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A WEIGHTED WEAK-TYPE INEQUALITY FOR THE ONE-SIDED MAXIMAL OPERATORS

ЗВАЖЕНА НЕРІВНІСТЬ СЛАБКОГО ТИПУ ДЛЯ ОДНОСТОРОННІХ МАКСИМАЛЬНИХ ОПЕРАТОРІВ

We obtain some necessary and sufficient conditions for a weighted weak-type inequality of the form

$$\int\limits_{\{M_g^+(f)>\lambda\}} \widetilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx \le C_1\int\limits_{-\infty}^{+\infty}\widetilde{\varphi}\left(C_1\frac{|f(x)|}{\omega_1(x)\omega_2(x)}\right)\omega_2(x)\,dx$$

to be true, which generalize some known results.

Наведено деякі необхідні й достатні умови для того, щоб виконувалась зважена нерівність слабкого типу

$$\int\limits_{\{M_g^+(f)>\lambda\}} \widetilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx \le C_1\int\limits_{-\infty}^{+\infty}\widetilde{\varphi}\left(C_1\frac{|f(x)|}{\omega_1(x)\omega_2(x)}\right)\omega_2(x)\,dx.$$

Результати, отримані в роботі, узагальнюють деякі відомі результати.

1. Introduction. In the past half century, the modern harmonic analysis theory has made a lot of great progress, the expansion of the two-sided operator theory and the demand of ergodic theory have greatly promoted the study of one-sided operator, among which the research on the weighted inequalities of one-sided operators has aroused the interest of many scholars (see [1, 2, 4, 5, 7 – 15]). Hardy – Littlewood maximal operators, one-sided maximal operators and their transformations have always been one of the main research objects in harmonic analysis, there are different technical requirements to deal with these operators, and seeking the correspondences between them is the core content in the study of one-sided maximal operators.

In 1994, A. Gogatishvili and V. Kokilashvili [6] studied the following weighted inequality:

$$\int\limits_{\{x\,:\,Mf(x)>\lambda\}} \widetilde{\varphi}\left(\frac{\lambda}{\omega(x)}\right)\omega(x)\,d\mu \le C\int\limits_X \widetilde{\varphi}\left(C\,\frac{f(x)}{\omega(x)}\right)\omega(x)\,d\mu, \tag{1.1}$$

where $\mathcal{M}f$ is a Hardy – Littlewood maximal operator in homogeneous space. Its four-weight generalization form was studied in [16]. The main aim of this paper is to give the corresponding form of the

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multiweight inequality (1.1) for the one-sided maximal operators. A weighted weak-type inequalities of the form

$$\int\limits_{\{M_g^+(f)>\lambda\}} \widetilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx \le C_1 \int\limits_{-\infty}^{+\infty} \widetilde{\varphi}\left(C_1\,\frac{|f(x)|}{\omega_1(x)\omega_2(x)}\right)\omega_2(x)\,dx$$

will be considered, and some necessary and sufficient conditions for it are obtained. Our work generalize some known results.

2. Preliminaries. This section mainly introduces some concepts, symbols and related knowledge which we will use.

In what follows the symbol Φ will be used to denote the set of all functions $\varphi\colon \mathbb{R}^1 \to \mathbb{R}^1$ which are nonnegative, even, and increasing on $(0,\infty)$ such that $\lim_{t\to 0^+}\varphi(t)=0$, $\lim_{t\to\infty}\varphi(t)=\infty$.

A function ω is called a Young function if $\omega\in\Phi$ and it is not identically zero or ∞ on $(0,\infty)$; at some point $t>0$ it may have a jump up to ∞ but in that case it must be left continuous at t.

A function $\varphi(t)$ is called quasiconvex if there exists a Young function ω and a constant $C>1$ such that $\omega(t)\le\varphi(t)\le\omega(Ct)$, $t\ge 0$. The function $\varphi(t)$ is said to be quasiincrease if there exists a constant $C>0$ such that

$$\varphi(t_1)\le C\varphi(Ct_2)$$

for each t_1 and t_2, $0<t_1<t_2$. For more information, one can refer to [3].

