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PRACTICAL SEMIGLOBAL UNIFORM EXPONENTIAL STABILITY OF NONLINEAR NONAUTONOMOUS SYSTEMS

ПРАКТИЧНА НАПІВГЛОБАЛЬНА РІВНОМІРНА ЕКСПОНЕНЦІАЛЬНА СТІЙКІСТЬ НЕЛІНІЙНИХ НЕАВТОНОМНИХ СИСТЕМ

We solve the following twofold problem: In the first part, we deduce Lyapunov sufficient conditions for practical uniform exponential stability of nonlinear perturbed systems under different conditions for the perturbed term. The second part presents a converse Lyapunov theorem for the notion of semiglobal uniform exponential stability for parametrized nonlinear time-varying systems. We establish the possibility of application of a perturbed parametrized system, by using Lyapunov theory, to the investigation of the robustness properties that may provide practical semiglobal uniform exponential stability with respect to perturbations.

Розглянуто двоїсту задачу. У першій частині отримано достатні умови Ляпунова для практичної рівномірної експоненціальної стійкості нелінійних збурених систем при різних умовах, що накладені на збурений член. У другій частині наведено обернену теорему Ляпунова для поняття напівглобальної рівномірної експоненціальної стійкості параметризованої нелінійної системи, що змінюється залежно від часу. Досліджено можливість застосування збуреної параметризованої системи з використанням теорії Ляпунова для дослідження властивостей стійкості, які може забезпечити практична напівглобальна рівномірна експоненціальна стабільність по відношенню до збурень.

1. Introduction. It is well-known that the stability may be a fundamental issue and it has been a crucial notion within the study of control systems. By a stable system, we broadly mean that little disturbances either within the system inputs or within the initial conditions don't result in large changes within the overall behavior of the system. To be of practical use, a system needs to be stable. For this reason, there are a motivating number of developments within the research of the stability criteria of nonlinear differential systems [1, 2, 5, 7, 13, 24] and a number of other works have presented many methods for analysing stability properties [8, 9, 14, 17, 18, 20, 21]. There are important sorts of stability of dynamical systems [3, 4, 11, 22]. In this work, we are interesting on the practical exponential stability and practical semiglobal exponential stability. In the case of exponential stability, it is required that each one solution starting near an equilibrium point not only stay nearby, but tend to the equilibrium point in no time with decline rate. For practical exponential stability (see [15]), one only has to stabilize a system in a very region of space, namely the system may oscillate near the state, within which the performance remains acceptable. In other word, the asymptotic behavior is studied in a sense that the trajectories converge to a small ball centred at the origin. This notion has been investigated by using Lyapunov-like functions and integral inequalities in [8]. In the case of semiglobal exponential stability, there exist several different versions in the literature for nonlinear time-varying systems [6, 10, 12, 19, 23]. For example, in [23], the authors derived sufficient conditions for uniform semiglobal exponential stability of parametrized nonlinear

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systems and they studied robustness properties of perturbed systems. The usual technique to study the stability of a perturbed system is the Lyapunov function associated to the nominal system as a Lyapunov candidate for the perturbed system. This method has been used extensively for stability analysis of nonlinear time-varying systems [9, 16, 21]. The concept utilized in [1] is to add within the Lyapunov function associated to the nominal system a special function which is chosen such the derivative along the trajectories of the system in presence of perturbations is negative. In [8], the authors provided some sufficient conditions for the exponential stability of a class of perturbed systems based on Lyapunov's technique and a new nonlinear inequality, and under some restrictions on the perturbed term, they proved that all state trajectories are bounded and converge to a small ball centred at the origin. The novelty of this work is to investigate the global practical uniform exponential stability of certain nonlinear perturbed systems under different upper bounds on the perturbed term using Lyapunov's theory. Furthermore, we established a new converse Lyapunov theorem in the case of semiglobal exponential stability of a parametrized system and use it to study the robustness properties with respect to nonvanishing perturbations.

This paper is organized as follows. In the next section, we present the concept of practical exponential stability and some technical lemmas used to proof the main results. Also, we consider a class of nonlinear nonautonomous systems with perturbation. We derive stability conditions, which are formulated in terms of the stability of the nominal system with some restrictions on the perturbed term. Our approach is based on a combined usage of the Lyapunov equations, new bounds on the perturbations and estimates on some scalar functions. In Subsection 2.3, practical semiglobal exponential stability is defined on the parametrized nonlinear system. A converse Lyapunov theorem is established by construction a differential Lyapunov function which satisfies certain properties and the robustness properties that practical semiglobal exponential stability may provide with respect to nonvanishing perturbations are examined. The conclusion is drawn in Section 3.

2. Basic results. ***2.1. Preliminaries and systems description.*** Throughout this paper, we adopt the following notations:

$\mathbb{R}_+^* =]0, +\infty[$,

$\mathbb{R}_+ = [0, +\infty[$ is the set of all nonnegative real numbers,

$\mathbb{R}^n$ is the real n-dimensional Euclidean space with the norm $\|\cdot\|$,

$\mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ denotes the space of all continuous functions from $\mathbb{R}_+ \times \mathbb{R}^n$ to $\mathbb{R}^n$,

for $r \geq 0$, $\mathcal{B}_r$ is the closed ball of $\mathbb{R}^n$ centred at zero, i.e., $\mathcal{B}_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$.

Consider the nonlinear time-varying system described by

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0, \quad t \geq t_0 \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x, uniformly in t.

Let $x(t, t_0, x_0) = x(t)$ be denoted by the unique solution of system (1) passing through $x_0 \in \mathbb{R}^n$, where $t = t_0$.

Firstly, we present the definition of global practical exponential stability of system (1), which is recently introduced in [3].

Definition 1. *$\mathcal{B}_r$ is called globally uniformly exponentially stable, if there exist positive numbers λ_1 and λ_2 such that, for all $t_0 \in \mathbb{R}_+$ and all $x_0 \in \mathbb{R}^n$,*

$$\|x(t)\| \leq \lambda_1 \|x_0\| e^{-\lambda_2 (t - t_0)} + r. \tag{2}$$

The system (1) *is said to be globally practically uniformly exponentially stable if there exists a ball* $\mathcal{B}_r \subset \mathbb{R}^n$ *such that* $\mathcal{B}_r$ *is globally uniformly exponentially stable.*

***Remark* 1.** The inequality (2) implies that $\|x(t)\|$ will be bounded by a small bound $r > 0$, that is, $\|x(t)\|$ will be small for sufficiently large t. In particular, Definition 1 generalizes the notion of uniform exponential stability when $r = 0$, see [17].

We use also the following lemmas to prove our results.

