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LEGENDRE SUPERCONVERGENT DEGENERATE KERNEL AND NYSTRÖM METHODS FOR NONLINEAR INTEGRAL EQUATIONS

СУПЕРЗБІЖНЕ ВИРОДЖЕНЕ ЯДРО ЛЕЖАНДРА І МЕТОДИ НІСТРЕМА ДЛЯ НЕЛІНІЙНИХ ІНТЕГРАЛЬНИХ РІВНЯНЬ

We study polynomially based superconvergent collocation methods for the approximation of solutions of nonlinear integral equations. The superconvergent degenerate kernel method is chosen for approximating the solutions of Hammerstein equations, while a superconvergent Nyström method is used for solving Urysohn equations. By applying interpolatory projections based on Legendre polynomials of degree $\leq n$, we analyze the superconvergence of these methods and their iterated versions. Numerical results are presented to validate the theoretical results.

Досліджено суперзбіжні методи колокації на поліноміальній основі для апроксимації розв'язків нелінійних інтегральних рівнянь. Для апроксимації розв'язків рівнянь Гаммерштейна використано суперзбіжний метод виродженого ядра, а для розв'язування рівнянь Урисона — суперзбіжний метод Ністрема. Застосовуючи інтерполяційні проєкції на основі поліномів Лежандра степеня $\leq n$, проаналізовано суперзбіжність цих методів та їхніх ітерованих версій. Дані числових розрахунків наведено для підтвердження теоретичних результатів.

1. Introduction. Nonlinear integral equations arise from different fields in mathematical physics like potential problems, electromagnetic fluid dynamics and transport problems (see [6]). In terms of nonlinear functional analysis we find two important special types including Hammerstein and Urysohn integral equations.

The Hammerstein integral equation is

$$x - \mathcal{K}x = f, \tag{1.1}$$

where $\mathcal{K}$ is the integral operator defined on $\mathscr{X} = \mathscr{L}^{\infty}[-1,1]$ by

$$(\mathcal{K}x)(s) = \int_{-1}^{1} \kappa(s,t)\psi(t,x(t))dt, \quad s \in [-1,1], \quad x \in \mathscr{X},$$

f and ψ are known functions, with $\psi(t,u)$ nonlinear in u and x is the function to be found. If the kernel κ is continuous, then $\mathcal{K}$ is a compact operator from $\mathscr{X}$ to $\mathscr{C}[-1,1]$.

The Urysohn integral equation

$$x - \mathcal{T}x = f, \tag{1.2}$$

where $\mathcal{T}$ is the nonlinear integral operator defined on $\mathscr{X}$ by

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$$(\mathcal{T}x)(s) = \int_{-1}^{1} \kappa(s,t,x(t))dt, \quad s \in [-1,1],$$

where the kernel $\kappa(s,t,u)$ is a real smooth function and u is the unknown function to be determined. This equation includes the Hammerstein equation and many other equations. As a consequence, a theory for it can also be based on generalizations of that of Hammerstein equations [5].

Various numerical methods for approximating the solutions of nonlinear integral equation with smooth or less smooth kernels have been extensively investigated in the literature. In [9], a degenerated kernel method for solving (1.1) that consists in approximating the kernel by several specific degenerate kernels was proposed and in [10], iterated degenerated kernel method was presented to obtain superconvergence results. Moreover, a variation of Nyström's method was proposed by Lardy [8]. A superconvergent Nyström method for Urysohn integral equations (1.2) was studied in [2]. Many authors have studied numerical methods to solve nonlinear integral equations with different kernels (see [12 – 15]). There have been many approaches to improve the accuracy of numerical solutions. In this framework, Kulkarni has introduced in [11] a new method (called modified projection or multiprojection method) which aims to improve the convergence of the classical methods. In [3], superconvergent Nyström and degenerate kernel methods for Hammerstein integral equation was discussed using the piecewise polynomial basis and established the rate of convergence of the approximate solution of (1.1). Recently, the following two approximation operators are inspired by the modified projection in [1]. They consist in approximating the operator $\mathcal{K}$ by one of the two finite rank operators:

$$\mathcal{K}_n = \pi_n\mathcal{K} + \mathcal{K}_{n,i} - \pi_n\mathcal{K}_{n,i}, \quad i = 1,2,$$

$$\mathcal{K} - \mathcal{K}_n = (I - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i}),$$

where π_n is a sequence of interpolatory projections, $\mathcal{K}_{n,1}$ is the degenerate kernel operator obtained by interpolating the kernel with respect to the second variable and $\mathcal{K}_{n,2}$ is the Nyström operator based on π_n.

The purpose of this paper is to approximate $\mathcal{K}$ by the degenerate kernel operator, defined in (2.6), to solve the Hammerstein integral equation (1.1) and $\mathcal{T}$ by the Nyström operator, established in (2.10) to solve the Urysohn integral equation (1.2). In particular, we use Legendre polynomials bases, which can be generated recursively with ease and possess nice property of orthogonality. Theses polynomials, are less expensive computationally compared to piecewise polynomial basis functions and to other orthogonal polynomials. Also the associated nonlinear systems which are needed to be solved to evaluate the approximate solutions are much smaller as compared to those obtained when using piecewise polynomials. It is shown that the Legendre superconvergent degenerate kernel and Nyström solutions converge with the order $\mathcal{O}\left(n^{-2r+\frac{1}{2}}\right)$ in infinity norm, where r denotes the smoothness of the kernel and n denotes the degree of the Legendre polynomials used. By using the Sloan iteration, we prove that the order of convergence of two methods can be improved to reach $\mathcal{O}(n^{-2r})$.

In the last few years, several polynomially based projection methods for nonlinear equations were studied. Legendre superconvergent Galerkin-collocation type methods for Hammerstein equations was proposed in [4]. Other important results on the numerical solutions of nonlinear integral equations using Legendre and Chebyshev polynomials can be found in [7, 13].

This paper is organized as follows. In Section 2, we set up notations and discuss superconvergent degenerate kernel and Nyström methods to obtain superconvergence results. Section 3 contains the convergence orders of the approximate solutions and their iterated versions. In Section 4, we present numerical examples, which illustrate the theoretical estimates.

2. Preliminaries and methods. Let $\mathbb{X}_n$ denote the space of all polynomials of degree $\leq n$ defined on $[-1,1]$. Then the dimension of $\mathbb{X}_n$ is $n+1$, and the Legendre polynomials $\{L_0, L_1, \ldots, L_n\}$, defined by

$$L_0(s) = 1, \quad L_1(s) = s, \quad s \in [-1,1],$$

$$(i+1)L_{i+1}(s) = (2i+1)sL_i(s) - iL_{i-1}(s), \quad i = 1, 2, \ldots, n-1,$$

form an orthogonal basis for $\mathbb{X}_n$. For $u, v \in \mathscr{C}[-1,1]$, the inner product is given by

$$\langle u, v\rangle = \int\limits_{-1}^{1} u(t)v(t)dt \quad \text{and norm is} \quad \|u\|_{\mathscr{L}^2} = \left(\int\limits_{-1}^{1} u(t)^2 dt\right)^{\frac{1}{2}}.$$

For $x \in \mathscr{C}[-1,1]$, let $\pi_n x : \mathscr{C}[-1,1] \to \mathbb{X}_n$ be the interpolatory operator, defined by

$$(\pi_n x)(\tau_i) = x(\tau_i), \quad i = 0, 1, \ldots, n, \tag{2.1}$$

where $\{\tau_0, \tau_1, \ldots, \tau_n\}$ are the zeros of the Legendre polynomial L_{n+1}. In the Lagrange form, $\pi_n x$ can be written

$$(\pi_n x)(s) = \sum_{j=0}^{n} x(\tau_j)\ell_j(s), \quad s \in [-1,1],$$

where ℓ_j is the unique polynomial of degree n that satisfies $\ell_j(\tau_i) = \delta_{ij}$. Clearly, the interpolatory projection operator π_n is a linear operator on $\mathscr{C}[-1,1]$. According to the analysis of Golberg and Chen[7], the crucial properties of π_n are given in the following lemma.