We can associate every quasiconvex function φ with its complementary function $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(t)=\sup_{s\ge 0}(st-\varphi(s)).$$

The subadditivity of the supremum readily implies that $\tilde{\varphi}$ is a Young function. We recall the Young inequality

$$st\le\varphi(s)+\tilde{\varphi}(t),\quad s,t\ge 0.$$

Given a positive and locally integrable function g on the real line, the one-sided maximal function M_g^+ is defined by

$$M_g^+f(x)=\sup_{h>0}\frac{1}{g(x,x+h)}\int\limits_x^{x+h}|f(y)|g(y)\,dy$$

for $f\in L_{1,\mathrm{loc}}(\mathbb{R}^1)$, where $g(x,x+h)=\displaystyle\int_x^{x+h} g(y)\,dy$.

The one-sided g-mean of f is defined as

$$\mu_g^+(f)=\mu_g^+(f,a,b,c)=\frac{1}{g(a,c)}\int\limits_b^c|f(x)|g(x)\,dx,\quad a<b<c.$$

For an interval (a,b) and a measurable function h, we denote $\displaystyle\int_a^b h(x)\,dx$ by $h(a,b)$. As usual, $\{M_g^+f>\lambda\}$ will stand for $\{x\in R\colon M_g^+f(x)>\lambda\}$.

Let $\varphi\in\Phi$ be a quasiconvex function, it was proved in [6] that there exists a constant $\delta>0$ such that, for an arbitrary $t>0$, we have

$$\tilde{\varphi}\left(\delta\,\frac{\varphi(t)}{t}\right) \leq \varphi(t) \leq \tilde{\varphi}\left(2\,\frac{\varphi(t)}{t}\right) \tag{2.1}$$

and

$$\varphi\left(\delta\,\frac{\tilde{\varphi}(t)}{t}\right) \leq \tilde{\varphi}(t) \leq \varphi\left(2\,\frac{\tilde{\varphi}(t)}{t}\right). \tag{2.2}$$

For a quasiconvex function φ, we obtain

$$\varepsilon\varphi(t) \leq \varphi(C\varepsilon t), \quad t > 0, \quad \varepsilon > 1, \tag{2.3}$$

and

$$\varphi(\gamma t) \leq \gamma\varphi(Ct), \quad t > 0, \quad \gamma < 1, \tag{2.4}$$

where the constants C do not depend on t, ε, and γ (see [6]).

Throughout this paper, C and C_i are used to denote positive constants, which may be different from one line to another.

3. Main result and proof. The main result of this paper is the following theorem.

Theorem 3.1. *Let $\varphi \in \Phi$ be a quasiconvex function and ω_i, $i = 1, 2, 3, 4$, be a weight. Then the following statements are equivalent:*

(i) *there exists a constant $C_1 > 0$ such that*

$$\int\limits_{\left\{M_g^+(f)>\lambda\right\}} \widetilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx \leq C_1 \int\limits_{-\infty}^{+\infty} \widetilde{\varphi}\left(C_1\frac{|f(x)|}{\omega_1(x)\,\omega_2(x)}\right)\omega_2(x)dx \tag{3.1}$$

holds for all f and $\lambda > 0$;

(ii) *there exists a constant $C_2 > 0$ such that*

$$\int\limits_a^b \tilde{\varphi}\left(\frac{\mu_g^+(f)}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx \leq C_2 \int\limits_b^c \tilde{\varphi}\left(C_2\frac{|f(x)|}{\omega_1(x)\omega_2(x)}\right)\omega_2(x)\,dx \tag{3.2}$$

holds for all f and $a < b < c$;

(iii) *there exist constants $C_3 > 0$ and $\varepsilon > 0$ such that*

$$\int\limits_b^c \varphi\left(\frac{\varepsilon}{g(a,c)}\int\limits_a^b \tilde{\varphi}\left(\frac{\lambda}{\omega_3(y)\omega_4(y)}\right)\frac{\omega_4(y)}{\lambda}\,dy\omega_1(x)g(x)\right)\omega_2(x)\,dx$$

$$\leq C_3 \int\limits_a^b \tilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx \tag{3.3}$$

holds for all $\lambda > 0$ and $a < b < c$.