Lemma 1 (see [9]). *Let* $a, b, d \in \mathbb{R}_+^*$ *and* $\omega\colon \mathbb{R}_+ \to \mathbb{R}^n$ *be continuously differentiable function such that*

$$\dot{\omega}(t) \leq -\Big(a - b\theta(t)\Big)\omega(t) + d\gamma(t).$$

For all $t \geq 0$ *and* $x \in \mathbb{R}^n$,

$$\theta(t) < \frac{a}{b}\cdot$$

Then, for all $t \geq t_0 \geq 0$, *we have*

$$\omega(t) \leq \omega(t_0)e^{bM_\theta}e^{-a(t-t_0)} + de^{bM_\theta}\sqrt{\frac{M_\gamma}{2a}},$$

where $M_\theta = \displaystyle\int_0^\infty \theta(s)ds$ *and* $M_\gamma = \displaystyle\int_0^\infty \gamma^2(s)ds.$

Lemma 2 (see [9]). *Let* $a \in \mathbb{R}_+^*$ *and* $b \in \mathbb{R}_+^*$, *then*

$$a^n + b^n \leq \frac{1}{n}(a+b)^n \quad \forall n \in \,]0, 1[.$$

Lemma 3 (see [24], generalized Gronwall – Bellman inequality). *Let* $\theta, \psi\colon \mathbb{R}_+ \to \mathbb{R}$ *be continuous functions and* $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a function such that*

$$\dot{\varphi}(t) \leq \theta(t)\varphi(t) + \psi(t) \quad \forall t \geq t_0.$$

Then, for any $t \geq t_0 \geq 0$, *we have the following inequality*:

$$\varphi(t) \leq \varphi(t_0)e^{\int_{t_0}^t \theta(s)ds} + \int\limits_{t_0}^{t} e^{\int_s^t \theta(v)dv}\psi(s)ds.$$

Note that the derivative of a function $V(t,x)$ along the solution of system (1) is given by

$$\dot{V}(t,x) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x).$$

2.2. Global practical exponential stability of perturbed systems. The uniform exponential stability of an equilibrium point of a system can be established by requiring the existence of a Lyapunov function that satisfies certain conditions. In as follows, we shall be interested in the relation between the solution of the unperturbed system (1) and the solution of the perturbed system

$$\dot{x} = f(t,x) + \psi(t,x) \quad \forall t \geq t_0, \tag{3}$$

where $\psi \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x, uniformly in t. Precisely, we will give sufficient conditions to show that if the nominal system (1) is globally uniformly exponentially stable then the perturbed system (3) is globally practically uniformly exponentially stable under different assumptions on the perturbed term using Lyapunov's direct method.

Now, we suppose the following assumptions:

$(\mathcal{H}_1)$ The nominal system (1) is globally uniformly exponentially stable and there exists a continuous differentiable function $V(\cdot,\cdot)\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ such that, for all $t \in \mathbb{R}_+$ and all $x \in \mathbb{R}^n$, we have

(i) $c_1\|x\|^2 \le V(t,x) \le c_2\|x\|^2$,

(ii) $\dot V(t,x) \le -c_3\|x\|^2$,

(iii) $\left\|\dfrac{\partial V}{\partial x}\right\| \le c_4\|x\|$,

where $c_1,\ c_2,\ c_3$, and c_4 are positive constants.

$(\mathcal{H}_2)$ The perturbed term $\psi(t,x)$ satisfies $\psi(t,0)=0$ for all $t \in \mathbb{R}_+$ and there exists a nonnegative constant $0<\alpha<1$ such that

$$\|\psi(t,x)-\psi(t,y)\| \le \gamma(t)\|x-y\|^\alpha + \varepsilon(t)\|x-y\| \quad \forall t \ge 0 \quad \forall x,y \in \mathbb{R}^n,$$

where $\gamma\colon \mathbb{R}_+ \to \mathbb{R}$ and $\varepsilon\colon \mathbb{R}_+ \to \mathbb{R}$ are continuous nonnegative functions with

$$\int\limits_0^{+\infty} \varepsilon(s)ds \le M_\varepsilon < +\infty,$$

$$\int\limits_0^{+\infty} \gamma^2(s)ds \le M_\gamma < +\infty$$

and the assumption

$$\varepsilon(t) < \frac{c_3c_1}{c_2c_4} \quad \forall t \ge t_0.$$

***Remark* 2.** The assumption $(\mathcal{H}_1)$ is first formulated in [18] as a criteria to show the exponential asymptotic behavior of solutions of system (3) under another condition on the perturbed term.

Now, the first theorem about global practical uniform exponential stability behavior of the perturbed system (3) is as follows.

Theorem 1. *Under assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, the perturbed system* (3) *is globally practically uniformly exponentially stable.*

Proof. Let $x(t)$ be the solution of system (3). Then the derivative of V along the trajectories of system (3) is given by

$$\begin{aligned}
\dot V(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x}\psi(t,x) \\
&\le -c_3\|x\|^2 + \left\|\frac{\partial V}{\partial x}\right\| \|\psi(t,x)\| \\
&\le -c_3\|x\|^2 + c_4\|x\| \Big(\gamma(t)\|x\|^\alpha + \varepsilon(t)\|x\|\Big) \\
&= -c_3\|x\|^2 + c_4\gamma(t)\|x\|^{\alpha+1} + c_4\varepsilon(t)\|x\|^2 \\
&= -(c_3 - c_4\varepsilon(t))\|x\|^2 + c_4\gamma(t)\|x\|^{\alpha+1}
\end{aligned}$$

$$\leq -\left(\frac{c_3}{c_2} - \frac{c_4}{c_1}\varepsilon(t)\right)V(t,x) + \frac{c_4}{c_1^{\frac{\alpha+1}{2}}}\gamma(t)V(t,x)^{\frac{\alpha+1}{2}}.$$

Let

$$Z(t) = V(t,x)^{\frac{1-\alpha}{2}} \Rightarrow \dot{Z}(t) = \frac{1-\alpha}{2}\dot{V}(t,x)V(t,x)^{-\frac{\alpha+1}{2}}.$$

Then

$$\dot{Z}(t) \leq -\frac{1-\alpha}{2}\left(\frac{c_3}{c_2} - \frac{c_4}{c_1}\varepsilon(t)\right)Z(t) + \frac{1-\alpha}{2}\frac{c_4}{c_1^{\frac{\alpha+1}{2}}}\gamma(t).$$

Hence, using Lemma 1 with

$$a = \frac{1-\alpha}{2}\frac{c_3}{c_2} \qquad b = \frac{1-\alpha}{2}\frac{c_4}{c_1} \qquad d = \frac{1-\alpha}{2}\frac{c_4}{c_1^{\frac{1+\alpha}{2}}},$$

we obtain, for all $t \geq t_0$,

$$Z(t) \leq Z(t_0)e^{\frac{1-\alpha}{2}\frac{c_4}{c_1}M_\varepsilon}e^{-\frac{1-\alpha}{2}\frac{c_3}{c_2}(t-t_0)} + \frac{1-\alpha}{2}\frac{c_4}{c_1^{\frac{1+\alpha}{2}}}e^{\frac{1-\alpha}{2}\frac{c_4}{c_2}M_\varepsilon}\sqrt{\frac{M_\gamma c_2}{(1-\alpha)c_3}}.$$

Therefore,

$$\|x(t)\|^{1-\alpha} \leq e^{\frac{1-\alpha}{2}\frac{c_4}{c_2}M_\varepsilon}\left[\left(\frac{c_2}{c_1}\right)^{\frac{1-\alpha}{2}}\|x_0\|^{1-\alpha}e^{-\frac{1-\alpha}{2}\frac{c_3}{c_2}(t-t_0)} + \frac{1-\alpha}{2}\frac{c_4}{c_1^{\frac{\alpha+1}{2}}}\sqrt{\frac{M_\gamma c_2}{(1-\alpha)c_3}}\right].$$

Thus, since $0 < 1-\alpha < 1$, using Lemma 2, we get, for all $t \geq t_0$ and all $x_0 \in \mathbb{R}^n$, the solution of the system is given by

$$\|x(t)\| \leq \left(\frac{1}{1-\alpha}\right)^{\frac{1}{1-\alpha}}e^{\frac{c_4}{c_2}M_\varepsilon}\left[\sqrt{\frac{c_2}{c_1}}\|x_0\|e^{-\frac{c_3}{2c_2}(t-t_0)} + \left(\frac{1-\alpha}{2}\frac{c_4}{c_1^{\frac{\alpha+1}{2}}}\sqrt{\frac{M_\gamma c_2}{(1-\alpha)c_3}}\right)^{\frac{1}{1-\alpha}}\right].$$

This yield the global uniform exponential stability of B_r with

$$r = \left(\frac{c_4}{2c_1^{\frac{\alpha+1}{2}}}\sqrt{\frac{M_\gamma c_2}{(1-\alpha)c_3}}\right)^{\frac{1}{1-\alpha}}e^{\frac{c_4}{c_2}M_\varepsilon}.$$

Hence, the system (3) is globally practically uniformly exponentially stable.