Lemma 2.1. *Let $\pi_n : \mathscr{C}[-1,1] \to \mathbb{X}_n$ be the interpolatory projection operator defined by* (2.1). *There exists a constant $p > 0$, independent of n, such that, for $x \in \mathscr{C}[-1,1]$,*

$$\|\pi_n x\|_{\mathscr{L}^2} \leq p\|x\|_{\mathscr{L}^2}, \tag{2.2}$$

$$\|x - \pi_n x\|_{\mathscr{L}^2} \leq (1+p) \inf_{\phi \in \mathbb{X}_n} \|x - \phi\|_{\mathscr{L}^2}. \tag{2.3}$$

Moreover, for any $x \in \mathscr{C}^r[-1,1]$,

$$\|x - \pi_n x\|_{\mathscr{L}^2} \leq c_1 n^{-r} \|x^{(r)}\|_{\mathscr{L}^2}, \tag{2.4}$$

$$\|x - \pi_n x\|_\infty \leq c_1 n^{\frac{1}{2}-r} \|x^{(r)}\|_\infty, \tag{2.5}$$

where c_1 is a constant, independent of n.

Let us consider the degenerate kernel

$$\pi_n \kappa(s,t) = \kappa_n(s,t) = \sum_{i=0}^{n} \kappa(s,\tau_i)\ell_i(t),$$

obtained by interpolating by π_n the kernel $\kappa(s,t)$ considered as a function of t. The associated Hammerstein operator is given by

$$(\mathcal{K}_n^D x)(s) = \int_{-1}^{1} \kappa_n(s,t)\psi(t,x(t))dt = \sum_{i=0}^{n} \kappa(s,\tau_i) \int_{-1}^{1} \ell_i(t)\psi(t,x(t))dt, \quad s \in [-1,1]. \tag{2.6}$$

We propose to approximate $\mathcal{K}$ by the following finite rank operator:

$$\begin{aligned} \mathcal{K}_n &= \pi_n \mathcal{K} + \mathcal{K}_n^D - \pi_n \mathcal{K}_n^D, \\ \mathcal{K} - \mathcal{K}_n &= (I - \pi_n)\big(\mathcal{K} - \mathcal{K}_n^D\big). \end{aligned} \tag{2.7}$$

The corresponding approximate of (1.1) becomes

$$x_n^D - \big(\pi_n \mathcal{K} + \mathcal{K}_n^D - \pi_n \mathcal{K}_n^D\big) x_n^D = f, \tag{2.8}$$

where x_n^D will be called the Legendre superconvergent degenerate kernel solution. The iterated solutions is defined by

$$\widetilde{x}_n^D = \mathcal{K} x_n^D + f. \tag{2.9}$$

On the other hand, the Nyström operator associated with $\mathcal{T}$ and based on π_n is defined by

$$(\mathcal{T}_n^N x)(s) := \int_{-1}^{1} \pi_n \kappa(s,.,x(.))(t)dt = \sum_{i=0}^{n} \omega_i \kappa(s,\tau_i,x(\tau_i)), \quad s \in [-1,1], \tag{2.10}$$

where $\omega_i = \displaystyle\int_{-1}^{1} \ell_i(t)dt \;\; \forall i = 0,1,\ldots,n.$

We propose to approximate $\mathcal{T}$ by the following two finite rank operators:

$$\begin{aligned} \mathcal{T}_n &= \pi_n \mathcal{T} + \mathcal{T}_n^N - \pi_n \mathcal{T}_n^N, \\ \mathcal{T} - \mathcal{T}_n &= (I - \pi_n)\big(\mathcal{T} - \mathcal{T}_n^N\big). \end{aligned} \tag{2.11}$$

The corresponding approximate of (1.2) becomes

$$x_n^N - \big(\pi_n \mathcal{T} + \mathcal{T}_n^N - \pi_n \mathcal{T}_n^N\big) x_n^N = f, \tag{2.12}$$

where x_n^N is the Legendre superconvergent Nyström solution. The iterated solutions is defined by

$$\widetilde{x}_n^N = \mathcal{T} x_n^N + f. \tag{2.13}$$

Implementation note. We consider the reduction of (2.8) to a system of nonlinear equations. Set $k_j := \kappa(., \tau_j)$, from equation (2.8), we can easily show that the approximate solution x_n^D has the following from:

$$x_n^D = f + \sum_{i=0}^{n} a_i \ell_i + \sum_{j=0}^{n} b_j k_j, \tag{2.14}$$

where the coefficients $\{a_i, b_i,\ i = 0, 1, \ldots, n\}$ are obtained by substituting x_n^D from equation (2.14) into equation (2.8). Then we successively have

$$\pi_n \mathcal{K} x_n^D = \sum_{i=0}^{n} (\mathcal{K} x_n^D)(\tau_i)\ell_i = \sum_{i=0}^{n} \left[\int_{-1}^{1} \kappa(\tau_i, t)\psi\left(t, \sum_{k=0}^{n} a_k \ell_k(t) + \sum_{l=0}^{n} b_l k_l(t) + f(t) \right) dt \right] \ell_i,$$

$$\mathcal{K}_n^D x_n^D = \sum_{j=0}^{n} k_j \int_{-1}^{1} \psi\left(t, \sum_{k=0}^{n} a_k \ell_k(t) + \sum_{l=0}^{n} b_l k_l(t) + f(t) \right) \ell_j(t) dt,$$

$$\begin{aligned} \pi_n \mathcal{K}_n^D x_n^D &= \sum_{i=0}^{n} \big(\mathcal{K}_n^D x_n^D\big)(\tau_i)\ell_i \\ &= \sum_{i=0}^{n} \left\{ \sum_{j=0}^{n} \left[\int_{-1}^{1} \psi\left(t, \sum_{k=0}^{n} a_k \ell_k(t) + \sum_{l=0}^{n} b_l k_l(t) + f(t) \right) \ell_j(t) dt \right] k_j(\tau_i) \right\} \ell_i. \end{aligned}$$

Except for some very specific situations, the family of functions $\{\ell_i, k_j\}$ are linearly independent, therefore, we can identify the coefficients of ℓ_i and k_j, respectively, and we obtain the nonlinear system of size $2n + 2$:

$$a_i = \int_{-1}^{1} \kappa(\tau_i, t)\psi\left(t, \sum_{k=0}^{n} a_k \ell_k(t) + \sum_{l=0}^{n} b_l k_l(t) + f(t) \right) dt - \sum_{j=0}^{n} b_j k_j(\tau_i), \quad i = 1, \ldots, n,$$

$$b_j = \int_{-1}^{1} \psi\left(t, \sum_{k=0}^{n} a_k \ell_k(t) + \sum_{l=0}^{n} b_l k_l(t) + f(t) \right) \ell_j(t) dt, \quad j = 1, \ldots, n.$$