Proof. The implication (i) $\Rightarrow$ (ii) is an easy consequence of the estimate

$$M_g^+ f(x) \ge \mu_g^+(f)\chi_{(a,b)}(x),$$

which is valid for all $f,\ x$ and $a < b < c.$

(ii) $\Rightarrow$ (iii). For $k \in \mathbb{N}$, we set $B_k = \left\{x \in (b,c) : \dfrac{1}{k} < \omega_1(x)g(x) < k\right\}$ and

$$A(x) = \left(\frac{1}{g(a,c)} \int_a^b \tilde{\varphi}\left(\frac{\lambda}{\omega_3(y)\omega_4(y)}\right)\frac{\omega_4(y)}{\lambda}\, dy\omega_1(x)g(x)\right)^{-1}$$

$$\times\, \varphi\left(\frac{\varepsilon}{g(a,c)} \int_a^b \tilde{\varphi}\left(\frac{\lambda}{\omega_3(y)\omega_4(y)}\right)\frac{\omega_4(y)}{\lambda}\, dy\omega_1(x)g(x)\right)\chi_{B_k},$$

where χ_{B_k} denotes the characteristic function of the set B_k, and ε will be specified later. Then

$$I = \int_{B_k} \varphi\left(\frac{\varepsilon}{g(a,c)} \int_a^b \tilde{\varphi}\left(\frac{\lambda}{\omega_3(y)\omega_4(y)}\right)\frac{\omega_4(y)}{\lambda}\, dy\omega_1(x)\, g(x)\right)\omega_2(x)\, dx$$

$$= \frac{1}{\lambda g(a,c)} \int_b^c A(x)\omega_1(x)\omega_2(x)g(x)\, dx \int_a^b \tilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\, dx.$$

If $\dfrac{1}{g(a,c)}\displaystyle\int_b^c A(x)\omega_1(x)\omega_2(x)g(x)\, dx < \lambda$, then we have

$$I < \int_a^b \tilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\, dx;$$

if $\dfrac{1}{g(a,c)}\displaystyle\int_b^c A(x)\omega_1(x)\omega_2(x)g(x)\, dx \ge \lambda$, then by (2.3) we get

$$I \le \int_a^b \tilde{\varphi}\left(\frac{C}{g(a,c)}\,\frac{\int_b^c A(y)\omega_1(y)\omega_2(y)g(y)\, dy}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\, dx.$$

Consequently,

$$I \le \int_a^b \tilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\, dx + \int_a^b \tilde{\varphi}\left(\frac{C}{g(a,c)}\,\frac{\int_b^c A(y)\omega_1(y)\omega_2(y)g(y)\, dy}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\, dx.$$

It follows from (3.2) that

$$I \le \int\limits_a^b \tilde\varphi\Big(\frac{\lambda}{\omega_3(x)\omega_4(x)}\Big)\omega_4(x)\,dx + C_2\int\limits_b^c \tilde\varphi(CC_2A(x))\omega_2(x)\,dx,$$

where C is the constant in (2.3). Then choose ε so small that $\dfrac{C^2C_2\varepsilon}{\delta} < 1$ and $\dfrac{C^2C_2^2\varepsilon}{\delta} < 1$, where δ is from (2.1). By the definition of $A(x)$, (2.4) and (2.1), we obtain

$$I \le \int\limits_a^b \tilde\varphi\Big(\frac{\lambda}{\omega_3(x)\omega_4(x)}\Big)\omega_4(x)\,dx$$

