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STUDY OF ANALYTIC FUNCTION RELATED TO THE LE ROY-TYPE MITTAG-LEFFLER FUNCTION

ДОСЛІДЖЕННЯ АНАЛІТИЧНОЇ ФУНКЦІЇ, ЩО ПОВ'ЯЗАНА З ФУНКЦІЄЮ МІТТАГ-ЛЕФФЛЕРА ТИПУ ЛЕ РУА

We study some geometric properties (such as univalence, starlikeness, convexity, and close-to-convexity) of Le Roy-type Mittag-Leffler function. In order to achieve our goal, we use new two-sided inequalities for the digamma function. Some examples are also provided to illustrate the obtained results. Interesting consequences are deduced to show that these results improve several results available in the literature for the two-parameter Mittag-Leffler function.

Досліджено деякі геометричні властивості (такі як однолистість, зіркоподібність, опуклість, близькість до опуклості) функції Міттаг-Леффлера типу Ле Руа. Для досягнення поставленої мети використано нові двосторонні нерівності для дигамма-функції. Також наведено деякі приклади для ілюстрації отриманих результатів. Виведено цікаві наслідки для підтвердження того, що ці результати покращують кілька результатів, відомих з літератури для двопараметричної функції Міттаг-Леффлера.

1. Introduction. Let $\mathcal{H}$ denote the class of all analytic functions inside the unit disk $\mathcal{D} = \{z : |z| < 1\}$. Suppose that $\mathcal{A}$ is the class of all functions $f \in \mathcal{H}$ which are normalized by $f(0) = f'(0) - 1 = 0$ such that

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{for all} \quad z \in \mathcal{D}.$$

A function $f \in \mathcal{A}$ is said to be a starlike function (with respect to the origin 0) in $\mathcal{D}$, if f is univalent in $\mathcal{D}$ and $f(\mathcal{D})$ is a starlike domain with respect to 0 in $\mathbb{C}$. This class of starlike functions is denoted by $\mathcal{S}^*$. The analytic characterization of $\mathcal{S}^*$ is given [5] below:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \forall z \in \mathcal{D} \quad \Longleftrightarrow \quad f \in \mathcal{S}^*.$$

If $f(z)$ is a univalent function in $\mathcal{D}$ and $f(\mathcal{D})$ is a convex domain in $\mathbb{C}$, then $f \in \mathcal{A}$ is said to be a convex function in $\mathcal{D}$. We denote this class of convex functions by $\mathcal{K}$. This class can be analytically characterized as follows:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad \forall z \in \mathcal{D} \quad \Longleftrightarrow \quad f \in \mathcal{K}.$$

It is well-known that zf' is starlike if and only if $f \in \mathcal{A}$ is convex (see [2]). A function $f(z) \in \mathcal{A}$ is said to be close-to-convex in $\mathcal{D}$ if there exists a starlike function $g(z)$ in $\mathcal{D}$ such that $\Re\left(\frac{zf'(z)}{g(z)}\right) > 0$ for all $z \in \mathcal{D}$. The class of all close-to-convex functions is denoted by $\mathcal{C}$. It can be easily verified that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C}$. It is well-known that every close-to-convex function in $\mathcal{D}$ is also univalent in $\mathcal{D}$.

Problems for investing geometric properties including starlikeness, closed-to-convexity, convexity or univalency of family of analytic functions in the unit disk $\mathcal{D}$, involving special functions have

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always been attracted by several researchers [3, 8, 9, 11, 12, 19, 20] and to the references therein. One can see the following papers in this direction for the Mittag-Leffler function [3, 8 – 10].

The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ (also known as the Wiman function [13]):

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha,\beta,z\in\mathbb{C},\quad \Re(\alpha)>0,$$

which was introduced by Mittag-Leffler [14, 15] in 1903 for the case $\beta=1$, where $\Gamma(z)$ denote the classical Euler gamma function, which integral expression reads

$$\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}dt,\quad z>0.$$

Bansal and Prajapat [3] presented some interesting geometric properties for the normalization of the function $\mathbb{E}_{\alpha,\beta}(z)$ defined by

$$\mathbb{E}_{\alpha,\beta}(z)=\Gamma(\beta)zE_{\alpha,\beta}(z),\quad z\in\mathcal{D}.$$

