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ON LOCAL PROPERTIES OF SINGULAR INTEGRAL

ПРО ЛОКАЛЬНІ ВЛАСТИВОСТІ СИНГУЛЯРНОГО ІНТЕГРАЛА

Let γ be a regular curve. We study the local properties of singular integrals in the $H_\alpha^{\alpha+\beta}(t_0,\gamma)$ class of functions. We obtain a strengthening of the Plemelj – Privalov theorem for functions from the class $H_\alpha^{\alpha+\beta}(t_0,\gamma)$. It is proved that, at the point t_0, of increased smoothness for $\alpha+\beta<1$, there is only a logarithmic loss.

Нехай γ — регулярна крива. Досліджуються локальні властивості сингулярних інтегралів у класі функцій $H_\alpha^{\alpha+\beta}(t_0,\gamma)$. Отримано посилення теореми Племеля – Привалова для функцій з класу $H_\alpha^{\alpha+\beta}(t_0,\gamma)$. Доведено, що в точці t_0 підвищеної гладкості для $\alpha+\beta<1$ є лише логарифмічні втрати.

1. Introduction and main results. Let γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$. We denote by $L^p(\gamma)$, $1<p<\infty$, the Lebesgue space of all measurable functions on γ for which the norm

$$\|f\|_p=\left(\int\limits_\gamma |f(t)|^p|dt|\right)^{1/p}<\infty.$$

We use $c,c_1,c_2,\ldots$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the question of interest. We also will use the relation $f=O(g)$ which means that $f\le cg$ for a constant c independent of f and g. We also write $a\preceq b$ if $a\le cb$, $a\asymp b$ if $a\preceq b$, and $b\preceq a$ at the same time. If $a\asymp b$ then we will say that a and b are equivalent.

Definition 1.1. *Let f be a function defined on γ. We denote by $H_\alpha(\gamma)$, $0<\alpha\le 1$, the class of functions* (*Hölder class of functions*) *satisfying the condition*

$$\big|f(\xi)-f(t)\big|=c(\alpha)|\xi-t|^\alpha$$

for all $\xi,t\in\gamma$.

Definition 1.2 [39]. *We denote by $H_\alpha^{\alpha+\beta}(t_0,\gamma)$ the class of functions belonging to $H_\alpha(\gamma)$, $0<\alpha\le 1$, and satisfying the condition*

$$|f(t_0)-f(t)|=O\Big(|t-t_0|^{\alpha+\beta}\Big),\quad \beta>0,\quad t\in\gamma.$$

Let B be a simply-connected domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve γ, and let $B^-:=\operatorname{ext}\gamma$. Further, let

$$T:=\{w\in\mathbb{C}:|w|=1\},\qquad D:=\operatorname{int}T,\qquad \text{and}\qquad D^-:=\operatorname{ext}T,$$

where by $\operatorname{int}T$ we understand the finite domain whose boundary coincides with T and $\operatorname{ext}T=\mathbb{C}\setminus\overline{\operatorname{int}T}$.

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Let $w = \varphi(t)$ be the conformal mapping of B^- onto D^- normalized by

$$\varphi(\infty) = \infty, \qquad \lim_{t\to\infty} \frac{\varphi(t)}{t} > 0$$

and ψ stands for the inverse of φ.

Let $w = \varphi_1(t)$ indicate a function that maps the domain B conformally onto the disk $|w| < 1$. The inverse mapping of φ_1 will be shown by ψ_1. Let γ_r be the image of the circle $|\varphi_1(t)| = r$, $0 < r < 1$, under the mapping $z = \psi_1(w)$.

Definition 1.3. *Let us denote by $E_p(B)$, where $p > 0$, the class of all functions $f(t) \neq 0$ that are analytic in B and have the property that the integral*

$$\int\limits_{\gamma_r} |f(t)|^p |dt|$$

is bounded for $0 < r < 1$.

We shall call the $E_p(B)$ class the *Smirnov class*.

If the function $f(t)$ belongs to E_p, then $f(t)$ has definite limiting values $f(t')$ almost everywhere (a.e.) on γ, over all nontangential paths; $|f(t')|$ is summable on γ, and

$$\lim_{r\to 1} \int\limits_{\gamma_r} |f(t)|^p |dt| = \int\limits_{\gamma} |f(t')|^p |dt'|.$$

For $p > 1, E_p(B)$ is a Banach space with respect to the norm

$$\|f\|_{E_p(B)} := \|f\|_{L_p(\gamma)} := \left(\int\limits_{\gamma} |f(t)|^p |dt| \right)^{\frac{1}{p}}.$$

It is known that $\varphi' \in E_1(B^-)$ and $\psi' \in E_1(D^-)$. Note that the general information about Smirnov classes can be found in the books [9, p. 168 – 185; 11, p. 438 – 453].