Theorem 1 is proved.

To investigate the global practical uniform exponential stability of the perturbed system (3) we shall suppose some assumptions more than considered in Theorem 1. This result of stability can be stated as follows.

Proposition 1. *Assume that there exist a continuous differentiable function* $V(\cdot,\cdot): \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ *and positive constants* c_i, $i = 1,2,3$, *such that, for all* $t \in \mathbb{R}_+$ *and all* $x \in \mathbb{R}^n$, *the next properties are satisfied*:

(1) *conditions* (i) *and* (ii) *in assumption* $(\mathcal{H}_1)$ *hold,*

(2) *there exists a continuous integrable function* $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\frac{\partial V}{\partial x}\psi(t,x) \le \sigma(t)\|x\|^{\alpha+1} \quad \forall x \in \mathbb{R}^n \quad \forall t \in \mathbb{R}_+, \quad 0<\alpha<1.$$

Then the system (3) *is globally practically uniformly exponentially stable.*

Proof. Let $x(t)$ be the solution of system (3). Then the derivative of V along the trajectories of system (3) is as follows:

$$\begin{aligned}\dot V(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x}\psi(t,x)\\ &\le -c_3\|x\|^2 + \sigma(t)\|x\|^{\alpha+1}\\ &\le -\frac{c_3}{c_2}V(t,x) + \frac{1}{c_1^{\frac{\alpha+1}{2}}}\sigma(t)V^{\frac{\alpha+1}{2}}(t,x).\end{aligned}$$

Put

$$Z(t) = V^{\frac{1-\alpha}{2}}(t,x) \Rightarrow \dot Z(t) = \frac{1-\alpha}{2}\dot V(t,x)V^{-\frac{\alpha+1}{2}}(t,x),$$

which implies that

$$\dot Z(t) \le -\frac{1-\alpha}{2}\frac{c_3}{c_2}Z(t) + \frac{1-\alpha}{2c_1^{\frac{\alpha+1}{2}}}\sigma(t).$$

Applying Lemma 3, we get, for all $t \ge t_0$,

$$\begin{aligned}Z(t) &\le Z(t_0)e^{-\frac{1-\alpha}{2}\frac{c_3}{c_2}(t-t_0)} + \frac{1-\alpha}{2c_1^{\frac{\alpha+1}{2}}}\int\limits_{t_0}^{t} e^{-\frac{1-\alpha}{2}\frac{c_3}{2}(t-s)}\sigma(s)ds\\ &\le Z(t_0)e^{-\frac{(1-\alpha)c_3}{2c_2}(t-t_0)} + \frac{1-\alpha}{2c_1^{\frac{\alpha+1}{2}}}M_\sigma,\end{aligned}$$

where $M_\sigma = \displaystyle\int_0^{+\infty}\sigma(s)ds$. Then

$$\|x(t)\|^{1-\alpha} \le \left(\frac{c_2}{c_1}\right)^{\frac{1-\alpha}{2}}\|x_0\|^{1-\alpha}e^{\frac{1-\alpha}{2}\frac{c_3}{c_2}(t-t_0)} + \frac{1-\alpha}{2c_1^{\frac{\alpha+1}{2}}}M_\sigma.$$

Using Lemma 2, we have, for all $t \ge t_0$ and all $x_0 \in \mathbb{R}^n$, the solution is given by

$$\|x(t)\| \le \left(\frac{1}{1-\alpha}\right)^{\frac{1}{1-\alpha}}\sqrt{\frac{c_2}{c_1}}\|x_0\|e^{-\frac{c_3}{2c_2}(t-t_0)} + \left(\frac{M_\sigma}{2c_1^{\frac{\alpha+1}{2}}}\right)^{\frac{1}{1-\alpha}}.$$

Consequently, the global practical uniform exponential stability of system (3) is fulfilled.

Proposition 1 is proved.

We make another assumption for the perturbed term $\psi(t,x)$ as follows:

$(\mathcal{H}_3)$ There exist a continuous integrable function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ and nonnegative continuous functions ϑ and δ such that

$$\|\psi(t,x)\| \le \rho(t)\vartheta(\|x\|) + \delta(t) \quad \forall t \ge t_0 \quad \forall x \in \mathbb{R}^n$$

with

$$\vartheta(\|x\|) \le k\|x\| \qquad \text{and} \qquad \int\limits_0^\infty \delta^2(s)ds = M_\delta < +\infty.$$

We are now in position to present the next result.

Theorem 2. *Under the assumptions* $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ *and the condition*

$$\rho(t) < \frac{c_3 c_1}{c_2 c_4 k}$$

the system (3) *is globally practically uniformly exponentially stable.*

Proof. Let $x(t)$ be the solution of system (3). Then the derivative of V along the trajectories of system (3) is given by

$$\begin{aligned}\dot V(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x}\psi(t,x)\\ &\le -c_3\|x\|^2 + c_4\|x\|\Big(\rho(t)\vartheta(\|x\|) + \delta(t)\Big)\\ &\le -c_3\|x\|^2 + c_4 k\rho(t)\|x\|^2 + c_4\delta(t)\|x\|\\ &\le -\left(\frac{c_3}{c_2} - \frac{c_4 k}{c_1}\rho(t)\right)V(t,x) + \frac{c_4}{\sqrt{c_1}}\delta(t)\sqrt{V(t,x)}.\end{aligned}$$

Let

$$Z(t) = \sqrt{V(t,x)} \Rightarrow \dot Z(t) = \frac{1}{2}\dot V(t,x)\sqrt{V(t,x)}.$$

Then we get

$$\dot Z(t) \le -\frac{1}{2}\left(\frac{c_3}{c_2} - \frac{c_4 k}{c_1}\rho(t)\right)Z(t) + \frac{c_4}{2\sqrt{c_1}}\delta(t).$$

Using Lemma 1, we get, for all $t \ge t_0$,

$$Z(t) \le Z(t_0) e^{\frac{c_4 k}{2c_1}M_\rho} e^{-\frac{c_3}{2c_2}(t-t_0)} + \frac{c_4}{2\sqrt{c_1}} e^{\frac{c_4 k}{c_1}M_\rho}\sqrt{\frac{M_\delta c_2}{2c_3}},$$

where $M_\rho = \displaystyle\int_0^\infty \rho(s)ds$. Therefore, for all $t \ge t_0$ and all $x_0 \in \mathbb{R}^n$, the solution of the system is as follows:

$$\|x(t)\| \le \sqrt{\frac{c_2}{c_1}}\|x_0\| e^{\frac{c_4 k}{c_3}M_\rho} e^{-\frac{c_3}{c_2}(t-t_0)} + \frac{c_4}{2c_1} e^{\frac{c_4 k}{c_1}M_\rho}\sqrt{\frac{M_\delta c_2}{c_3}}.$$

Consequently, the system (3) is globally practically uniformly exponentially stable.