Now to get the solution x_n^N, we apply π_n and $(I - \pi_n)$ to equation (2.12). Then we obtain

$$\pi_n x_n^N - \pi_n \mathcal{T}_n x_n^N = \pi_n f, \tag{2.15}$$

$$(I - \pi_n)x_n^N - (I - \pi_n)\mathcal{T}_n x_n^N = (I - \pi_n)f. \tag{2.16}$$

By writing

$$\mathcal{T} x_n^N = \mathcal{T}(I - \pi_n)x_n^N + \mathcal{T}\pi_n x_n^N \tag{2.17}$$

and replacing $(I - \pi_n)x_n^N$ by its expression from equation (2.16), equation (2.17) becomes

$$\mathcal{T}x_n^N = \mathcal{T}\big((I-\pi_n)\mathcal{T}_n^N x_n^N + \pi_n x_n^N + (I-\pi_n)f\big).$$

Now, by replacing $\mathcal{T}x_n^N$ in equation (2.15), we obtain

$$\pi_n x_n^N - \pi_n\mathcal{T}\big((I-\pi_n)\mathcal{T}_n^N x_n^N + \pi_n x_n^N + (I-\pi_n)f\big) = \pi_n f.$$

Then, for $i = 0, 1, \ldots, n$, we have

$$x_n^N(\tau_i) - \mathcal{T}\big((I-\pi_n)\mathcal{T}_n^N x_n^N + \pi_n x_n^N + (I-\pi_n)f\big)(\tau_i) = f(\tau_i).$$

Now using the expressions of the operators π_n, $\mathcal{T}$, and $\mathcal{T}_n^N$, we obtain the following nonlinear system of size $n+1$:

$$a_i - \int\limits_{-1}^{1} \kappa\left(\tau_i, t, \sum_{l=0}^{n}(a_l - f_l)\ell_l(t) + \sum_{l=0}^{n}\omega_l\kappa(t,\tau_i,a_l) - \sum_{l=0}^{n}\sum_{j=0}^{n}\omega_j\kappa(\tau_l,\tau_j,a_j)\ell_l(t) + f(t)\right)dt = f_i,$$

where $f_i := f(\tau_i)$ and the unknowns are $a_i = x_n^N(\tau_i)$, $i = 0, 1, \ldots, n$. From (2.16), the approximate solution is given by

$$\begin{aligned} x_n^N &= \pi_n x_n^N + (I-\pi_n)\mathcal{T}_n^N x_n^N + (I-\pi_n)f \\ &= f + \sum_{i=0}^{n}(a_i - f_i)\ell_i + \sum_{i=0}^{n}\omega_i\kappa(.,\tau_i,a_i) - \sum_{i=0}^{n}\sum_{j=0}^{n}\omega_j\kappa(\tau_i,\tau_j,a_j)\ell_i. \end{aligned} \tag{2.18}$$

Let x_0 be an isolated solution of (1.1) and (1.2), a and b be real numbers such that

$$\left[\min_{s\in[-1,1]} x_0(s), \max_{s\in[-1,1]} x_0(s)\right] \subset [a,b].$$

For $\delta_0 > 0$, let

$$\mathcal{B}(x_0,\delta_0) = \big\{y \in \mathscr{X} : \|x_0 - y\|_\infty < \delta_0\big\}.$$

***Remark* 2.1.** The iterated solutions $\widetilde{x}_n^D$ and $\widetilde{x}_n^N$ are obtained by substituting (2.14) and (2.18) into the definition (2.9) and (2.13), respectively. Now, applying π_n to both sides of equations (2.8), (2.12), (2.9) and (2.13), we obtain

$$\pi_n x_n^D = \pi_n\mathcal{K}x_n^D + \pi_n f = \pi_n\widetilde{x}_n^D,$$

$$\pi_n x_n^N = \pi_n\mathcal{T}x_n^N + \pi_n f = \pi_n\widetilde{x}_n^N,$$

and this yields, for $j = 0, 1, \ldots, n$,

$$x_n^D(\tau_j) = \widetilde{x}_n^D(\tau_j) \qquad \text{and} \qquad x_n^N(\tau_j) = \widetilde{x}_n^N(\tau_j).$$

The above formula proves that at the collocation points the convergence of x_n^D and x_n^N to x_0 are as rapid as that of $\widetilde{x}_n^D$ and $\widetilde{x}_n^N$ to x_0.

3. Convergence rates. In this section, we analyse the existence and uniqueness of the approximate solutions of (1.1), (1.2) and we discuss the superconvergence results.

3.1. Superconvergent degenerate kernel method. Define $\Omega_1 = [-1,1] \times [a,b]$ and assume that otherwise, the following conditions on κ, f, and ψ:

(i) $f \in [-1,1]$ and $\psi \in \mathscr{C}(\Omega_1)$;

(ii) $M_1 = \sup_{s\in[-1,1]} \displaystyle\int_{-1}^{1} |\kappa(s,t)|dt < \infty$;

(iii) the function $\psi(t,u)$ is Lipschitz continuous in $u \in [a,b]$, i.e., there exists a constant $\delta_1 > 0$, for which $|\psi(t,u_1) - \psi(t,u_2)| \le \delta_1 |u_1 - u_2|$ for all $u_1, u_2 \in [a,b]$;

(iv) the partial derivative $\partial\psi/\partial u$ of ψ with respect to the second variable exists and is Lipschitz continuous, that is, there exists a constant $\delta_2 > 0$ such that

$$\left|\frac{\partial\psi}{\partial u}(t,u_1) - \frac{\partial\psi}{\partial u}(t,u_2)\right| \le \delta_2 |u_1 - u_2| \quad \text{for all} \quad u_1, u_2 \in [a,b].$$

Using the assumption (iv), we see that the operator $\mathcal{K}$ is Fréchet differentiable and $\mathcal{K}'(x_0)$ is $M_1\delta_2$-Lipschitz. The Fréchet derivative at $x_0 \in \mathscr{C}[-1,1]$ is given by

$$(\mathcal{K}'(x_0)g)(s) = \int_{-1}^{1} \kappa(s,t)\frac{\partial\psi}{\partial u}(t,x_0(t))g(t)dt, \qquad s \in [-1,1], \quad g \in \mathscr{C}[-1,1].$$

Then $\mathcal{K}'(x_0)$ is a compact operator on $\mathscr{C}[-1,1]$. For $j = 0,1,\ldots,r$, we have

$$\begin{aligned}
\|[\mathcal{K}'(x_0)g]^{(j)}\|_\infty &= \max_{s\in[-1,1]}\left|\int_{-1}^{1} \frac{\partial^j\kappa}{\partial s^j}(s,t)\frac{\partial\psi}{\partial u}(t,x_0(t))g(t)dt\right| \\
&\le \max_{s,t\in[-1,1]}\left|\frac{\partial^j\kappa}{\partial s^j}(s,t)\right|\left|\frac{\partial\psi}{\partial u}(t,x_0(t))\right|\int_{-1}^{1}|g(t)|dt \\
&\le 2\|\kappa\|_{j,\infty}\Psi_1\|g\|_\infty,
\end{aligned}$$

where

$$\|\kappa\|_{r,\infty} = \max_{s,t\in[-1,1]}\left|\frac{\partial^r\kappa}{\partial s^r}(s,t)\right|, \qquad \Psi_1 = \max_{t\in[-1,1]}\left|\frac{\partial\psi}{\partial u}(t,x_0(t))\right|.$$

Hence, using condition (ii), we deduce that $\|\mathcal{K}'(x_0)g\|_\infty \le 2M_1\Psi_1\|g\|_\infty$. This implies that

$$\|\mathcal{K}'(x_0)\|_\infty \le 2M_1\Psi_1. \tag{3.1}$$

For the rest of paper, we set

$$\overline{\kappa}_s(t) = \kappa(s,t), \qquad \overline{\ell}_s(t) = \frac{\partial^r\kappa}{\partial s^r}(s,t), \qquad \psi_r(t) = \frac{\partial^r\psi}{\partial u^r}(t,x_0(t)) \quad s,t \in [-1,1].$$

The following lemma, which can be shown easily, will be used to prove the main results of this section.