Let $f \in L_1(\gamma)$. Then the functions f^+ and f^- defined by

$$f^+(t) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(\zeta)}{\zeta - t} d\zeta, \quad t \in B,$$

and

$$f^-(t) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(\zeta)}{\zeta - t} d\zeta, \quad t \in B^-,$$

are analytic in B and B^-, respectively, and $f^-(\infty) = 0$. Thus, the limit

$$\widetilde{f}(t) = S_\gamma(f)(t) := \lim_{\varepsilon\to 0} \frac{1}{\pi i} \int\limits_{\gamma\setminus\{\zeta\,:\,|\zeta - t|<\varepsilon\}} \frac{f(\zeta) - f(t)}{\zeta - t} d\zeta + f(t)$$

exists and is finite for almost all $t \in \gamma$.

The limit $S_\gamma(f)(t)$ is called the *Cauchy singular integral* of f at $t \in \gamma$.

According to the Privalov theorem [11, p. 431], if one of the functions $f^+(t)$ and $f^-(t)$ has a nontangential limit on γ a.e., then Cauchy singular integral $S_\gamma(f)(t)$ exists a.e. on γ and also the other one of the functions $f^+(t)$ and $f^-(t)$ has a nontangential limit on γ a.e. Conversely, if $S_\gamma(f)(t)$ exists a.e. on γ, then the functions $f^+(t)$ and $f^-(t)$ have nontangential limits a.e. on γ. In both cases, the formulae

$$f^+(t) = S_\gamma(f)(t) + \frac{1}{2}f(t), \qquad f^-(t) = S_\gamma(f)(t) - \frac{1}{2}f(t)$$

hold a.e. on γ. From this it follows that

$$f(t) = f^+(t) - f^-(t)$$

a.e. on γ.

Let γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$ and $D(z,r)$ be an open disk with radius r and center at z. We define by l length of curve γ. Let $t = t(s),\ 0 \le s \le l$, be the equation of the curve γ in arcual coordinates.Also we define by d diameter of curve γ $(d = \sup_{t,\tau} |t-\tau|)$.

Definition 1.4. *γ Is called a regular curve (or Carleson curve) if there exists a constant $c(\gamma) > 0$ depending on only γ such that, for every $r > 0$, $\theta(r) = \sup\{|\gamma \cap D(z,r)| : z \in \gamma\} \le c(\gamma)r$, where $|\gamma \cap D(z,r)|$ is the length of the set $\gamma \cap D(z,r)$.*

We denote by S the set of all regular Jordan curves in the complex plane. Note that the class of all regular curves is very wide. G. David proved [8] that γ is a regular curve if and only if, for every $f \in L_p(\gamma)$, $S_\gamma(f)$ exists and belongs to $L_p(\gamma)$ and the singular operator $S_\gamma(f)\colon L_p(\gamma) \to L_p(\gamma)$ is bounded, that is, there exists a constant $C(p,\gamma)$ such that

$$\|S_\gamma(f)\|_{L_p(\gamma)} \le C(p,\gamma)\|f\|_{L_p(\gamma)}$$

for all $f \in L_p(\gamma)$. In [14], V. Havin proved that if the singular operator $S_\gamma(f)\colon L_p(\gamma) \to L_p(\gamma)$ is bounded, for every $f \in L_p(\gamma)$, the functions f^+ and f^- belong to Smirnov's classes $E_p(B)$ and $E_p(B^-)$, respectively. For instance, convex curves, Ljapunov curves, chord arcs, smooth curves and Lipschitz curves are all regular. Note that boundedness of the Cauchy singular integral in the different spaces were investigated by several authors (see, for example, [8, 17 – 23, 45]).

We denote by Φ a class of functions $\omega = \omega(\delta)$, $\delta \in (0, d]$, such that $\omega(\delta)$ does not decrease, $\omega(\delta) \to 0$ when $\delta \to 0$, $\omega(\delta)\delta^{-1}$ does not increase.

According to classical Plemelj – Privalov's theorem [34, 35] if γ is a circle, then, for $0 < \alpha < 1$, there is an inclusion

$$f \in H_\alpha(\gamma) \Rightarrow S_\gamma(f) \in H_\alpha(\gamma).$$

Later this theorem was proved by I. I. Privalov [35] for piecewise smooth curve without cusps, N. I. Muskhelishvili [29] for the whole class of piecewise smooth curves, N. A. Davydov [7] for K-curves (quasismooth curves).

In this paper, if $\gamma \in S$ the local properties of the singular integral in the $H_\alpha^{\alpha+\beta}(t_0,\gamma)$ class of functions are studied. We obtain a strengthening of Plemelj – Privalov's theorem for functions from the class $H_\alpha^{\alpha+\beta}(t_0,\gamma)$. It is proved that at the point t_0, of increased smoothness for $\alpha + \beta < 1$ there is only a logarithmic loss. We note that similar results for different curves in the complex plane and different function classes have been obtained by several authors (see, for example, [1 – 7, 10 – 15, 24 – 44]).

The main result of this paper is the following theorem.