Theorem 2 is proved.

The following example is an illustrative of the applicability of the previous result.

***Example* 1.** We consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1 &= x_1^2 x_2 - \frac{1}{4}x_1 + \frac{x_1}{4(1+t)^2\sqrt{1+x_2^2}} + \frac{1}{\sqrt{1+t^2}},\\ \dot{x}_2 &= -x_1^3 - \frac{1}{4}x_2 + \frac{x_2}{4(1+t)^2\sqrt{1+x_1^2}} + \frac{1}{\sqrt{1+t^2}},\end{aligned} \tag{4}$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$ and $t \in \mathbb{R}_+$. This system has the same form of (3) with

$$f(t,x) = \begin{pmatrix} x_1^2 x_2 - \dfrac{1}{4}x_1 \\ -x_1^3 - \dfrac{1}{4}x_2 \end{pmatrix}, \qquad \psi(t,x) = \begin{pmatrix} \dfrac{x_1}{4(1+t)^2\sqrt{1+x_2^2}} + \dfrac{1}{\sqrt{1+t^2}} \\ \dfrac{x_2}{4(1+t)^2\sqrt{1+x_1^2}} + \dfrac{1}{\sqrt{1+t^2}} \end{pmatrix}.$$

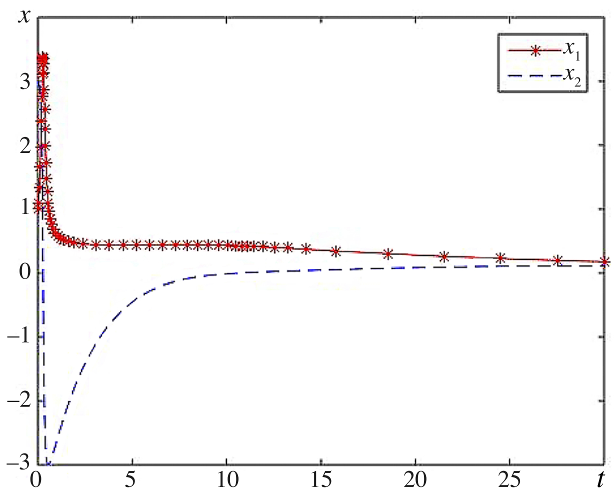

Fig. 1. Time evolution of the state $x(t)$ of system (4).

We set $V(t,x) = \frac{1}{2}(x_1^2 + x_2^2)$ as a Lyapunov function for the nominal system witch is continuously differentiable. It is clear that assumption $(\mathcal{H}_1)$ is satisfied with $c_1 = c_2 = \frac{1}{2}$, $c_4 = 2$ and $c_3 = \frac{1}{2}\cdot$ The function $\psi(t,x)$ verifies the assumption $(\mathcal{H}_3)$, just take $\vartheta(\|x\|) = \frac{1}{\sqrt{2}}\|x\|$, $k = \frac{1}{\sqrt{2}}$, $\rho(t) = \frac{1}{(1+t)^2}$ and $\delta(t) = \frac{2}{\sqrt{1+t^2}}$ are nonnegative and continuous functions with

$$\int_0^\infty \rho(t)dt = 1 := M_\rho < +\infty,$$

$$\int_0^\infty \delta^2(t)dt = 2\pi := M_\delta < +\infty$$

and

$$\rho(t) < \frac{c_3 c_1}{c_2 c_4 k}\cdot$$

Therefore, from Theorem 2 we deduce that the system (4) can be globally practically uniformly exponentially stable. In Fig. 1, it can be seen that the trajectories of system (4) practically converge to zero where the initial state is $\big(x_1(0), x_2(0)\big) = (1, 3)$.

To obtain more general results, we shall introduce the following assumptions:

$(\mathcal{H}_4)$ Assume that the nominal system (1) is globally uniformly exponentially stable and there exists a continuous differentiable function $V(\cdot,\cdot)\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ such that, for all $t \in \mathbb{R}_+$ and all $x \in \mathbb{R}^n$, we have

(i) $c_1\|x\|^b \le V(t,x) \le c_2\|x\|^b$;

(ii) $\dot V(t,x) \le -c_3\|x\|^b$,

where $c_1,\ c_2,\ c_3$ are positive constants and $b \ge 1$.

$(\mathcal{H}_5)$ There exists an integrable continuous function $\delta\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\frac{\partial V}{\partial x}\psi(t,x) \le \delta(t) \quad \forall x \in \mathbb{R}^n \quad \forall t \in \mathbb{R}_+.$$

Now, under the above assumptions we are ready to present the next theorem.

Theorem 3. *If the assumptions* $(\mathcal{H}_4)$ *and* $(\mathcal{H}_5)$ *are hold*, *then system* (3) *is globally practically uniformly exponentially stable.*

Proof. Let $x(t)$ be the solution of system (3). Then, the derivative of V along the trajectories of the system is as follows:

$$\dot V(t,x) \le -c_3\|x\|^b + \delta(t) \le -\frac{c_3}{c_2}V(t,x) + \delta(t).$$

For all $t \ge t_0$, we get

$$V(t,x) \le V(t_0,x_0)e^{-\frac{c_3}{c_2}(t-t_0)} + M_\delta,$$

where $M_\delta = \displaystyle\int_0^\infty \delta(s)ds$. Then, for all $t \ge t_0$ and all $x_0 \in \mathbb{R}^n$, the solution of the system is as follows:

$$\|x(t)\| \le \sqrt{\frac{c_2}{c_1}}\|x_0\|e^{-\frac{c_3}{2c_2}(t-t_0)} + \sqrt{\frac{M_\delta}{c_1}}\cdot$$

Theorem 3 is proved.

2.3. Practical semiglobal uniform exponential stability of perturbed systems. In this section, we consider the parametrized nonlinear time-varying system

$$\dot x = f(t,x,\epsilon), \quad x(t_0) = x_0, \tag{5}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $\epsilon \in \Theta \subset \mathbb{R}^m$ is a constant parameter and $f(t,x,\epsilon)$ is locally Lipschitz in x and piecewise continuous in t for all $\epsilon \in \Theta$. The origin is an equilibrium point of system (5).

A definition of semiglobal uniform exponential stability for nonlinear time-varying systems has been presented in [19]. In order to explicitly show the impact that system parameters may have on the semiglobal uniform exponential stability property. Here, we will use the following definition. This explicitly shows the parameter dependency that practical semiglobal uniform exponential stability may involve and is thus in line with the definition of uniform semiglobal practical asymptotic stability in [6]. In as follows, we present a definition of practical semiglobal uniform exponential stability of system (5) given in [12] (with $\delta = 0$).

First, we define the closed ball $\mathcal{B}_\xi$ as follows: $\mathcal{B}_\xi = \{x \in \mathbb{R}^n : \|x\| \leq \xi\}$.