Lemma 3.1. *Let $x_0 \in \mathscr{C}[-1,1]$ be the unique solution of* (1.1). *Assume that $\kappa \in \mathscr{C}^r[-1,1]^2$ and 1 is not an eigenvalue of $\mathcal{K}'(x_0)$. Then, for n large enough, $\big(I - \mathcal{K}_n'(x_0)\big)^{-1}$ exists and it is a bounded linear operator, i.e., there exists a constant $A_1 > 0$ such that*

$$\big\|\big(I - \mathcal{K}_n'(x_0)\big)^{-1}\big\|_\infty \leq A_1. \tag{3.2}$$

Proof. For each $g \in \mathscr{C}[-1,1]$ and each $t \in [-1,1]$ it follows from the Cauchy–Schwarz inequality and estimate (2.5) that

$$\big\|(\mathcal{K}'(x_0)g - \mathcal{K}_n'(x_0)g\big\|_\infty = \max_{-1\leq s\leq 1}\Big|(I-\pi_n)\big(\mathcal{K}'(x_0) - \mathcal{K}_n^{D'}(x_0)\big)g(s)\Big|$$

$$= \max_{-1\leq s\leq 1}\int_{-1}^{1}\big|(I-\pi_n)\kappa(s,t)\big[(I-\pi_n)\psi_1\big]g(t)\big|dt.$$

Then

$$\big\|(\mathcal{K}'(x_0)g - \mathcal{K}_n'(x_0)g\big\|_\infty \leq \|(I-\pi_n)\overline{\kappa}_s\|_{\mathscr{L}^2}\|[(I-\pi_n)\psi_1]g\|_{\mathscr{L}^2}$$

$$\leq 2\|(I-\pi_n)\overline{\kappa}_s\|_\infty\|(I-\pi_n)\psi_1\|_{\mathscr{L}^2}\|g\|_\infty$$

$$\leq 2c_1 n^{\frac{1}{2}-r}\|\kappa\|_{r,\infty}\|(I-\pi_n)\psi_1\|_{\mathscr{L}^2}\|g\|_\infty.$$

Since $\psi_1 \in \mathscr{C}[-1,1]$, we have $\|\psi_1 - \pi_n\psi_1\|_{\mathscr{L}^2} \to 0$ as $n \to \infty$, which implies that $\mathcal{K}_n'(x_0) \to \mathcal{K}'(x_0)$ pointwise in $\mathscr{C}[-1,1]$ as $n \to \infty$. Hence, by Lemma 2.6 in [6], the operators $\big(I - \mathcal{K}_n'(x_0)\big)^{-1}$ exists and are uniformly bounded, for some sufficiently large n.

Lemma 3.1 is proved.

The following theorem can be proved by using Theorem 2 given in [16].

Theorem 3.1. *Let $x_0 \in \mathscr{C}[-1,1]$ be an isolated solution of* (1.1). *Assume that 1 is not an eigenvalue of $\mathcal{K}'(x_0)$. Then there exists a real number $\delta_0 > 0$ such that the approximate equation* (2.8) *has a unique solution x_n^D in $\mathcal{B}(x_0, \delta_0)$ for a sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n, such that*

$$\frac{\alpha_n}{1+q} \leq \big\|x_0 - x_n^D\big\|_\infty \leq \frac{\alpha_n}{1-q},$$

where $\alpha_n = \big\|(I - \mathcal{K}_n'(x_0))^{-1}(\mathcal{K}(x_0) - \mathcal{K}_n(x_0))\big\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$.

The next theorem establish the rate of convergence of the approximation x_n^D to the exact solution x_0.

Theorem 3.2. *Let $x_0 \in \mathscr{C}[-1,1]$ be an isolated solution of* (1.1). *Assume that $\kappa \in \mathscr{C}^r[-1,1]^2$, $\psi \in \mathscr{C}^r(\Omega)$ and $f \in \mathscr{C}[-1,1]$. Let x_n^D be the unique solution of* (2.8) *in $\mathcal{B}(x_0, \delta_0)$. Then*

$$\|x_0 - x_n^D\|_\infty = \mathcal{O}\big(n^{-2r+\frac{1}{2}}\big). \tag{3.3}$$

Proof. We see from Theorem 3.1 that to estimate $\|x_0 - x_n^D\|_\infty$ we need to estimate $\|\mathcal{K}(x_0) - \mathcal{K}_n(x_0)\|_\infty$. Using estimates (2.5), (2.7), and Lemma 3.1, we have

$$\|x_0 - x_n^D\|_\infty \leq A_1\|(I-\pi_n)\big(\mathcal{K}(x_0) - \mathcal{K}_n^D(x_0)\big)\|_\infty$$

$$\leq c_1 A_1 n^{\frac{1}{2}-r}\|[\mathcal{K}(x_0) - \mathcal{K}_n^D(x_0)]^{(r)}\|_\infty. \tag{3.4}$$

For each $s \in [-1,1]$, we obtain

$$\big[\mathcal{K}(x_0) - \mathcal{K}_n^D(x_0)\big]^{(r)}(s) = \int_{-1}^{1} \psi(t, x_0(t))(I - \pi_n)\overline{\ell}_s(t)dt.$$

Then, taking supremum and using the Cauchy – Schwarz inequality, we get

$$\big\|\big[\mathcal{K}(x_0) - \mathcal{K}_n^D(x_0)\big]^{(r)}\big\|_\infty \le \max_{s\in[-1,1]} \|\psi_0\|_{\mathscr{L}^2}\big\|(I-\pi_n)\overline{\ell}_s\big\|_{\mathscr{L}^2}$$

$$\le \sqrt{2}c_1 n^{-r} \max_{s\in[-1,1]} \|\psi_0\|_\infty \big\|\overline{\ell}_s^{(r)}\big\|_{\mathscr{L}^2} \le 2c_1 n^{-r}\Psi_0\|\kappa\|_{2r,\infty}. \qquad (3.5)$$

Therefore, the estimate (3.3) follows from (3.4) and (3.5).

Theorem 3.2 is proved.

The following lemma will be used to obtain the rate of convergence of $\widetilde{x}_n^D$ to x_0.