Theorem 1.1. *Let* $\gamma \in S,\ t_0 \in \gamma,\ f \in H_\alpha^{\alpha+\beta}(t_0),\ 0 < \alpha \le 1$ *and* $\beta \ge 0$. *Then, for the singular integral* $\widetilde{f}(t)$, *the following inequalities hold*:

(1) *for* $\alpha + \beta \le 1$,

$$\left|\widetilde{f}(t) - \widetilde{f}(t_0)\right| \le c_1(f,\gamma,\alpha,\beta)|t - t_0|^{\alpha+\beta} \ln \frac{2d}{|t - t_0|},$$

(2) *for* $\alpha + \beta > 1$,

$$\left|\widetilde{f}(t) - \widetilde{f}(t_0)\right| \le c_2(f,\gamma,\alpha,\beta)|t - t_0|.$$

2. Auxiliary results. In the proof of the main result we need the following lemmas.

Lemma 2.1 [28]. *Let* $\gamma \in S,\ \omega \in \Phi$ *and* $0 \le 2\varepsilon \le \eta \le d$. *Then the inequlities*

$$\int\limits_{\gamma_\eta(t)\setminus\gamma_\varepsilon(t)} \frac{\omega(|\xi - t|)}{|\xi - t|} \le c_3(\gamma) \int\limits_{\varepsilon}^{\eta} \frac{\omega(\xi)}{\xi} d\xi,$$

$$\int\limits_{\gamma_\eta(t)\setminus\gamma_\varepsilon(t)} \frac{\omega(|\xi - t|)}{|\xi - t|^2} \le c_4(\gamma) \int\limits_{\varepsilon}^{\eta} \frac{\omega(\xi)}{\xi^2} d\xi$$

hold, where $\gamma_\eta(t) := \{\xi \in \gamma : |\xi - t| \le \eta\}$ *and* $\gamma_\varepsilon(t) := \{\xi \in \gamma : |\xi - t| \le \varepsilon\}$.

Lemma 2.2 [28]. *Let* $\gamma \in S$. *Then, for any* $0 < \varepsilon < \eta$, *the inequality*

$$\left| \int_{\gamma\setminus\gamma_\varepsilon(t)} \frac{d\xi}{\xi - t} \right| \le 2\pi$$

holds.

We give necessary and sufficient conditions for a function f to belong to the class $H_\alpha^{\alpha+\beta}(t_0)$.

Lemma 2.3. *Let* $\gamma \in S$ *and* $t_0 \in \gamma$. *Then* $f \in H_\alpha^{\alpha+\beta}(t_0),\ 0 < \alpha \le 1,$ *and* $\beta \ge 0 \Longleftrightarrow$

(1) $\left|f(\xi) - f(t)\right| \le c_5(f)|\xi - t|^\alpha$ *for* $\xi \in \gamma_{\varepsilon|t-t_0|^{1+\beta/\alpha}}(t)$, $\left(\varepsilon = \dfrac{1}{2d^{\frac{\alpha+\beta}{2}}}\right)$,

(2) $\left|f(\xi) - f(t)\right| \le c_6(f)|t - t_0|^{\alpha+\beta}$ *for* $\xi \in \gamma_{\frac{3}{2}|t-t_0|}(t_0)\setminus\gamma_{\varepsilon|t-t_0|^{1+\beta/\alpha}}(t)$,

(3) $\left|f(\xi) - f(t)\right| \le c_7(f)|\xi - t|^{\alpha+\beta}$ *for* $\xi \in \gamma\setminus\gamma_{\frac{3}{2}|t-t_0|}(t_0)$.

Proof. *Sufficiency.* Let us assume that conditions (1)–(3) are satisfied. We prove that $f \in H_\alpha^{\alpha+\beta}(t_0)$. For this, it should be shown that $f \in H_\alpha(\gamma)$ and for $t_0 \in \gamma$ the inequality

$$\left|f(t) - f(t_0)\right| \le c_8(f)|t - t_0|^{\alpha+\beta} \tag{2.1}$$

holds.

First of all we shown that the inequality (2.1) holds. Let $t \in \gamma$. We set $\xi = t_0$. For such ξ the case (2) is satisfied. Then the inequality

$$\left|f(t) - f(t_0)\right| = |f(t) - f(\xi)| \le c_9(f)|t - t_0|^{\alpha+\beta}$$

holds. That is the estimation (2.1) is proved.

Now let us shown that $f \in H_\alpha(\gamma)$. That is, we shown that, for all $t_1, t_2 \in \gamma$, the following inequality holds:

$$|f(t_1) - f(t_2)| \le c_{10}(f)|t_1 - t_2|^\alpha. \tag{2.2}$$

We set $t_1 = t,\ t_2 = \dot{\xi}$. Then the following cases are possible:

(a) Let $\xi \in \gamma_{\varepsilon|t-t_0|^{1+\beta/\alpha}}(t)$. According to the case (1) the estimation

$$\big|f(\xi) - f(t)\big| \le c_{11}(f)|\xi - t|^{\alpha}$$

holds. That is, in this case the estimation (2.2) is valid.