Definition 2. *Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. System* (5) *is said to be practically semi-globally uniformly exponentially stable on Θ, if, for all $\xi > 0$, there exist a parameter $\epsilon^*(\xi) \in \Theta$ and positive constants $\lambda_{1,\xi}$ and $\lambda_{2,\xi}$, independent of t_0, such that, for all $x_0 \in \mathcal{B}_\xi$, we have*

$$\|x(t, t_0, x_0, \epsilon^*)\| \leq \lambda_{1,\xi}\|x_0\|e^{-\lambda_{2,\xi})(t-t_0)} + \xi \quad \forall t \geq t_0 \geq 0.$$

***Remark* 3.** In other words, system (5) is practically semiglobally uniformly exponentially stable if we can choose a parameter value ϵ^* and find the overshoot and convergence parameters $\lambda_{1,\xi}$ and $\lambda_{2,\xi}$, such that the region of attraction in which the system has practical exponential convergence, $\mathcal{B}_\xi$ can be made arbitrarily large.

***Remark* 4.** We could have given a stronger definition of practical semiglobal uniform exponential stability by requiring that the overshoot and convergence parameters λ_1 and λ_2 should be uniform in ξ, i.e., should not be allowed to depend on the size of the region of attraction. Since both overshoot and convergence in practice typically depend on the tuning of the system, we have chosen the more relaxed definition allowing a dependence on ξ, which is in line with the definition of uniform semiglobal practical asymptotic stability in [6].

The next definition is introduced in [12] to define semiglobal uniform exponential stability of the parametrized system (5).

Definition 3. *Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. System* (5) *is said to be semiglobally uniformly exponentially stable on Θ, if, for all $\xi > 0$, there exist a parameter $\epsilon^*(\xi) \in \Theta$ and positive constants $\lambda_{1,\xi}$ and $\lambda_{2,\xi}$, independent of t_0, such that, for all $x_0 \in \mathcal{B}_\xi$, we have*

$$\|x(t, t_0, x_0, \epsilon^*)\| \leq \lambda_{1,\xi}\|x_0\|e^{-\lambda_{2,\xi}(t-t_0)} \quad \forall t \geq t_0 \geq 0.$$

We define the derivative of a function $V_\xi(t, x)$ along the solution of system (5) by

$$\dot{V}_\xi(t, x) := \frac{\partial V_\xi}{\partial t} + \frac{\partial V_\xi}{\partial x} f(t, x, \epsilon^*).$$

2.3.1. A converse semiglobal exponential stability theorem. In this subsection, we establish a new converse Lyapunov theorem when the system (5) is semiglobally uniformly exponentially stable by requiring the existence of a Lyapunov function depending on a parameter satisfies certain conditions.

The use of Lyapunov functions provides criteria for concluding the asymptotic stability of an equilibrium point without the integration of equations of the considered system being necessary. The next theorem spells out some sufficient conditions to obtain semiglobal uniform exponential stability of system (5) by employing a Lyapunov function.

Theorem 4 (see [23]). *Assume that, for any $\xi > 0$, there exist a parameter $\epsilon^*(\xi) \in \Theta$, a continuously differentiable function $V_\xi(\cdot, \cdot) : \mathbb{R}_+ \times \mathcal{B}_\xi \to \mathbb{R}$ and positive constants $c_{1,\xi}$, $c_{2,\xi}$, $c_{3,\xi}$ such that the next properties are hold*:

(1) $c_{1,\xi}\|x\|^p \leq V_\xi(t, x) \leq c_{2,\xi}\|x\|^p$,
(2) $\dot{V}_\xi(t, x) \leq -c_{3,\xi}\|x\|^p$

for all $t \in \mathbb{R}_+$ *and all* $x \in \mathcal{B}_\xi$ *with* $p \geq 1$. *Then the origin of system* (5) *is semiglobally uniformly exponentially stable on* Θ.

This result deserves the following question: if system (5) is semiglobally uniformly exponentially stable, is there a function V_ξ which satisfies the hypothesis of the earlier theorem? We will show that under some conditions there is a function V_ξ that satisfies properties similar but not the same to those of Theorem 4.

We prove first the following lemma which will be used later.

Lemma 4. *For any* $\xi > 0$, *there exists a parameter* $\epsilon^*(\xi) \in \Theta$ *and let* $\phi(\tau; t, x, \epsilon^*)$ *be a solution of the system* (5) *that starts at* $(t, x, \epsilon^*) \in \mathbb{R}_+ \times \mathcal{B}_\xi \times \Theta$. *Suppose that* $\|f(t, x, \epsilon)\| \leq L_\xi \|x\|$, *where* L_ξ *is a positive constant. Then*

$$\|\phi(\tau; t, x, \epsilon^*)\| \geq \|x\| e^{-L_\xi(\tau - t)}.$$

Proof. Let $\phi(\tau; t, x, \epsilon^*)$ be the solution of the equation

$$\frac{\partial}{\partial \tau}\phi(\tau; t, x, \epsilon^*) = f(\tau, \phi(\tau; t, x, \epsilon^*), \epsilon^*), \quad \phi(t; t, x, \epsilon^*) = x.$$

We have

$$\begin{aligned}
&\left\| \frac{\partial}{\partial \tau}\phi^\top(\tau; t, x, \epsilon^*)\phi(\tau; t, x, \epsilon^*) \right\| \\
&\quad = \left\| \frac{\partial}{\partial \tau}\Big(\phi^\top(\tau; t, x, \epsilon^*)\Big)\phi(\tau; t, x, \epsilon^*) + \phi^\top(\tau; t, x, \epsilon^*)\frac{\partial}{\partial \tau}\phi(\tau; t, x, \epsilon^*) \right\| \\
&\quad = \Big\| f(\tau, \phi(\tau; t, x, \epsilon^*), \epsilon^*)\phi(\tau; t, x, \epsilon^*) + \phi^\top(\tau; t, x, \epsilon^*) f(\tau, \phi(\tau; t, x, \epsilon^*), \epsilon^*) \Big\| \\
&\quad \leq 2\|f(\tau, \phi(\tau; t, x, \epsilon^*), \epsilon^*)\| \|\phi(\tau; t, x, \epsilon^*)\| \\
&\quad \leq 2L_\xi \|\phi(\tau; t, x, \epsilon^*)\|^2.
\end{aligned}$$

Therefore,

$$-2L_\xi \|\phi(\tau; t, x, \epsilon^*)\|^2 \leq \frac{\partial}{\partial \tau}\|\phi(\tau; t, x, \epsilon^*)\|^2 \leq 2L_\xi \|\phi(\tau; t, x, \epsilon^*)\|^2.$$

Integrating the above inequality from t to τ, we get

$$\int\limits_t^\tau -2L_\xi ds \leq \int\limits_t^\tau \frac{\frac{\partial}{\partial s}\|\phi(s; t, x, \epsilon^*)\|^2}{\|\phi(s; t, x, \epsilon^*)\|^2} ds \leq \int\limits_t^\tau 2L_\xi ds.$$

Thus,

$$-2L_\xi(\tau - t) \leq \log\Big(\|\phi(\tau; t, x, \epsilon^*)\|^2\Big) - \log\Big(\|\phi(t; t, x, \epsilon^*)\|^2\Big) \leq 2L_\xi(\tau - t).$$

Since that $\|\phi(t; t, x, \epsilon^*)\| = \|x\|$, then

$$\|\phi(\tau; t, x, \epsilon^*)\| \geq \|x\| e^{-L_\xi(\tau - t)}.$$

Lemma 4 is proved.

Now, we can establish the following theorem.