$$+ C_2\int\limits_{B_k}\tilde\varphi\left(C^2C_2\varepsilon\,\frac{\varphi\Big(\dfrac{\varepsilon}{g(a,c)}\displaystyle\int_a^b\tilde\varphi\Big(\frac{\lambda}{\omega_3(y)\omega_4(y)}\Big)\frac{\omega_4(y)}{\lambda}\,dy\omega_1(x)g(x)\Big)}{\dfrac{C\varepsilon}{g(a,c)}\displaystyle\int_a^b\tilde\varphi\Big(\frac{\lambda}{\omega_3(y)\omega_4(y)}\Big)\frac{\omega_4(y)}{\lambda}\,dy\,\omega_1(x)g(x)}\right)\omega_2(x)\,dx$$

$$\le \int\limits_a^b \tilde\varphi\Big(\frac{\lambda}{\omega_3(x)\omega_4(x)}\Big)\omega_4(x)\,dx$$

$$+ \frac{C^2C_2^2\varepsilon}{\delta}\int\limits_{B_k}\tilde\varphi\left(C\delta\,\frac{\varphi\Big(\dfrac{\varepsilon}{g(a,c)}\displaystyle\int_a^b\tilde\varphi\Big(\frac{\lambda}{\omega_3(y)\omega_4(y)}\Big)\frac{\omega_4(y)}{\lambda}\,dy\omega_1(x)g(x)\Big)}{\dfrac{C\varepsilon}{g(a,c)}\displaystyle\int_a^b\tilde\varphi\Big(\frac{\lambda}{\omega_3(y)\omega_4(y)}\Big)\frac{\omega_4(y)}{\lambda}\,dy\omega_1(x)g(x)}\right)\omega_2(x)\,dx$$

$$\le \int\limits_a^b \tilde\varphi\Big(\frac{\lambda}{\omega_3(x)\,\omega_4(x)}\Big)\omega_4(x)\,dx$$

$$+ \frac{C^2C_2^2\varepsilon}{\delta}\int\limits_{B_k}\varphi\left(\frac{\varepsilon}{g(a,c)}\int\limits_a^b\tilde\varphi\Big(\frac{\lambda}{\omega_3(y)\,\omega_4(y)}\Big)\frac{\omega_4(y)}{\lambda}\,dy\omega_1(x)g(x)\right)\omega_2(x)\,dx,$$

that is,

$$I \le \int\limits_a^b \tilde\varphi\Big(\frac{\lambda}{\omega_3(x)\omega_4(x)}\Big)\omega_4(x)\,dx + \frac{C^2\,C_2^2\varepsilon}{\delta}\,I.$$

Since I is finite, it follows from the above inequality that

$$\int\limits_{B_k}\varphi\left(\frac{\varepsilon}{g(a,c)}\int\limits_a^b\tilde\varphi\Big(\frac{\lambda}{\omega_3(y)\omega_4(y)}\Big)\frac{\omega_4(y)}{\lambda}\,dy\omega_1(x)g(x)\right)\omega_2(x)\,dx$$

$$\le \frac{\delta}{\delta - C^2C_2^2\varepsilon}\int\limits_a^b\tilde\varphi\Big(\frac{\lambda}{\omega_3(x)\omega_4(x)}\Big)\omega_4(x)\,dx.$$

Now let $k \to \infty$, so that (iii) is valid.