Lemma 3.2. *Assume that* $\kappa \in \mathscr{C}^r[-1,1]^2$ *and* $\dfrac{\partial\psi}{\partial u} \in \mathscr{C}^r(\Omega_1)$. *Then the linear operator* $\mathcal{K}_n'(x_0)$ *is Lipschitz continuous*, *that is*, *there exists a constant* $\delta_3 > 0$, *independent of* n, *such that*

$$\big\|\mathcal{K}_n'(x_0) - \mathcal{K}_n'(x)\big\|_\infty \le \delta_3\|x_0 - x\|_\infty, \quad x \in \mathcal{B}(x_0, \delta_0).$$

Proof. From equation (2.7), we have

$$\mathcal{K}_n'(y) = \pi_n\mathcal{K}'(y) + (I - \pi_n)\mathcal{K}_n^{D'}(y), \quad y \in \mathscr{C}[-1,1].$$

Using the above result, we obtain, for any $g \in \mathscr{C}[-1,1]$,

$$\big\|\big[\mathcal{K}_n'(x_0) - \mathcal{K}_n'(x)\big]g\big\|_\infty = \big\|\pi_n\big(\mathcal{K}'(x_0) - \mathcal{K}'(x)\big)g\big\|_\infty + \Big\|(I-\pi_n)\big(\mathcal{K}_n^{D'}(x_0) - \mathcal{K}_n^{D'}\big)g\Big\|_\infty.$$

Now using the Lipschitz continuity of $\dfrac{\partial\psi}{\partial u}$ and estimates (2.5), we get

$$\big\|\pi_n\big(\mathcal{K}'(x_0) - \mathcal{K}'(x)\big)g\big\|_\infty \le \big\|\big(\pi_n - I\big)(\mathcal{K}'(x_0) - \mathcal{K}'(x)\big)g\big\|_\infty + \big\|\big(\mathcal{K}'(x_0) - \mathcal{K}'(x)\big)g\big\|_\infty$$

$$\le c_1 n^{\frac{1}{2}-r}\big\|\big[\big(\mathcal{K}'(x_0) - \mathcal{K}'(x)\big)g\big]^{(r)}\big\|_\infty + \big\|\big(\mathcal{K}'(x_0) - \mathcal{K}'(x)\big)g\big\|_\infty$$

$$\le 2c_1 n^{\frac{1}{2}-r}\|\kappa\|_{r,\infty}\delta_2\|x_0 - x\|_\infty\|g\|_\infty + 2M_1\delta_2\|x_0 - x\|_\infty\|g\|_\infty.$$

Similarly, using (2.2) it can be shown that

$$\big\|(I-\pi_n)\big(\mathcal{K}_n^{D'}(x_0) - \mathcal{K}_n^{D'}(x)\big)g\big\|_\infty \le c_1 n^{\frac{1}{2}-r}\big\|\big[\big(\mathcal{K}_n^{D'}(x_0) - \mathcal{K}_n^{D'}(x)\big)g\big]^{(r)}\big\|_\infty$$

$$\le c_1 p n^{\frac{1}{2}-r}\|\kappa\|_{r,\infty}\delta_2\|x_0 - x\|_{\mathscr{L}^2}\|g\|_{\mathscr{L}^2}$$

$$\le 2c_1 p\delta_2 n^{\frac{1}{2}-r}\|\kappa\|_{r,\infty}\|x_0 - x\|_\infty\|g\|_\infty.$$

Hence, by the above bounds the desired result follows with

$$\delta_3 = \big[M_1 + (1+p)c_1 n^{\frac{1}{2}-r}\|\kappa\|_{r,\infty}\big]2\delta_2.$$

Lemma 3.2 is proved.

The following theorem give the superconvergence of the iterated Legendre superconvergent degenerate kernel solution $\widetilde{x}_n^D$ to x_0.

Theorem 3.3. *Let* $x_0 \in \mathscr{C}[-1,1]$ *be an isolated solution of* (1.1) *and assume that* $\dfrac{\partial \psi}{\partial u} \in \mathscr{C}^r(\Omega_1)$. *Then, for* n *sufficiently large, the iterated solution* $\widetilde{x}_n^D$, *given by* (2.9), *satisfies*

$$\left\| x_0 - \widetilde{x}_n^D \right\|_\infty = \mathcal{O}(n^{-2r}). \tag{3.6}$$

Proof. Note that, from (1.1) and (2.9), we have

$$x_0 - \widetilde{x}_n^D = \mathcal{K}x_0 - \mathcal{K}x_n^D.$$

Therefore, for some $0 < \theta < 1$, we get

$$\begin{aligned}\mathcal{K}x_0 - \mathcal{K}x_n^D &= \mathcal{K}'\big(x_0 + \theta\big(x_0 - x_n^D\big)\big)\big(x_0 - x_n^D\big)\\ &= \Big[\mathcal{K}'\big(x_0 + \theta\big(x_0 - x_n^D\big)\big) - \mathcal{K}'(x_0) + \mathcal{K}'(x_0)\Big]\big(x_0 - x_n^D\big).\end{aligned}$$

Taking the norm on both sides of the above equation and applying the Lipschitz continuity of $\mathcal{K}'$, we can show that

$$\left\| x_0 - \widetilde{x}_n^D \right\|_\infty \le \delta_2 M_1 \theta \left\| x_0 - x_n^D \right\|_\infty^2 + \left\| \mathcal{K}'(x_0)\big(x_0 - x_n^D\big) \right\|_\infty. \tag{3.7}$$

For the second term of the estimate (3.7), we obtain

$$\big(I - \mathcal{K}_n'(x_0)\big)\big(x_n^D - x_0\big) = \mathcal{K}(x_0) - \mathcal{K}_n(x_0) - \mathcal{K}_n'(x_0)\big(x_0 - x_n^D\big) + \mathcal{K}_n(x_0) - \mathcal{K}_n(x_n^D).$$

Applying $\mathcal{K}'(x_0)$ to both sides and using the mean value theorem, we deduce that

$$\begin{aligned}\mathcal{K}'(x_0)\big(x_0 - x_n^D\big) &= \mathcal{K}'(x_0)\big(I - \mathcal{K}_n'(x_0)\big)^{-1}\\ &\qquad \times \Big[\mathcal{K}(x_0) - \mathcal{K}_n(x_0) - \mathcal{K}_n'(x_0)\big(x_0 - x_n^D\big) + \mathcal{K}_n(x_0) - \mathcal{K}_n(x_n^D)\Big]\\ &= \mathcal{K}'(x_0)\big(I - \mathcal{K}_n'(x_0)\big)^{-1}\big[\mathcal{K}(x_0) - \mathcal{K}_n(x_0)\big]\\ &\qquad + \mathcal{K}'(x_0)\big(I - \mathcal{K}_n'(x_0)\big)^{-1}\big[\mathcal{K}_n'(x_0 + \theta(x_0 - x_n^D)) - \mathcal{K}_n'(x_0)\big](x_0 - x_n^D),\end{aligned}$$

where $0 < \theta < 1$. Now from estimates (3.1), (3.2) and Lemma 3.2 one has

$$\left\| \mathcal{K}'(x_0)\big(x_0 - x_n^D\big) \right\|_\infty \le A_1 \left\| \mathcal{K}'(x_0)\big[\mathcal{K}(x_0) - \mathcal{K}_n(x_0)\big] \right\|_\infty + 2A_1 M_1 \Psi_1 \theta \delta_3 \left\| x_0 - x_n^D \right\|_\infty^2. \tag{3.8}$$

Combining (3.7) with (3.8), we get

$$\left\| x_0 - \widetilde{x}_n^D \right\|_\infty \le c_3 \left\| x_0 - x_n^D \right\|_\infty^2 + A_1 \left\| \mathcal{K}'(x_0)\big[\mathcal{K}(x_0) - \mathcal{K}_n(x_0)\big] \right\|_\infty \tag{3.9}$$

with $c_3 = M_1\theta(\delta_2 + 2A_1\Psi_1\delta_3)$. Using the Cauchy–Schwarz inequality and estimates (2.4), (3.1), we obtain

$$\begin{aligned}\left\| \mathcal{K}'(x_0)[\mathcal{K}(x_0) - \mathcal{K}_n(x_0)] \right\|_\infty &\le \max_{s\in[-1,1]} \|\overline{\kappa}_s \psi_1\|_{\mathscr{L}^2} \left\| (I - \pi_n)\big(\mathcal{K}(x_0) - \mathcal{K}_n^D(x_0)\big) \right\|_{\mathscr{L}^2}\\ &\le 2\sqrt{2} c_1 M_1 \Psi_1 n^{-r} \left\| \big[\mathcal{K}(x_0) - \mathcal{K}_n^D(x_0)\big]^{(r)} \right\|_{\mathscr{L}^2}\end{aligned}$$

$$\leq \big[8(c_1)^2 M_1 \Psi_1 \Psi_0 \|\kappa\|_{2r,\infty}\big] n^{-2r}.$$

This result, together with (3.9), proves (3.6).