Theorem 5. *Let in the nonlinear system* (5) f *is continuously differentiable. For any* $\xi > 0$, *there exists a parameter* $\epsilon^*(\xi) \in \Theta$ *such that the Jacobian matrix* $\left[\dfrac{\partial f}{\partial x}\right]$ *is bounded on* $\mathcal{B}_\xi$, *uniformly in* t. *If the system is semiglobally uniformly exponentially stable on* Θ, *then there exists a function* $V_\xi(\cdot,\cdot) : \mathbb{R}_+ \times \mathcal{B}_\xi \to \mathbb{R}$, *continuously differentiable, that satisfies the following inequalities*:

(1) $c_{1,\xi}\|x\|^p \leq V_\xi(t,x) \leq c_{2,\xi}\|x\|^p,\ (t,x) \in \mathbb{R}_+ \times \mathcal{B}_\xi$,

(2) $\dot{V}_\xi(t,x) \leq -c_{3,\xi}\|x\|^p,\ (t,x) \in \mathbb{R}_+ \times \mathcal{B}_\xi$,

(3) $\left\|\dfrac{\partial V_\xi}{\partial x}\right\| \leq c_{4,\xi}\|x\|^{p-1},\ (t,x) \in \mathbb{R}_+ \times \mathcal{B}_\xi$,

for some positive constants $c_{1,\xi},\ c_{2,\xi},\ c_{3,\xi},\ c_{4,\xi}$ *and* $p \geq 2$.

Proof. Let $\phi(\tau; t, x, \epsilon^*)$ denotes the solution of the system that starts at (t, x, ϵ^*), that is, $\phi(t; t, x, \epsilon^*) = x$, and let L_ξ denotes the bound of $\left[\dfrac{\partial f}{\partial x}\right]$. Let

$$V_\xi(t,x) = \int\limits_t^{t+T} \Big(\phi^\top(\tau; t, x, \epsilon^*)\phi(\tau; t, x, \epsilon^*)\Big)^{\frac{p}{2}} d\tau,$$

where T is a positive constant to be chosen. Due to the exponentially decaying bound on the trajectories, on one side, we have

$$\begin{aligned} \dot{V}_\xi(t,x) &= \int\limits_t^{t+T} \|\phi(\tau; t, x, \epsilon^*)\|^p d\tau \\ &\leq \int\limits_t^{t+T} \lambda_{1,\xi}^p e^{-p\lambda_{2,\xi}(\tau - t)} d\tau \|x\|^p = \frac{\lambda_{1,\xi}^p}{p\lambda_{2,\xi}}\Big(1 - e^{-p\lambda_{2,\xi}T}\Big)\|x\|^p. \end{aligned}$$

Therefore,

$$V_\xi(t,x) \leq c_{2,\xi}\|x\|^p$$

with $c_{2,\xi} = \dfrac{\lambda_{1,\xi}^p}{p\lambda_{2,\xi}}\big(1 - e^{-p\lambda_{2,\xi}T}\big)$.

On the other side, the Jacobian matrix $\left[\dfrac{\partial f}{\partial x}\right]$ is bounded on $\mathcal{B}_\xi$. Then, from Lemma 4, we get

$$V_\xi(t,x) \geq \int\limits_t^{t+T} e^{-pL_\xi(\tau - t)} d\tau \|x\|^p = \frac{1}{pL_\xi}(1 - e^{-pL_\xi T})\|x\|^p.$$

Thus,

$$V_\xi(t,x) \geq c_{1,\xi}\|x\|^p$$

with $c_{1,\xi} = \dfrac{1 - e^{-pL_\xi T}}{pL_\xi}$. Hence, the first inequality of the theorem is hold.

We define now the functions $\phi_t(\tau;t,x,\epsilon^*)$ and $\phi_x(\tau;t,x,\epsilon^*)$ as follows:

$$\phi_t(\tau;t,x,\epsilon^*)=\frac{\partial}{\partial t}\phi(\tau;t,x,\epsilon^*),\qquad \phi_x(\tau;t,x,\epsilon^*)=\frac{\partial}{\partial x}\phi(\tau;t,x,\epsilon^*).$$

Then

$$\begin{aligned}\frac{\partial V_\xi}{\partial t}+\frac{\partial V_\xi}{\partial x}f(t,x,\epsilon^*)&=\phi^\top(t+T;t,x,\epsilon^*)^p\phi(t+T;t,x,\epsilon^*)^p-\phi^\top(t;t,x,\epsilon^*)^p\phi(t;t,x,\epsilon^*)^p\\&\quad+p\int\limits_t^{t+T}\phi^\top(\tau;t,x,\epsilon^*)^{p-1}\phi_t(\tau;t,x,\epsilon^*)d\tau\\&\quad+p\int\limits_t^{t+T}\phi^\top(\tau;t,x,\epsilon^*)^{p-1}\phi_x(\tau;t,x,\epsilon^*)d\tau f(t,x,\epsilon^*)\\&=\left\|\phi^\top(t+T;t,x,\epsilon^*)\right\|^p-\|x\|^p\\&\quad+p\int\limits_t^{t+T}\phi^\top(\tau;t,x,\epsilon^*)^{p-1}\Big(\phi_t(\tau;t,x,\epsilon^*)+\phi_x(\tau;t,x,\epsilon^*)f(t,x,\epsilon^*)\Big)d\tau.\end{aligned}$$

By the composition rule of flow, we have

$$\phi(\tau;t,\phi(t;\tau,u,\epsilon^*))=u.$$

We consider the identity $\phi(t;\tau,x,\epsilon^*)=x\ \forall\tau\geq t$. Differentiating both sides of the previous identity with respect to t gives

$$\phi_t(\tau;t,x,\epsilon^*)+\phi_x(\tau;t,x,\epsilon^*)\phi_t(t;\tau,x,\epsilon^*)\equiv 0\quad\forall\tau\geq t.$$

Then

$$\phi_t(\tau;t,x,\epsilon^*)+\phi_x(\tau;t,x,\epsilon^*)f(t,\phi(t;\tau,x,\epsilon^*),\epsilon^*)\equiv 0\quad\forall\tau\geq t.$$

Hence,

$$\phi_t(\tau;t,x,\epsilon^*)+\phi_x(\tau;t,x,\epsilon^*)f(t,x,\epsilon^*)\equiv 0\quad\forall\tau\geq t.$$

Therefore,

$$\dot V_\xi(t,x)=\|\phi^\top(t+T;t,x,\epsilon^*)\|^p-\|x\|^p\leq-\Big(1-\lambda_{1,\xi}^p e^{-pT\lambda_{2,\xi}}\Big)\|x\|^p.$$

By choosing $T=\dfrac{\log(p\lambda_{1,\xi}^p)}{p\lambda_{2,\xi}}$, we get

$$\dot V_\xi(t,x)\leq -c_{3,\xi}\|x\|^p,$$

where $c_{3,\xi}=\dfrac{p-1}{p}$. Thus, the second inequality of Theorem 5 is satisfied.