(b) Let $\xi \in \gamma_{\frac{3}{2}|t-t_0|}(t_0)\backslash\gamma_{\varepsilon|t-t_0|^{1+\beta/\alpha}}(t)$. Then by the case (2) we obtain

$$\big|f(\xi) - f(t)\big| \le c_{12}(f)|t - t_0|^{\alpha+\beta}. \tag{2.3}$$

Since $|\xi - t| \ge \varepsilon|t - t_0|^{1+\beta/\alpha}$ we have $|t - t_0|^{\alpha+\beta} \le \varepsilon^{-\alpha}|\xi - t|^{\alpha}$. Then, using (2.3), we find the estimation (2.2).

(c) Let $\xi \in \gamma\backslash\gamma_{\frac{3}{2}|t-t_0|}(t_0)$. In this case we have

$$|\xi - t| \ge |\xi - t_0| - |t - t_0| \ge \frac{3}{2}|t - t_0| - |t - t_0| = \frac{1}{2}|t - t_0|$$

and

$$|\xi - t| \le |\xi - t| + |t - t_0| \le 3|\xi - t|.$$

Therefore, we obtain

$$|\xi - t|^{\alpha+\beta} \le 3^{\alpha+\beta}|\xi - t|^{\alpha+\beta} \le 3^{\alpha+\beta}d^{\beta}|\xi - t|^{\alpha}.$$

Using the last inequality, according to the case (3), we obtain estimation (2.2). So, it is proved that $f \in H_{\alpha}(\gamma)$.

Necessity. Let $f \in H_{\alpha}^{\alpha+\beta}(t_0)$, $0 < \alpha \le 1$, $\beta \ge 0$, and $\xi, t \in \gamma$. We consider all possible cases given in the right-hand side of the estimation of the lemma.

1. Let $\xi \in \gamma_{\varepsilon|t-t_0|^{1+\beta/\alpha}}(t)$. Since $f \in H_{\alpha}$ we get

$$\big|f(\xi) - f(t)\big| \le c_{13}(f)|\xi - t|^{\alpha},$$

that is, we have the relation required in the lemma.

2. Let $\xi \in \gamma_{\frac{3}{2}|t-t_0|}(t_0)\backslash\gamma_{\varepsilon|t-t_0|^{1+\beta/\alpha}}(t_0)$. Since $f \in H_{\alpha}^{\alpha+\beta}(t_0)$, then in this case we obtain

$$|f(\xi) - f(t_0)| \le c_{14}(f)|\xi - t_0|^{\alpha+\beta}.$$

But, since $\xi \in \gamma_{\frac{3}{2}|t-t_0|}(t_0)$ then $|\xi - t_0| \le \dfrac{3}{2}|t - t_0|$. We get

$$|f(\xi) - f(t_0)| \le c_{15}(f)\left(\frac{3}{2}\right)^{\alpha+\beta}|t - t_0|^{\alpha+\beta}. \tag{2.4}$$

In the other hand, since $f \in H_{\alpha}^{\alpha+\beta}(t_0)$ the estimation

$$\big|f(t) - f(t_0)\big| \le c_{16}(f)|t - t_0|^{\alpha+\beta} \tag{2.5}$$

holds. Taking into account the relations (2.4) and (2.5), we have

$$\big|f(\xi) - f(t)\big| \le |f(\xi) - f(t_0)| + |f(t_0) - f(t)|$$

$$\le c_{17}(f)\left(\frac{3}{2}\right)^{\alpha+\beta}|t-t_0|^{\alpha+\beta}+c_{18}(f)|t-t_0|^{\alpha+\beta}$$
$$= c_{19}(f)|t-t_0|^{\alpha+\beta},$$

that is, we obtain the estimation in Lemma 2.3.

3. Let $\xi\in\gamma\backslash\gamma_{\frac{3}{2}|t-t_0|}(t_0)$. In this case since $f\in H_\alpha^{\alpha+\beta}(t_0)$ the inequality

$$\big|f(\xi)-f(t)\big|\le|f(\xi)-f(t_0)|+|f(t_0)-f(t)|$$
$$\le c_{20}(f)|\xi-t_0|^{\alpha+\beta}+c_{21}(f)|t-t_0|^{\alpha+\beta} \tag{2.6}$$

holds. Taking account of $|\xi-t_0|\ge\dfrac{3}{2}|t-t_0|$ or $|t-t_0|\le\dfrac{3}{2}|\xi-t_0|$, we obtain

$$|t-t_0|^{\alpha+\beta}\le\left(\frac{2}{3}\right)^{\alpha+\beta}|\xi-t_0|^{\alpha+\beta}.$$

The last inequality and (2.6) gives us

$$\big|f(\xi)-f(t)\big|\le c_{22}(f)|\xi-t_0|^{\alpha+\beta}+c_{23}(f)\left(\frac{2}{3}\right)^{\alpha+\beta}|\xi-t_0|^{\alpha+\beta}=c_{24}(f)|\xi-t_0|^{\alpha+\beta}.$$

Therefore, we obtain the last estimation in Lemma 2.3.

Lemma 2.3 is proved.

Now let us give another lemma expressing the necessary and sufficient condition for $f\in H_\alpha^{\alpha+\beta}(t_0,\gamma)$.