Theorem 3.3 is proved.

3.2. Superconvergent Nyström method. In this subsection, we prove the existence and uniqueness of the solution of equation (1.2). Likewise, we give the results relative to the rate of convergence.

The operator $\mathcal{T}$ is Fréchet differentiable and the Fréchet derivative at $x_0 \in \mathscr{C}[-1,1]$ is given by

$$(\mathcal{T}'(x_0)g)(s) = \int\limits_{-1}^{1} \frac{\partial \kappa}{\partial u}(s,t,x_0(t))g(t)dt.$$

Define $\Omega_2 = [-1,1] \times [-1,1] \times [a,b]$. Throughout this section, the following conditions are made on f and κ:

(i) $f \in [-1,1]$;

(ii) $M_2 = \sup_{s,t\in[-1,1]}\left|\dfrac{\partial \kappa}{\partial u}(s,t,x_0(t))\right| < \infty$;

(iii) the kernel $\kappa(s,t,u)$ and $\dfrac{\partial \kappa}{\partial u}(s,t,u)$ are Lipschitz continuous in $u \in [a,b]$, i.e., for any $u_1, u_2 \in [a,b]$, there exists $\gamma_1, \gamma_2 > 0$, for which

$$\big|\kappa(s,t,u_1) - \kappa(s,t,u_2)\big| \leq \gamma_1 |u_1 - u_2|$$

and

$$\left|\frac{\partial \kappa}{\partial u}(s,t,u_1) - \frac{\partial \kappa}{\partial u}(s,t,u_2)\right| \leq \gamma_2 |u_1 - u_2|.$$

Then the operator $\mathcal{T}'(x_0)$ is compact. For $j = 0,1,\ldots,r$, we have

$$\left\|\big[\mathcal{T}'(x_0)g\big]^{(j)}\right\|_\infty = \max_{s\in[-1,1]}\left|\int\limits_{-1}^{1} \frac{\partial^{j+1}\kappa}{\partial s^j \partial u}(s,t,x_0(t))g(t)dt\right|$$

$$\leq \max_{s,t\in[-1,1]}\left|\frac{\partial^{j+1}\kappa}{\partial s^j \partial u}(s,t,x_0(t))\right| \int\limits_{-1}^{1} |g(t)|dt \leq 2\|\kappa^*\|_{j,\infty}\|g\|_\infty,$$

where

$$\|\kappa^*\|_{r,\infty} = \max_{\substack{0\leq i,j\leq r\\ s,t\in[-1,1]}}\left|\frac{\partial^{i+j}\kappa}{\partial s^i \partial u^j}(s,t,x_0(t))\right| \qquad \text{and} \qquad \ell_s^*(t) = \frac{\partial^r \kappa}{\partial s^r}(s,t,x_0(t)).$$

Hence, using condition (ii) we obtain

$$\big\|\mathcal{T}'(x_0)\big\|_\infty \leq 2M_2. \tag{3.10}$$

We first establish the invertibility of the linear operators $\big(I - \mathcal{T}_n'(x_0)\big)$ in the following lemma, which will be used to prove the main results of this section.

Lemma 3.3. *Let $x_0 \in \mathscr{C}[-1,1]$ be the unique solution of* (1.2). *Assume that $\kappa \in \mathscr{C}^r(\Omega_2)$ and 1 is not an eigenvalue of $\mathcal{T}'(x_0)$. Then, for n large enough, $\big(I - \mathcal{T}_n'(x_0)\big)^{-1}$ exists and it is a bounded linear operator, i.e., there exists a constant $A_2 > 0$ such that*

$$\big\|\big(I - \mathcal{T}_n'(x_0)\big)^{-1}\big\|_\infty \le A_2.$$

Proof. Using estimate (2.5), we have for each $g \in \mathscr{C}[-1,1]$ and each $t \in [-1,1]$

$$\begin{aligned}\big\|(\mathcal{T}'(x_0)g) - (\mathcal{T}_n'(x_0)g)\big\|_\infty &= \max_{-1\le t\le 1}\Big|(I-\pi_n)\big(\mathcal{T}'(x_0) - \mathcal{T}_n^{N'}(x_0)\big)g(t)\Big| \\ &\le (1+p)\big\|(I-\pi_n)\mathcal{T}'(x_0)g\big\|_\infty \\ &\le (1+p)c_1 n^{\frac{1}{2}-r}\big\|[\mathcal{T}'(x_0)g]^{(r)}\big\|_\infty \\ &\le 2c_1(1+p)n^{\frac{1}{2}-r}\|\kappa^*\|_{r,\infty}\|g\|_\infty.\end{aligned}$$

For $r \ge 1$, it follows that

$$\big\|\mathcal{T}'(x_0) - \mathcal{T}_n'(x_0)\big\|_\infty = \mathcal{O}\big(n^{\frac{1}{2}-r}\big) \to 0 \quad \text{as} \quad n \to \infty.$$

Since 1 is not an eigenvalue of $\mathcal{T}'(x_0)$, it then follows from the results of Lemma 2.6 in [6] that the operators $\big(I - \mathcal{T}_n'(x_0)\big)^{-1}$ exists and are uniformly bounded.

Theorem 3.4. *Let $x_0 \in \mathscr{C}[-1,1]$ be the unique solution of* (1.2). *Assume that 1 is not an eigenvalue of $\mathcal{T}'(x_0)$. Then there exists a real number $\delta_0 > 0$ such that the approximate equation* (2.12) *has a unique solution x_n^N in $\mathcal{B}(x_0,\delta_0)$ for a sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n, such that*

$$\frac{\alpha_n}{1+q} \le \big\|x_0 - x_n^N\big\|_\infty \le \frac{\alpha_n}{1-q},$$

where $\alpha_n = \big\|(I - \mathcal{T}_n'(x_0))^{-1}(\mathcal{T}(x_0) - \mathcal{T}_n(x_0))\big\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$.

In the next theorem we give the error estimation between the Legendre superconvergent Nyström solution x_n^N defined by (2.12) and the exact solution x_0.