To show the last property, We note that $\phi_x(\tau,t,x,\epsilon^*)$ satisfies the equation

$$\frac{\partial}{\partial t}\phi_x(\tau;t,x,\epsilon^*) = \frac{\partial f}{\partial x}(\tau,\phi(\tau,t,x,\epsilon^*),\epsilon^*)\phi_x, \quad \phi_x(t;t,x,\epsilon^*) = I.$$

Now, since

$$\left\|\frac{\partial f}{\partial x}(t,x,\epsilon^*)\right\| \le L_\xi,$$

then

$$\|\phi_x(\tau;t,x,\epsilon^*)\| \le e^{L_\xi(\tau-t)}. \tag{6}$$

The proof of inequality (6) is just the same of proof Lemma 4. Hence,

$$\begin{aligned}\left\|\frac{\partial V_\xi}{\partial x}\right\| &= \left\|p\int_t^{t+T}\phi^\top(\tau;t,x,\epsilon^*)^{p-1}\phi_x(\tau;t,x,\epsilon^*)d\tau\right\| \\ &\le p\int_t^{t+T}\|\phi(\tau;t,x,\epsilon^*)\|^{p-1}\|\phi_x(\tau;t,x,\epsilon^*)\|d\tau \\ &\le p\lambda_{1,\xi}^{p-1}\int_t^{t+T}e^{-(p-1)\lambda_{2,\xi}(\tau-t)}e^{L_\xi(\tau-t)}d\tau\|x\|^{p-1} \\ &= \frac{p\lambda_{1,\xi}^{p-1}}{(p-1)\lambda_{2,\xi}-L_\xi}\Big(1-e^{-\big((p-1)\lambda_{2,\xi}-L_\xi\big)T}\Big)\|x\|^{p-1}.\end{aligned}$$

Thus, the last inequality of Theorem 5 is satisfied with

$$c_{4,\xi} = \frac{p\lambda_{1,\xi}^{p-1}}{(p-1)\lambda_{2,\xi}-L_\xi}\Big(1-e^{-\big((p-1)\lambda_{2,\xi}-L_\xi\big)T}\Big).$$

Theorem 5 is proved.

2.3.2. Robustness to nonvanishing perturbations. In this subsection, we will apply the converse theorem to perturbed nonlinear parametrized systems. We consider the following system:

$$\dot{x} = f(t,x,\epsilon) + h(t,x,\epsilon), \tag{7}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $\epsilon \in \Theta \subset \mathbb{R}^m$ is a constant parameter, $f(t,x,\epsilon)$ and $h(t,x,\epsilon)$ are locally Lipschitz in x and piecewise continuous in t for all $\epsilon \in \Theta$. The system (7) is the perturbed system of the nonlinear parametrized system (5).

In as follows, we will study the practical semiglobal uniform exponential stability of system (7) under different conditions on the perturbed term.

Firstly, we state the following assumption:

$(\mathcal{H}_6)$ For any $\xi > 0$, there exist a parameter $\epsilon^*(\xi) \in \Theta$, α and γ are nonnegative continuous functions on $\mathbb{R}_+$ such that the perturbation $h(t,x,\epsilon^*)$ verifies:

$$\|h(t,x,\epsilon^*)\| \le \alpha(t,\epsilon^*)\|x\| + \gamma(t,\epsilon^*) \quad \forall x \in \mathcal{B}_\xi \quad \forall t \in \mathbb{R}_+$$

with

$$\int_0^{+\infty} \alpha(s,\epsilon^*)ds \le M_\alpha < +\infty,$$

$$\int_0^{+\infty} \gamma^2(s,\epsilon^*)ds \le M_\gamma < +\infty$$

and

$$\alpha(t,\epsilon^*) < \frac{c_{3,\xi}c_{2,\xi}}{c_{1,\xi}c_{4,\xi}} \quad \forall t \ge t_0.$$

Now, we are ready to present the fundamental result of this subsection.

Theorem 6. *Let in the perturbed system* (7) f *is continuously differentiable and, for any* $\xi > 0$, *there exists a parameter* $\epsilon^*(\xi) \in \Theta$ *such that the Jacobian matrix* $\left[\dfrac{\partial f}{\partial x}\right]$ *is bounded on* $\mathcal{B}_\xi$, *uniformly in* t. *Assume that the system* (5) *is semiglobally uniformly exponentially stable and the perturbation* h *satisfies the assumption* $(\mathcal{H}_6)$. *Then the system* (7) *is practically semiglobally uniformly exponentially stable on* Θ.

Proof. By Theorem 5, there exists a Lyapunov function $V_\xi(t,x)$ having the three properties (1)–(3). The derivative along the trajectory of system (7) is given by

$$\begin{aligned}\dot V_\xi(t,x) &= \frac{\partial V_\xi}{\partial t} + \frac{\partial V_\xi}{\partial x} f(t,x,\epsilon^*) + \frac{\partial V_\xi}{\partial x} h(t,x,\epsilon^*)\\ &\le -c_{3,\xi}\|x\|^p + c_{4,\xi}\|x\|^{p-1}\Big(\alpha(t,\epsilon^*)\|x\| + \gamma(t,\epsilon^*)\Big)\\ &= -c_{3,\xi}\|x\|^p + c_{4,\xi}\alpha(t,\epsilon^*)\|x\|^p + c_{4,\xi}\gamma(t,\epsilon^*)\|x\|^{p-1}\\ &\le -\left(\frac{c_{3,\xi}}{c_{2,\xi}} - \frac{c_{4,\xi}}{c_{1,\xi}}\alpha(t,\epsilon^*)\right)V_\xi(t,x) + c_{4,\xi}\xi^{p-1}\gamma(t,\epsilon^*).\end{aligned}$$

Then, by using Lemma 1, we get, for all $t \ge t_0$,

$$V_\xi(t,x) \le V_\xi(t_0,x_0)e^{\frac{c_{4,\xi}}{c_{1,\xi}}M_\alpha} e^{-\frac{c_{3,\xi}}{c_{2,\xi}}(t-t_0)} + c_{4,\xi}\xi^{p-1}e^{\frac{c_{4,\xi}}{c_{1,\xi}}M_\alpha}\sqrt{\frac{M_\gamma c_{2,\xi}}{2c_{3,\xi}}}.$$

Consequently, for all $t \ge t_0$ and all $x_0 \in \mathcal{B}_\xi$, the solution of the system verifies the following estimation:

$$\|x(t,t_0,x_0,\epsilon^*)\| \le \left(\frac{c_{2,\xi}}{c_{1,\xi}}e^{\frac{c_{4,\xi}}{c_{1,\xi}}M_\alpha}\right)^{\frac{1}{p}}\|x_0\|e^{-\frac{c_{3,\xi}}{pc_{2,\xi}}(t-t_0)} + \left(\frac{c_{4,\xi}}{c_{1,\xi}}\xi^{p-1}e^{\frac{c_{4,\xi}}{c_{1,\xi}}M_\alpha}\right)^{\frac{1}{p}}\left(\frac{M_\gamma c_{2,\xi}}{2c_{3,\xi}}\right)^{\frac{1}{2p}}.$$

Therefore, the system (7) is practically semiglobally uniformly exponentially stable on Θ.

Theorem 6 is proved.

We also make the following assumption:

$(\mathcal{H}_7)$ For any $\xi > 0$, there exist a parameter $\epsilon^*(\xi) \in \Theta$ and an integrable continuous function γ such that

$$\|h(t, x, \epsilon^*)\| \leq \gamma(t, \epsilon^*) \quad \forall x \in \mathcal{B}_\xi \quad \forall t \in \mathbb{R}_+.$$

We are now in position to present the following result.

Theorem 7. *Let in the perturbed system* (7) f *is continuously differentiable and, for any* $\xi > 0$, *there exists a parameter* $\epsilon^*(\xi) \in \Theta$ *such that the Jacobian matrix* $\left[\dfrac{\partial f}{\partial x}\right]$ *is bounded on* $\mathcal{B}_\xi$, *uniformly in* t. *Assume that the system* (5) *is semiglobally uniformly exponentially stable on* Θ *and the perturbation* $h(t, x, \epsilon^*)$ *satisfies the assumption* $(\mathcal{H}_7)$. *Then, the system* (7) *is practically semiglobally uniformly exponentially stable on* Θ.

