially*, *for* $M = R$.

***Proof.*** Parts (i), (ii) and (iii) follow from Theorem 3.1, Grothendieck's vanishing theorem, and [9, Corollary 3.3.3].

(iv) First, note that by Corollary 3.3, $U^{(\Gamma_\Phi, \{f\cdot g\})}(R)$ is finite and in addition, by part (i), $U^{(\Gamma_\Phi, \{f\cdot g\})}(M) = \operatorname{cd}(\Phi, M)$ and $U^{(\Gamma_\Phi, \{f, g\})}(R) = \operatorname{cd}(\Phi, R)$. Now, by Grothendieck's vanishing theorem and Theorem 3.1, both $H^{\operatorname{cd}(\Phi,R)}_\Phi(-)$ and $H^d_\Phi(-)$ are right exact functors, and so by [9, Exercise 6.1.8], we have the following isomorphisms:

$$H^{\operatorname{cd}(\Phi,R)}_\Phi(R) \otimes_R N \cong H^{\operatorname{cd}(\Phi,R)}_\Phi(N), \qquad H^d_\Phi(R) \otimes_R N \cong H^d_\Phi(N)$$

for all finitely generated $R$-modules $N$. Now, the assertion follows as $H^j_\Phi(M)$ is $\Phi$-torsion $R$-module for all $j \geq 0$.

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