Theorem 3.5. *Let $x_0 \in \mathscr{C}[-1,1]$ be the unique solution of* (1.2). *Assume that $\kappa, \dfrac{\partial\kappa}{\partial u} \in \mathscr{C}^r(\Omega_2)$, and $f \in \mathscr{C}[-1,1]$. Let x_n^N be the unique solution of* (2.12) *in $\mathcal{B}(x_0,\delta_0)$. Then, for a sufficiently large n, we have*

$$\big\|x_0 - x_n^N\big\|_\infty = \mathcal{O}\big(n^{-2r+\frac{1}{2}}\big). \tag{3.11}$$

Proof. From estimate (2.5) and Theorem 3.4, we obtain

$$\begin{aligned}\|x_0 - x_n^N\|_\infty &\le A_2\|(I-\pi_n)(\mathcal{T}(x_0) - \mathcal{T}_n^N(x_0))\|_\infty \\ &\le c_1 A_2 n^{\frac{1}{2}-r}\big\|\big[\mathcal{T}(x_0) - \mathcal{T}_n^N(x_0)\big]^{(r)}\big\|_\infty.\end{aligned} \tag{3.12}$$

For any $p \in \mathbb{X}_n$, we get $2 = \displaystyle\int_{-1}^1 dt = \sum\nolimits_{i=1}^n w_i$,

$$\big\| \big[\mathcal{T}(x_0) - \mathcal{T}_n^N(x_0)\big]^{(r)} \big\|_\infty = \max_{-1\le s\le 1} \left| \int_{-1}^{1} \big[\ell_s^*(t) - p(t)\big] dt - \sum_{i=0}^{n} w_i \big[\ell_s^*(t_i) - p(t_i)\big] \right|$$

$$\le \|\ell_s^* - p\|_\infty \left[\int_{-1}^{1} dt + \sum_{i=0}^{n} w_i \right] \le 4\|\ell_s^* - p\|_\infty.$$

According to the Jackson theorem, we have, for all $x \in \mathscr{C}^r[-1,1]$,

$$\inf_{\phi\in\mathbb{X}_n} \|x - \phi\|_\infty \le c_1 n^{-r} \|x^{(r)}\|_\infty,$$

where c_1 is a constant independent of n. Thus,

$$\big\| \big[\mathcal{T}(x_0) - \mathcal{T}_n^N(x_0)\big]^{(r)} \big\|_\infty \le 4 \inf_{p\in\mathbb{X}_d} \|\ell_s^* - p\|_\infty \le 4c_1 n^{-r} \|\kappa^*\|_{2r,\infty}.$$

Now combining the obove bound with (3.12), we get (3.11).

Theorem 3.5 is proved.

Lemma 3.4. *Assume that* κ, $\dfrac{\partial\kappa}{\partial u} \in \mathscr{C}^r(\Omega_2)$. *Then the oprtator* $\mathcal{T}_n'(x_0)$ *is Lipschitz continuous, that is, there exists a constant* $\gamma_3 > 0$, *independent of* n, *such that*

$$\|\mathcal{T}_n'(x_0) - \mathcal{T}_n'(x)\|_\infty \le \gamma_3 \|x_0 - x\|_\infty, \quad x \in \mathcal{B}(x_0, \delta_0). \tag{3.13}$$

Proof. From (2.11) we have, for any $g \in \mathscr{C}[-1,1]$,

$$\|[\mathcal{T}_n'(x_0) - \mathcal{T}_n'(x)]g\|_\infty = \big\|\pi_n(\mathcal{T}'(x_0) - \mathcal{T}'(x))g\big\|_\infty + \big\|(I - \pi_n)\big(\mathcal{T}_n^{N'}(x_0) - \mathcal{T}_n^{N'}(x)\big)g\big\|_\infty. \tag{3.14}$$

Using the Lipschitz continuity of $\dfrac{\partial\kappa}{\partial u}(s,t,x_0(t))$ and the estimate (2.6), we obtain

$$\big\|\pi_n(\mathcal{T}'(x_0) - \mathcal{T}'(x))g\big\|_\infty \le \big\|(\pi_n - I)(\mathcal{T}'(x_0) - \mathcal{T}'(x))g\big\|_\infty + \big\|(\mathcal{T}'(x_0) - \mathcal{T}'(x))g\big\|_\infty$$

$$\le 2c_1 n^{\frac{1}{2}-r} \gamma_2 \|x_0 - x\|_\infty \|g\|_\infty + 2\gamma_2 \|x_0 - x\|_\infty \|g\|_\infty.$$

The second term in (3.14) becomes

$$\big\|(I - \pi_n)\big(\mathcal{T}_n^{N'}(x_0) - \mathcal{T}_n^{N'}(x)\big)g\big\|_\infty \le c_1 n^{\frac{1}{2}-r} \big\| \big[\big(\mathcal{T}_n^{N'}(x_0) - \mathcal{T}_n^{N'}(x)\big)g\big]^{(r)} \big\|_\infty$$

$$\le c_1 n^{\frac{1}{2}-r} \gamma_2 \|\pi_n(x_0 - x)\|_{\mathscr{L}^2} \|g\|_{\mathscr{L}^2}$$

$$\le 2c_1 \gamma_2 n^{\frac{1}{2}-r} \|x_0 - x\|_\infty \|g\|_\infty.$$

Thus, estimate (3.13) follows with $\gamma_3 = \big[1 + c_1 n^{\frac{1}{2}-r}\big] 2\gamma_2$.

Lemma 3.4 is proved.

The resuts below state that the iterated Legendre superconvergent Nyström solution defined by (2.13) converge to x_0 faster than x_n^N.

Theorem 3.6. *Let* $x_0 \in \mathscr{C}[-1,1]$ *be an isolated solution of* (1.2) *and* $\dfrac{\partial \kappa}{\partial u} \in \mathscr{C}^r(\Omega_2)$. *Then, for* n *sufficiently large, the iterated solution* $\widetilde{x}_n^N$, *given by* (2.13), *satisfies*

$$\left\| x_0 - \widetilde{x}_n^N \right\|_\infty = O(n^{-2r}).$$

Proof. Since $\mathfrak{T}_n$ is a nonlinear operator. Then, similarly to (3.7), we obtain

$$\left\| x_0 - \widetilde{x}_n^N \right\|_\infty \le \theta \gamma_2 \left\| x_0 - x_n^N \right\|_\infty^2 + \left\| \mathfrak{T}'(x_0)(x_0 - x_n^N) \right\|_\infty, \tag{3.15}$$

where $0 < \theta < 1$. By using the mean value theorem and Lemma 3.4, we have

$$\left\| x_0 - \widetilde{x}_n^N \right\|_\infty \le c_4 \left\| x_0 - x_n^N \right\|_\infty^2 + A_2 \left\| \mathfrak{T}'(x_0)\big[\mathfrak{T}(x_0) - \mathfrak{T}_n(x_0)\big] \right\|_\infty$$

with $c_4 = \theta(\gamma_2 + 2A_2M_2\gamma_3)$. Then, using the Cauchy–Schwarz inequality and estimates (2.4), (3.10), we get

$$\begin{aligned} \left\| \mathfrak{T}'(x_0)\big[\mathfrak{T}(x_0) - \mathfrak{T}_n(x_0)\big] \right\|_\infty &\le \max_{s \in [-1,1]} \left\| \frac{\partial \kappa}{\partial u}(s, ., x_0(.)) \|_{\mathscr{L}^2} \|(I - \pi_n)\big(\mathfrak{T}(x_0) - \mathfrak{T}_n^N(x_0)\big) \right\|_{\mathscr{L}^2} \\ &\le 2\sqrt{2} c_1 M_2 n^{-r} \left\| \big[\mathfrak{T}(x_0) - \mathfrak{T}_n^N(x_0)\big]^{(r)} \right\|_{\mathscr{L}^2} \\ &\le \big[16(c_1)^2 M_2 \|\kappa^*\|_{2r,\infty}\big] n^{-2r}. \end{aligned}$$

This together with (3.15) proves the desired result.

Theorem 3.6 is proved.