Lemma 2.4. *Let $\gamma\in S$ and $t_0\in\gamma$. Then*

$$f\in H_\alpha^{\alpha+\beta}(t_0)\Leftrightarrow|f(z_1)-f(z_2)|$$
$$\le\operatorname{const}\min\Big\{(\max[|z_1-t_0|,\ |z_2-t_0|])^{\alpha+\beta},\ |z_1-z_2|^\alpha\Big\}.$$

Proof. We denote by $A(z_1,z_2)$ the right-hand side of the estimate of the lemma. It follows from Lemma 2.3 that for the proof the lemma it is sufficient to show that $A(z_1,z_2)$ is equivalent to the right-hand sides of cases (1)–(3) of Lemma 2.3.

We investigate the following cases:

1. Let $t\in\gamma$ and $z_1\in\gamma_{\varepsilon|z_2-t_0|^{1+\beta/\alpha}}(t)$ $\left(\varepsilon=\dfrac{1}{2d^{\frac{\alpha+\beta}{2}}}\right)$. Then we obtain

$$|z_1-z_2|\le\varepsilon|z_2-t_0|.$$

The last inequality yields

$$|z_1-z_2|^\alpha\le\varepsilon^\alpha|z_2-t_0|^{\alpha+\beta}.$$

Taking account of $\alpha\le1$ and using the inequality

$$|z_1-t_0|\ge|z_2-t_0|-|z_1-z_2|\ge|z_2-t_0|-\varepsilon|z_2-t_0|^{1+\frac{\beta}{\alpha}}\ge\frac{1}{2}|z_2-t_0|,$$

we obtain

$$|z_1 - z_2|^\alpha \le \varepsilon^\alpha |z_2 - z_0|^{\alpha+\beta} \le 2^{\alpha+\beta}\varepsilon^\alpha |z_1 - t_0|^{\alpha+\beta}.$$

Then we have

$$\begin{aligned} A(z_1, z_2) &= \min\Big\{\max(|z_1 - t_0|^{\alpha+\beta}, |z_2 - t_0|^{\alpha+\beta}), |z_1 - z_2|^\alpha\Big\} \\ &\ge \min\Big\{\max(2^{-\alpha-\beta}\varepsilon^\alpha |z_1 - z_2|^\alpha, \varepsilon^{-\alpha}|z_1 - z_2|^\alpha), |z_1 - z_2|^\alpha\Big\} \\ &= |z_1 - z_2|^\alpha \min\Big\{\max\left(2^{-\alpha-\beta}\varepsilon^{-\alpha}\right), 1\Big\} = c(\alpha, \beta)|z_1 - z_2|^\alpha. \end{aligned}$$

In the other hand, it is clear that the inequality

$$A(z_1, z_2) \le |z_1 - z_2|^\alpha$$

holds. Eventually we obtain

$$A(z_1, z_2) \asymp |z_1 - z_2|^\alpha.$$

2. Let $z_1 \in \gamma_{\frac{3}{2}|z_2 - t_0|}(t_0) \backslash \gamma_{\varepsilon|z_2 - t_{t0}|^{1+\beta/\alpha}}(t)$. It is clear that

$$A(z_1, z_2) \le \max\Big\{|z_1 - t_0|^{\alpha+\beta},\ |z_2 - t_0|^{\alpha+\beta}\Big\}.$$

Since

$$|z_1 - t_0| \le \frac{3}{2}|z_2 - t_0|,$$

we obtain

$$A(z_1, z_2) \le \max\Big\{\left(\frac{3}{2}|z_2 - t_0|^{\alpha+\beta}\right),\ |z_2 - t_0|^{\alpha+\beta}\Big\} = c(\alpha, \beta,)|z_2 - t_0|^{\alpha+\beta}.$$

In the other hand, the inequality

$$A(z_1, z_2) \ge \min\Big\{\left(|z_2 - t_0|^{\alpha+\beta}\right),\ |z_1 - z_2|^\alpha\Big\}$$

holds. Since

$$|z_1 - z_2|^\alpha \ge \left(\varepsilon|z_2 - t_0|^{1+\beta/\alpha}\right)^\alpha = \varepsilon^\alpha |z_2 - t_0|^{\alpha+\beta},$$

we have

$$A(z_1, z_2) \ge \min\Big\{\left(|z_2 - t_0|^{\alpha+\beta}\right),\ \varepsilon^\alpha |z_2 - t_0|^{\alpha+\beta}\Big\} = c(\alpha, \beta)|z_2 - t_0|^{\alpha+\beta},$$

that is, in this case

$$A(z_1, z_2) \asymp |z_2 - t_0|^{\alpha+\beta}.$$

3. Let $z_1 \in \gamma \backslash \gamma_{\frac{3}{2}|z_2 - t_0|}(t_0)$. Then we obtain

$$\begin{aligned} A(z_1, z_2) &\leq \max\Big\{|z_1 - t_0|^{\alpha+\beta}, |z_2 - t_0|^{\alpha+\beta}\Big\} \\ &\leq \max\Big\{|z_1 - t_0|^{\alpha+\beta}, \frac{2}{3}|z_1 - t_0|^{\alpha+\beta}\Big\} \\ &= c_{26}(\alpha, \beta)|z_1 - t_0|^{\alpha+\beta}. \end{aligned}$$