(iii) $\Rightarrow$ (i). For a fixed $\lambda > 0$, it is known that

$$\left\{M_g^+(f) > \frac{8}{\varepsilon}\,\lambda\right\} = \bigcup_{i=1}^{\infty}(a_i, b_i),$$

where

$$\frac{8}{\varepsilon}\,\lambda \le \frac{1}{g(x, b_i)} \int\limits_x^{b_i} |f(t)|g(t)\,dt \quad \forall x \in (a_i, b_i). \tag{3.4}$$

Now using the "cutting method" introduced by F. J. Martín-Reyes [5], we assume that (a, b) is one of the intervals (a_i, b_i), and set $x_0 = a$ and $x_k,\ x_k \nearrow b$ such that the equality

$$\int\limits_{x_k}^{b} |f(x)|g(x)\,dx = 2 \int\limits_{x_{k+1}}^{b} |f(x)|g(x)\,dx \tag{3.5}$$

holds for any $k \in \mathbb{N}$. From (3.4) and (3.5), we obtain

$$\frac{2}{\varepsilon}\lambda \le \frac{1}{g(x_{k-1}, x_{k+1})} \int\limits_{x_k}^{x_{k+1}} |f(x)|g(x)\,dx,$$

from which we have

$$\int\limits_{x_{k-1}}^{x_k} \tilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx$$

$$\le \frac{\varepsilon}{2\lambda g(x_{k-1}, x_{k+1})} \int\limits_{x_{k-1}}^{x_k} \tilde{\varphi}\left(\frac{\lambda}{\omega_3(y)\omega_4(y)}\right)\omega_4(y)\,dy \int\limits_{x_k}^{x_{k+1}} |f(x)|g(x)\,dx$$

$$\le \frac{1}{2C_3} \int\limits_{x_k}^{x_{k+1}} \frac{\varepsilon}{g(x_{k-1}, x_{k+1})} \int\limits_{x_{k-1}}^{x_k} \tilde{\varphi}\left(\frac{\lambda}{\omega_3(y)\omega_4(y)}\right)\frac{\omega_4(y)}{\lambda}\,dy\,\omega_1(x)g(x)\,\frac{C_3|f(x)|}{\omega_1(x)\omega_2(x)}\,\omega_2(x)\,dx.$$

By the Young inequality, we get

$$\int\limits_{x_{k-1}}^{x_k} \tilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx$$

$$\le \frac{1}{2C_3} \int\limits_{x_k}^{x_{k+1}} \varphi\left(\frac{\varepsilon}{g(x_{k-1}, x_{k+1})} \int\limits_{x_{k-1}}^{x_k} \tilde{\varphi}\left(\frac{\lambda}{\omega_3(y)\omega_4(y)}\right)\frac{\omega_4(y)}{\lambda}\,dy\omega_1(x)g(x)\right)\omega_2(x)\,dx$$

$$+\frac{1}{2C_3}\int\limits_{x_k}^{x_{k+1}}\tilde{\varphi}\left(\frac{C_3|f(x)|}{\omega_1(x)\omega_2(x)}\right)\omega_2(x)\,dx.$$

It follows from (3.3) that

$$\int\limits_{x_{k-1}}^{x_k}\tilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx\le\frac{1}{C_3}\int\limits_{x_k}^{x_{k+1}}\tilde{\varphi}\left(\frac{C_3|f(x)|}{\omega_1(x)\omega_2(x)}\right)\omega_2(x)\,dx.$$

By summing up k, we have

$$\int\limits_a^b\tilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx\le C_4\int\limits_a^b\tilde{\varphi}\left(\frac{C_4|f(x)|}{\omega_1(x)\omega_2(x)}\right)\omega_2(x)\,dx.$$

Consequently, we obtain

$$\int\limits_{\left\{M_g^+(f)>\frac{8}{\varepsilon}\lambda\right\}}\tilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx=\sum_{i=1}^{\infty}\int\limits_{a_i}^{b_i}\tilde{\varphi}\left(\frac{\lambda}{\omega_3(x)\omega_4(x)}\right)\omega_4(x)\,dx$$

$$\le C_4\sum_{i=1}^{\infty}\int\limits_{a_i}^{b_i}\tilde{\varphi}\left(\frac{C_4|f(x)|}{\omega_1(x)\omega_2(x)}\right)\omega_2(x)\,dx$$

$$\le C_4\int\limits_{-\infty}^{+\infty}\tilde{\varphi}\left(\frac{C_4|f(x)|}{\omega_1(x)\omega_2(x)}\right)\omega_2(x)\,dx.$$

From the above estimates, we conclude that (i) is valid.