4. Numerical results. In this section, two examples are given to illustrate the results obtained in the previous sections. Note that all required integrals were calculated by high accurate *Gauss* quadrature rule. Moreover, the *Newton–Raphson* method was used to solve the nonlinear systems. Note that the numerical algorithms are compiled by using `WOLFRAM MATHEMATICA`.

Let $\mathbb{X}_n$ denote the space of polynomials of degree $\le n$. We present the errors of the approximate and iterated approximate solutions in the infinity norm. Moreover, we give the maximum of the error of the solutions x_n^D and x_n^N, defined as

$$\max_{0 \le i \le n} \left| x_0(\tau_i) - x_n^D(\tau_i) \right| = \max_i \left| x_{0,i} - x_{n,i}^D \right|,$$

$$\max_{0 \le i \le n} \left| x_0(\tau_i) - x_n^N(\tau_i) \right| = \max_i \left| x_{0,i} - x_{n,i}^N \right|.$$

We compare our results with the piecewise polynomial based degenerate kernel and Nyström methods proposed in [3, 10]. To do this, we consider a uniform partition of $[0,1]$:

$$0 = s_0 < s_1 < s_2 < \ldots < s_{n-1} < s_n = 1,$$

where

$$s_i = \frac{i-1}{n}, \quad i = 0, 1, \ldots, n.$$

We choose the approximating subspace to be the space of piecewise constant functions, which has dimension n. The collocation points are the midpoints

Table 1. Legendre degenerate kernel method

n	$\left\|x_0 - x_n^D\right\|_\infty$	$\max_i \left|x_{0,i} - x_{n,i}^D\right|$	$\left\|x_0 - \widetilde{x}_n^D\right\|_\infty$
2	3.68×10^{-2}	4.67×10^{-5}	4.67×10^{-5}
3	2.30×10^{-3}	1.02×10^{-5}	1.13×10^{-5}
4	1.11×10^{-4}	3.48×10^{-7}	3.48×10^{-7}
5	1.01×10^{-5}	2.15×10^{-8}	2.20×10^{-8}
6	7.00×10^{-7}	4.32×10^{-10}	4.32×10^{-10}
7	3.39×10^{-9}	1.00×10^{-11}	1.01×10^{-11}

Table 2. Legendre – Nyström method

n	$\left\|x_0 - x_n^N\right\|_\infty$	$\max_i \left|x_{0,i} - x_{n,i}^N\right|$	$\left\|x_0 - \widetilde{x}_n^N\right\|_\infty$
1	1.69×10^{-3}	6.47×10^{-5}	9.05×10^{-5}
2	3.70×10^{-4}	5.00×10^{-6}	6.10×10^{-6}
3	4.79×10^{-6}	1.24×10^{-9}	1.41×10^{-9}
4	1.32×10^{-8}	9.31×10^{-12}	1.01×10^{-11}
5	7.93×10^{-13}	4.44×10^{-16}	2.22×10^{-15}

$$t_i = (2i-1)/n, \quad i = 1, 2, \ldots, n.$$

We denote, the maximum of the error of the solutions x_n^D and x_n^N at the collocation points as

$$\max_{0 \le i \le n} \left|x_0(t_i) - x_n^D(t_i)\right| = \max_i \left|x_{0,i} - x_{n,i}^D\right|,$$

$$\max_{0 \le i \le n} \left|x_0(t_i) - x_n^N(t_i)\right| = \max_i \left|x_{0,i} - x_{n,i}^N\right|.$$

In Tables 1 and 2, we present the errors of the approximation solutions, obtained by using the Legendre degenerate kernel and Legendre – Nyström methods, while in Tables 3 and 4, we give the corresponding ones obtained by using piecewise constant functions. Note that in Tables 1 and 2, n denotes the highest degree of the Legendre polynomial employed in the computation, while in Tables 3 and 4, n denotes the dimension of the approximating subspace.

***Example* 1.** We consider the Hammerstein equation with a degenerate kernel

$$x(s) - \int_{-1}^{1} \frac{1}{10} \sin(\pi t) \cos(\pi s) \psi(t, x(t)) dt = f(s), \quad s \in [-1, 1],$$

where $\psi(t, x(t)) = [x(t)]^3$ and $f(s)$ is selected so that $x_0(s) = \sin(\pi s) + \dfrac{1}{3}\big(20 - \sqrt{391}\,\big) \cos(\pi s)$.

***Example* 2.** We consider the Urysohn integral equation

Table 3. Piecewise polynomial based degenerate kernel method

n	$\left\|x_0 - x_n^D\right\|_\infty$	$\max_i \left\|x_{0,i} - x_{n,i}^D\right\|$	$\left\|x_0 - \widetilde{x}_n^D\right\|_\infty$
2	7.34×10^{-3}	2.11×10^{-3}	3.19×10^{-5}
4	3.68×10^{-4}	1.24×10^{-7}	1.34×10^{-7}
8	4.71×10^{-5}	8.57×10^{-9}	8.42×10^{-9}
16	5.93×10^{-6}	5.49×10^{-10}	5.61×10^{-10}
32	7.43×10^{-7}	3.45×10^{-11}	3.37×10^{-11}
64	9.29×10^{-8}	2.16×10^{-12}	2.09×10^{-11}

Table 4. Piecewise polynomial based Nyström method

n	$\left\|x_0 - x_n^N\right\|_\infty$	$\max_i \left\|x_{0,i} - x_{n,i}^N\right\|$	$\left\|x_0 - \widetilde{x}_n^N\right\|_\infty$
2	2.18×10^{-3}	6.71×10^{-5}	7.04×10^{-5}
4	3.06×10^{-4}	4.02×10^{-6}	4.12×10^{-6}
8	4.07×10^{-5}	2.49×10^{-7}	2.53×10^{-7}
16	5.32×10^{-6}	4.64×10^{-8}	4.67×10^{-8}
32	6.78×10^{-7}	7.98×10^{-9}	8.01×10^{-9}

$$x(s) - \int_{-1}^{1} \frac{3\sqrt{2}\pi}{16} \cosh(s+1) e^{t-s} [x(t)]^3 dt = f(s), \quad s \in [-1, 1],$$

where $f(s) = \cos\left(\frac{\pi s}{4}\right) - \frac{10431.6 e^{-1-s} \cosh(s+1)}{256 + 160\pi^2 + 9\pi^4}$ and the exact solution is given by $x_0(s) = \cos\left(\frac{\pi s}{4}\right)$.

The results illustrated in the tables above show that a high precision is obtained even if the degree of polynomials does not exceed 7. This is due to the advantage of using Legendre polynomials which represent a low computational cost.

From Tables 1 – 4, it can be seen that a high accuracy is obtained by the Legendre degenerate kernel and Nyström methods whereas the size of nonlinear systems are much smaller as compared to the case of piecewise polynomials. For example, to obtain the error of order 10^{-9}, in the iterated Legendre Nyström method a system of size 3×3 is needed to be solved, while in the case of piecewise polynomials, we need to solve a system of size 32×32 to obtain an accuracy of comparable order.

5. Conclusion. This paper presents an application of Legendre polynomials for approximating the solution of nonlinear integral equations via superconvergent degenerate kernel and Nyström methods. Theoretically, a complete study of the error associated with each method is given. Moreover, we have provided convergence order for each method, and we have showed that, for the iterated versions, the convergence order is $O(n^{-2r})$. The integer r denotes the smoothness of the kernel. Finally, in order to validate our theoretical study, numerical examples are given. It is to be noted that the analysis given in this paper will hold for Chebyshev polynomials basis. That will be considered in future papers.

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