In the other hand, the inequality

$$A(z_1, z_2) \geq \min\Big\{|z_1 - t_0|^{\alpha+\beta}, |z_1 - z_2|^{\alpha}\Big\} \tag{2.7}$$

holds. Also, we have

$$|z_1 - z_2| \geq |z_1 - t_0| - |z_2 - t_0| \geq |z_1 - t_0| - \frac{2}{3}|z_1 - t_0| = \frac{1}{3}|z_1 - t_0|. \tag{2.8}$$

From the relations of (2.7) and (2.8), we finally conclude that

$$\begin{aligned} A(z_1, z_2) &\geq \min\Big\{|z_1 - t_0|^{\alpha+\beta}, \frac{1}{3}|z_1 - t_0|^{\alpha}\Big\} \\ &= |z_1 - t_0|^{\alpha+\beta} \min\Big\{1, \left(\frac{1}{3}\right)^{\alpha} \frac{1}{|z_1 - t_0|^{\beta}}\Big\} \\ &\geq |z_1 - t_0|^{\alpha+\beta} \min\Big\{1, \left(\frac{1}{3}\right)^{\alpha} . d^{-\beta}\Big\} = c_{27}(\alpha, \beta)|z_1 - t_0|^{\alpha+\beta}. \end{aligned}$$

Lemma 2.4 is proved.

3. Proof of the main result. ***Proof of Theorem*** **1.1.** Let $t \in \gamma$, $t \neq t_0$, and $\varepsilon = \frac{1}{2}d^{-\frac{\alpha+\beta}{2}}$. The following expression holds:

$$\begin{aligned} \widetilde{f}(t) - \widetilde{f}(t_0) &= \frac{t - t_0}{\pi i} \int\limits_{\gamma \backslash \gamma_{\frac{3}{2}|t - t_0|}(t_0)} \frac{f(\xi) - f(t_0)}{(\xi - t)(\xi - t_0)} d\xi \\ &+ \frac{f(t_0) - f(t)}{\pi i} \int\limits_{\gamma \backslash \gamma_{\frac{3}{2}|t - t_0|}(t_0)} \frac{d\xi}{\xi - t} \\ &+ \frac{1}{\pi i} \int\limits_{\gamma_{\varepsilon|t - t_0|^{1+\beta/\alpha}}(t)} \frac{f(\xi) - f(t)}{\xi - t} d\xi \\ &+ \frac{1}{\pi i} \int\limits_{\gamma_{\frac{3}{2}|t - t_0|}(t_0) \backslash \gamma_{\varepsilon|t - t_0|^{1+\beta/\alpha}}(t)} \frac{f(\xi) - f(t)}{\xi - t} d\xi \\ &- \frac{1}{\pi i} \int\limits_{\gamma_{\frac{3}{2}|t - t_0|}(t_0)} \frac{f(\xi) - f(t_0)}{\xi - t} d\xi + (f(t) - f(t_0)) \end{aligned}$$

$$= A_1 + A_2 + \ldots + A_6. \tag{3.1}$$

We estimate the quantity

$$A_1 = \frac{t - t_0}{\pi i} \int\limits_{\gamma \setminus \gamma_{\frac{3}{2}|t-t_0|}(t_0)} \frac{|f(\xi) - f(t_0)|}{(\xi - t)(\xi - t_0)} d\xi. \tag{3.2}$$

Using the inequality $|\xi - t_0| \geq \dfrac{3}{2}|t - t_0|$ for $\xi \in \gamma \setminus \gamma_{\frac{3}{2}|t-t_0|}(t_0)$, we have

$$|\xi - t| \geq |\xi - t_0| - |t - t_0| \geq |\xi - t_0| - \frac{2}{3}|\xi - t_0| = \frac{1}{3}|\xi - t_0|. \tag{3.3}$$

In the other hand, since $f \in H_\alpha^{\alpha+\beta}(t_0)$ the inequality

$$|f(\xi) - f(t_0)| \leq c_{28}(f)|\xi - t_0|^{\alpha+\beta} \tag{3.4}$$

holds. Consideration of (3.2), (3.3) and (3.4) gives us

$$|A_1| \leq \frac{1}{\pi}|t - t_0| \int\limits_{\gamma \setminus \gamma_{\frac{3}{2}|t-t_0|}(t_0)} \frac{|f(\xi) - f(t_0)|}{|\xi - t||\xi - t_0|} |d\xi|$$

$$\leq c_{29}(f)|t - t_0| \int\limits_{\gamma \setminus \gamma_{\frac{3}{2}|t-t_0|}(t_0)} \frac{|\xi - t_0|}{1} |d\xi|.$$