Theorem 3.1 is proved.

***Remark* 3.1.** If we interchange φ and $\widetilde{\varphi}$ in Theorem 3.1, then the conclusions still hold.

If we put $\omega_1=\omega_3=1,\ \omega_2=\omega_4=\omega$ in Theorem 3.1, then we have the following corollary.

***Corollary* 3.1.** *Let $\varphi\in\Phi$ be a quasiconvex function and ω be a weight. Then the following statements are equivalent*:

(i) *there exists a constant $C_1>0$ such that*

$$\int\limits_{\left\{M_g^+(f)>\lambda\right\}}\varphi\left(\frac{\lambda}{\omega(x)}\right)\omega(x)\,dx\le C_1\int\limits_{-\infty}^{+\infty}\varphi\left(C_1\frac{|f(x)|}{\omega(x)}\right)\omega(x)\,dx$$

holds for all f and $\lambda>0$;

(ii) *there exists a constant $C_2>0$ such that*

$$\int\limits_a^b\varphi\left(\frac{\mu_g^+(f)}{\omega(x)}\right)\omega(x)\,dx\le C_2\int\limits_b^c\varphi\left(C_2\frac{|f(x)|}{\omega(x)}\right)\omega(x)\,dx$$

holds for all f and $a<b<c$;

(iii) *there exist constants* $C_3 > 0$ *and* $\varepsilon > 0$ *such that*

$$\int\limits_b^c \widetilde{\varphi}\left(\frac{\varepsilon}{g(a,c)}\int\limits_a^b \varphi\left(\frac{\lambda}{\omega(y)}\right)\frac{\omega(y)}{\lambda}\,dyg(x)\right)\omega(x)\,dx \le C_3\int\limits_a^b \varphi\left(\frac{\lambda}{\omega(x)}\right)\omega(x)\,dx$$

holds for all $\lambda > 0$ *and* $a < b < c$.

If we put $\omega_1 = \omega_3 = \dfrac{1}{\omega}$, $\omega_2 = \omega_4 = \omega$ in Theorem 3.1, then we have the following corollary.

***Corollary* 3.2.** *Let* $\varphi \in \Phi$ *be a quasiconvex function and* ω *be a weight. Then the following statements are equivalent*:

(i) *there exists a constant* $C_1 > 0$ *such that*

$$\varphi(\lambda)\int\limits_{\{M_g^+(f)>\lambda\}} \omega(x)\,dx \le C_1\int\limits_{-\infty}^{+\infty} \varphi(C_1|f(x)|)\omega(x)\,dx \tag{3.6}$$

holds for all f *and* $\lambda > 0$;

(ii) *there exists a constant* $C_2 > 0$ *such that*

$$\int\limits_a^b \varphi\big(\mu_g^+(f)\big)\omega(x)\,dx \le C_2\int\limits_b^c \varphi(C_2|f(x)|)\omega(x)\,dx$$

holds for all f *and* $a < b < c$;

(iii) *there exist constants* C_3 *and* $\varepsilon > 0$ *such that*

$$\int\limits_b^c \varphi\left(\frac{\varepsilon}{g(a,c)}\int\limits_a^b \frac{\varphi(\lambda)\omega(y)}{\lambda}\,dy\,\frac{g(x)}{\omega(x)}\right)\omega(x)\,dx \le C_3\varphi(\lambda)\int\limits_a^b \omega(x)\,dx$$

holds for all $\lambda > 0$ *and* $a < b < c$.

We should point out that the weighted inequality (3.6) has been studied by P. Ortega Salvador in [9]. Here we obtain its new characterizations by Theorem 3.1.

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