Proof. By Theorem 5, there exists a Lyapunov function $V_\xi(t, x)$ having the three properties (1)–(3). The derivative along the trajectory of system (7) is as follows:

$$\begin{aligned}\dot{V}_\xi(t, x) &\leq \frac{\partial V_\xi}{\partial t} + \frac{\partial V_\xi}{\partial x} f(t, x, \epsilon^*) + \frac{\partial V_\xi}{\partial x} h(t, x, \epsilon^*) \\ &\leq c_{3,\xi}\|x\|^p + c_{4,\xi}\|x\|^{p-1}\gamma(t, \epsilon^*) \\ &\leq -\frac{c_{3,\xi}}{c_{2,\xi}} V_\xi(t, x) + c_{4,\xi}\xi^{p-1}\gamma(t, \epsilon^*).\end{aligned}$$

Using Lemma 3, we get, for all $t \leq t_0$,

$$V_\xi(t, x) \leq V_\xi(t_0, x_0) e^{-\frac{c_{3,\xi}}{c_{2,\xi}}(t-t_0)} + c_{4,\xi}\xi^{p-1} \int\limits_{t_0}^{t} e^{-\frac{c_{3,\xi}}{c_{2,\xi}}(t-s)} \gamma(s, \epsilon^*) ds$$

$$\leq V(t_0, x_0) e^{-\frac{c_{3,\xi}}{c_{2,\xi}}(t-t_0)} + c_{4,\xi}\xi^{p-1} M_\gamma,$$

where $M_\gamma = \displaystyle\int_0^\infty \gamma(s, \epsilon^*) ds$. Hence, for all $t \geq t_0$ and all $x_0 \in \mathcal{B}_\xi$, the solution of the system is given by

$$\|x(t, t_0, x_0, \epsilon^*)\| \leq \left(\frac{c_{2,\xi}}{c_{1,\xi}}\right)^{\frac{1}{p}} \|x_0\| e^{-\frac{c_{3,\xi}}{pc_{2,\xi}}(t-t_0)} + \left(\frac{c_{4,\xi}}{c_{1,\xi}}\xi^{p-1} M_\gamma\right)^{\frac{1}{p}}.$$

Theorem 7 is proved.

Next, we establish another assumption to investigate the practical semiglobal uniform exponential stability of system (7).

Proposition 2. *Let in the perturbed system* (7) f *is continuously differentiable and, for any* $\xi > 0$, *there exists a parameter* $\epsilon^*(\xi) \in \Theta$ *such that the Jacobian matrix* $\left[\dfrac{\partial f}{\partial x}\right]$ *is bounded on* $\mathcal{B}_\xi$, *uniformly in* t. *Assume that the system* (5) *is semiglobally uniformly exponentially stable and the perturbation* $h(t, x, \epsilon^*)$ *satisfies the following assumption*:

$(\mathcal{A})$ *for any* $\xi > 0$, *there exist a parameter* $\epsilon^*(\xi) \in \Theta$ *and a nonnegative continuous integrable function* σ *such that*

$$\frac{\partial V_\xi}{\partial x} h(t, x, \epsilon^*) \leq \sigma(t, \epsilon^*)\|x\|^{\alpha+1} \quad \forall x \in \mathcal{B}_\xi \quad \forall t \in \mathbb{R}_+, \quad 0 \leq \alpha < 1.$$

Then the system (7) *is practically semiglobally uniformly exponentially stable on* Θ.

Proof. By Theorem 5, there exists a Lyapunov function $V_\xi(t,x)$ satisfying the properties (1) and (2) with $p=2$. Then the derivative of $V_\xi(t,x)$ along the trajectories of system (7) is as follows:

$$\begin{aligned}\dot{V}_\xi(t,x) &= \frac{\partial V_\xi}{\partial t} + \frac{\partial V_\xi}{\partial x} f(t,x,\epsilon^*) + \frac{\partial V_\xi}{\partial x} h(t,x,\epsilon^*)\\ &= -c_{3,\xi}\|x\|^2 + \sigma(t,\epsilon^*)\|x\|^{\alpha+1}\\ &= -\frac{c_{3,\xi}}{c_{2,\xi}} V_\xi(t,x) + \frac{\sigma(t,\epsilon^*)}{c_{1,\xi}^{\frac{\alpha+1}{2}}} V_\xi(t,x)^{\frac{\alpha+1}{2}}.\end{aligned}$$

Let

$$\vartheta_\xi(t) = V_\xi(t,x)^{\frac{\alpha+1}{2}} \Rightarrow \dot{\vartheta}_\xi(t) = \frac{1-\alpha}{2}\dot{V}_\xi(t,x)V_\xi(t,x)^{-\frac{\alpha+1}{2}}.$$

Hence,

$$\dot{\vartheta}_\xi(t) \le -\frac{1-\alpha}{2}\frac{c_{3,\xi}}{c_{2,\xi}}\vartheta_\xi(t) + \frac{1-\alpha}{2}\frac{\sigma(t,\epsilon^*)}{c_{1,\xi}^{\frac{\alpha+1}{2}}}.$$

Applying Lemma 3, we get, for all $t \ge t_0$,

$$\begin{aligned}\vartheta_\xi(t) \le \vartheta_\xi(t_0) & e^{-\frac{1-\alpha}{2}\frac{c_{3,\xi}}{c_{2,\xi}}(t-t_0)}\\ &+ \frac{1-\alpha}{2}\frac{1}{c_{1,\xi}^{\frac{\alpha+1}{2}}}\int\limits_{t_0}^{t} e^{-\frac{1-\alpha}{2}\frac{c_{3,\xi}}{c_{2,\xi}}(t-s)}\sigma(s)ds\\ &\le \vartheta_\xi(t_0)e^{-\frac{1-\alpha}{2}\frac{c_{3,\xi}}{c_{2,\xi}}(t-t_0)} + \frac{1-\alpha}{2}\frac{1}{c_{1,\xi}^{\frac{\alpha+1}{2}}}M_\sigma,\end{aligned}$$

where $M_\sigma = \displaystyle\int_0^\infty \sigma(s,\epsilon^*)ds$. Therefore, for all $t \ge t_0$ and all $x_0 \in \mathcal{B}_\xi$, the solution of the system satisfies

$$\|x(t,t_0,x_0,\epsilon^*)\| \le \left(\frac{1}{1-\alpha}\right)^{\frac{1}{1-\alpha}}\sqrt{\frac{c_{2,\xi}}{c_{1,\xi}}}\|x_0\|e^{-\frac{c_{3,\xi}}{c_{2,\xi}}(t-t_0)} + \left(\frac{1-\alpha}{2}\right)^{\frac{1}{1-\alpha}}\left(\frac{M_\sigma}{c_{1,\xi}}\right)^{\frac{1}{1-\alpha}}.$$

Proposition 2 is proved.

3. Conclusion. In this paper, we considered the problem of stability of nonlinear systems with perturbations and we studied the asymptotic behavior in a sense that the trajectories converge to a small ball centred at the origin. Practical semiglobal exponential stability is studied using a new converse stability theorem for time-varying perturbed parametrized systems. In addition, sufficient conditions for the global practical uniform exponential stability of perturbed systems are obtained under various assumptions on the perturbed term by using Lyapunov techniques and the integral inequalities approach.

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