If $\alpha + \beta < 2$, the last inequality and Lemma 2.1 imply that

$$|A_1| \leq c_{30}(f, \gamma)|t - t_0| \int\limits_{\frac{3}{2}|t-t_0|}^{d} y^{\alpha+\beta-2} dy$$

$$\leq c_{31}(f, \gamma)|t - t_0| \int\limits_{\frac{3}{2}|t-t_0|}^{d} (y + |t - t_0|)^{\alpha+\beta-2} dy$$

$$\leq c_{32}(f, \gamma)|t - t_0| \int\limits_{0}^{d} (y + |t - t_0|)^{\alpha+\beta-2} dy$$

$$\leq c_{33}(f, \gamma, \alpha, \beta) \begin{cases} |t - t_0|^{\alpha+\beta}, & \text{if} \quad \alpha + \beta < 1, \\ |t - t_0| \ln \dfrac{2d}{|t - t_0|}, & \text{if} \quad \alpha + \beta = 1, \\ |t - t_0|, & \text{if} \quad 1 < \alpha + \beta < 2. \end{cases}$$

Let $\alpha + \beta \geq 2$. Then we obtain

$$|A_1| \leq c_{34}(f, \gamma)|t - t_0| d^{\alpha+\beta-2} \operatorname{mes} \gamma = c_{35}(f, \gamma)|t - t_0|.$$

Thus, we finally get

$$|A_1| \leq c_{36}(f,\gamma)\begin{cases} |t-t_0|^{\alpha+\beta}, & \text{if} \quad \alpha+\beta<1, \\ |t-t_0| \ln \dfrac{2d}{|t-t_0|}, & \text{if} \quad \alpha+\beta=1, \\ |t-t_0|, & \text{if} \quad \alpha+\beta>1. \end{cases} \tag{3.5}$$

Now we estimate the quantity

$$A_2 = \frac{f(t_0)-f(t)}{\pi i} \int\limits_{\gamma\setminus\gamma_{\frac{3}{2}|t-t_0|}(t_0)} \frac{d\xi}{\xi-t}. \tag{3.6}$$

It is clear that

$$\int\limits_{\gamma\setminus\gamma_{\frac{3}{2}|t-t_0|}(t_0)} \frac{d\xi}{\xi-t} = \int\limits_{\gamma\setminus\gamma_{\frac{|t-t_0|}{2}}(t)} \frac{d\xi}{\xi-z} - \int\limits_{\gamma_{\frac{3}{2}|t-t_0|}(t_0)\setminus\gamma_{\frac{1}{2}|t-t_0|}(t)\setminus} \frac{d\xi}{\xi-z}. \tag{3.7}$$

According to Lemma 2.1, for the first integral on the right-hand side in the above equality, we obtain

$$\left| \int\limits_{\gamma\setminus\gamma_{\frac{|t-t_0|}{2}}(t)} \frac{d\xi}{\xi-z} \right| \leq 2\pi. \tag{3.8}$$

Now, we estimate the second integral on the right-hand side of equality (3.7). Since $|\xi-t| \geq \dfrac{|t-t_0|}{2}$ we have

$$\int\limits_{\gamma_{\frac{3}{2}|t-t_0|}(t_0)\setminus\gamma_{\frac{1}{2}|t-t_0|}(t)\setminus} \frac{d\xi}{\xi-z} \leq \frac{2}{|t-t_0|} \operatorname{mes} \gamma_{\frac{3}{2}|t-t_0|}(t_0) \leq c_{37}(\gamma). \tag{3.9}$$

Use of (3.7), (3.8) and (3.9) gives us

$$\left| \int\limits_{\gamma\setminus\gamma_{\frac{3}{2}|t-t_0|}(t_0)} \frac{d\xi}{\xi-t} \right| \leq c_{38}(\gamma).$$

Then, for any α and β, the last inequality and (3.6) imply that

$$|A_2| \leq c_{39}(\gamma)|f(t_0)-f(t)| \leq c_{40}(f,\gamma)|t-t_0|^{\alpha+\beta}. \tag{3.10}$$

Now we estimate the the integral

$$A_3 = \frac{1}{\pi i} \int\limits_{\gamma_{\varepsilon|t-t_0|^{1+\beta/\alpha}}(t)} \frac{f(\xi)-f(t)}{\xi-t} d\xi. \tag{3.11}$$

According to property (1) of Lemma 2.3, we obtain

$$\big|f(\xi)-f(t)\big|\le c_{41}(f)|\xi-t|. \tag{3.12}$$

Use of (3.11) and (3.12) gives us

$$|A_3|\le c_{42}(f)\int\limits_{\gamma_{\varepsilon|t-t_0|^{1+\beta/\alpha}}(t)}|\xi-t|^{\alpha-1}|d\xi|.$$

Then, for any α and β, the last inequality and Lemma 2.1 imply that

$$|A_3|\le c_{43}(f,\gamma)\int\limits_0^{\varepsilon|t-t_0|^{1+\beta/\alpha}}y^{\alpha-1}dy$$

$$=c_{44}(f,\gamma)\frac{1}{\alpha}\Big(\varepsilon|t-t_0|^{1+\beta/\alpha}\Big)^{\alpha}=c_{45}(f,\gamma)|t-t_0|^{\alpha+\beta}. \tag{3.13}$$

Now we estimate the quantity

$$A_4=\frac{1}{\pi i}\int\limits_{\gamma_{\frac{3}{2}|t-t_0|}(t_0)\setminus\gamma_{\varepsilon|t-t_0|^{1+\beta/\alpha}}(t)}\frac{f(\xi)-f(t)}{\xi-t}d\xi.$$

According to property (2) of Lemma 2.3, we have

$$|A_4|\le c_{46}(f)|t-t_0|^{\alpha+\beta}\int\limits_{\gamma_{\frac{3}{2}|t-t_0|}(t_0)\setminus\gamma_{\varepsilon|t-t_0|^{1+\beta/\alpha}}(t)}\frac{|d\xi|}{|\xi-t|}d\xi$$

$$\le c_{47}(f)|t-t_0|^{\alpha+\beta}\int\limits_{\gamma_{\frac{5}{2}|t-t_0|}(t)\setminus\gamma_{\varepsilon|t-t_0|^{1+\beta/\alpha}}(t)}\frac{|d\xi|}{|\xi-t|}d\xi.$$

Then from the last inequality and Lemma 2.1 we conclude that

$$|A_4|\le c_{48}(f)|t-t_0|^{\alpha+\beta}\int\limits_{\varepsilon|t-t_0|^{1+\beta/\alpha}}^{\frac{5}{2}|t-t_0|}\frac{dy}{y}=c_{49}(f,\gamma)|t-t_0|^{\alpha+\beta}\ln\frac{2d}{|t-t_0|} \tag{3.14}$$

for any α and β.

Now we estimate the expression

$$A_5=\frac{1}{\pi i}\int\limits_{\gamma_{\frac{3}{2}|t-t_0|}(t_0)}\frac{f(\xi)-f(t)}{\xi-t}d\xi. \tag{3.15}$$

Since $f\in H_\alpha^{\alpha+\beta}(t_0)$ the inequality

$$|f(\xi)-f(t_0)|\le c_{50}(f)|\xi-t_0|^{\alpha+\beta} \tag{3.16}$$

holds. Use of (3.15) and (3.16) gives us

$$|A_5| \le c_{51}(f) \int\limits_{\gamma_{\frac{3}{2}|t-t_0|}(t_0)} |\xi - t_0|^{\alpha+\beta} |d\xi|.$$

If $\alpha+\beta-1 \le 0$ from the last inequality and Lemma 2.1 we obtain

$$|A_5| \le c_{52}(f,\gamma) \int\limits_0^{\frac{3}{2}|t-t_0|} y^{\alpha+\beta-1} dy = c_{53}(f,\gamma)|t-t_0|^{\alpha+\beta}. \tag{3.17}$$

Let $\alpha+\beta-1 > 0$. Since $|\xi - t_0| \le \dfrac{3}{2}|t-t_0|$, this gives

$$|A_5| \le c_{54}(f)\left(\frac{3}{2}\right)^{\alpha+\beta-1} |t-t_0|^{\alpha+\beta-1} \operatorname{mes} \gamma_{\frac{3}{2}|t-t_0|}(t_0)$$
$$= c_{55}(f)|t-t_0|^{\alpha+\beta-1} \theta_{t_0}\left(\frac{3}{2}|t-t_0|\right).$$

Noticing that $\theta_t(\delta) \le \theta(\delta) \le c(\gamma)\delta$ from the last inequality we conclude that

$$|A_5| \le c_{56}(f,\gamma)|t-t_0|^{\alpha+\beta}. \tag{3.18}$$

It is clear that the inequality

$$|A_6| = \big|f(t) - f(t_0)\big| \le c_{57}(f)|t-t_0|^{\alpha+\beta} \tag{3.19}$$

holds for any α and β.

Taking into account the relations (3.1), (3.5), (3.10), (3.13), (3.14), (3.17) – (3.19), we have

$$\left|\widetilde{f}(t) - \widetilde{f}(t_0)\right| \le c_{58}(f,\gamma)\left[|t-t_0|^{\alpha+\beta} + |t-t_0|^{\alpha+\beta} \ln \frac{2d}{|t-t_0|}\right]$$
$$+ c_{58}(f,\gamma)\begin{cases} |t-t_0|^{\alpha+\beta}, & \text{if } \alpha+\beta < 1, \\ |t-t_0| \ln \dfrac{2d}{|t-t_0|}, & \text{if } \alpha+\beta = 1, \\ |t-t_0|, & \text{if } \alpha+\beta > 1. \end{cases}$$

Thus, we finally obtain that, for $\alpha+\beta \le 1$,

$$\left|\widetilde{f}(t) - \widetilde{f}(t_0)\right| \le c_{59}(f,\gamma)|t-t_0|^{\alpha+\beta} \ln \frac{2d}{|t-t_0|}$$

and, for $\alpha+\beta > 1$,

$$\left|\widetilde{f}(t) - \widetilde{f}(t_0)\right| \le c_{60}(f,\gamma)|t-t_0|.$$

Theorem 1.1 is proved.

Remark. The validity of the proposition $\widetilde{f} \in H_\alpha(\gamma)$ was proved in [40].

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