Recently, in [8 – 10] geometric properties of normalized form of $E_{\alpha,\beta}(z)$ were studied, which improve some results of [3]. The above results inspire us to study the geometric properties of the normalized form of Le Roy-type Mittag-Leffler function and improve the results available in the literature. Here, and in what follows, we use $\mathcal{F}^{(\gamma)}_{\alpha,\beta}$ to denote Le Roy-type Mittag-Leffler function, defined by [16, 17]:

$$F^{(\gamma)}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{[\Gamma(\alpha k+\beta)]^{\gamma}},\quad \alpha,\beta,z\in\mathbb{C},\quad \gamma>0,\quad \Re(\alpha)>0.$$

Here, in our present investigation, we use the normalized form of Le Roy-type Mittag-Leffler function $\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)$ given as

$$\begin{aligned}\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)&=z[\Gamma(\beta)]^{\gamma}\,E^{(\gamma)}_{\alpha,\beta}(z)\\&=\sum_{k=1}^{\infty}\left[\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}\right]^{\gamma}z^k,\quad z\in\mathcal{D}.\end{aligned}$$

The main focus of the present paper is to establish sufficient conditions for the parameters of the normalized form of Le Roy-type Mittag-Leffler function to be starlike, close-to-convex and convex on the open unit disc. Interesting consequences and examples are derived to support that these results are better than the existing ones and improve several results available in the literature.

In the end of this section, each of the following definition will be used in our investigation. We recall here that a function $f(x)$ is said to be completely monotonic on an interval $I\subseteq\mathbb{R}$ if $f(x)$ has derivatives of all orders on I and

$$(-1)^k f^{(k)}(x)\geq 0,\quad k\in\mathbb{N}:=\{0,1,2,\ldots\},\quad x\in I.$$

The Bernstein – Widder theorem [18, p. 161] states that a function $f(x)$ is completely monotonic on $(0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} d\lambda(t), \tag{1.1}$$

where λ is a nonnegative measure on $[0, \infty)$ such that the integral (1.1) converges for all $x > 0$. For further details, one may consult, for example, the book by Widder [18, Chapter IV].

2. Two useful lemmas: on a new two-sided inequalities for digamma function. The main focus in this section is to present new inequalities of digamma function $\Psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$, which was the chief tool in the proof of main results.

Lemma 2.1. *If* $0 < a \leq b$, *then the function* $z \mapsto F_{a,b}(z) := \log(z+b) - \psi(z+a)$ *is completely monotonic on* $(0, \infty)$. *Furthermore, the inequalities*

$$\psi(a) - \log(b) + \log(z+b) \leq \psi(z+a) \leq \log(z+b) \tag{2.1}$$

is true for all $z > 0$ *such that* $0 < a \leq b$.

Proof. By the means of the integral representation of the digamma function [1, Chapter 6]

$$\psi(z) = \log(z) - \int_0^\infty \frac{1 - t - e^{-t}}{t(1 - e^{-t})} e^{-zt} dt, \tag{2.2}$$

we have

$$F_{a,b}(z) = \log\left(1 + \frac{b-a}{z+a}\right) + \int_0^\infty u_a(t) e^{-zt} dt,$$

where

$$u_a(t) = \frac{(-1 + t + e^{-t})e^{-at}}{t(1 - e^{-t})}.$$

By using the fact that the function

$$z \mapsto \log\left(1 + \frac{b-a}{z+a}\right)$$

is completely monotonic on $(0, \infty)$ for all $0 < a \leq b$ and the function $u_a(t)$ is nonnegative, we deduce that the function $F_{a,b}(z)$ is completely monotonic on $(0, \infty)$ if $0 < a \leq b$. In particular, the function $F_{a,b}(z)$ is decreasing on $(0, \infty)$ which readily implies

$$0 = \lim_{z \to \infty} F_{a,b}(z) \leq F_{a,b}(z) \leq F_{a,b}(0) = \log(b) - \psi(a).$$

Lemma 2.1 is proved.

Lemma 2.2. *The function* $G_{a,b}(z) := -F_{a,b}(z)$ *is completely monotonic on* $(0, \infty)$ *if and only if* $a - b \geq 1/2$. *Moreover, the inequalities*

$$\log(z+b) \leq \psi(z+a) \leq \log(z+b) + \psi(a) - \log(b) \tag{2.3}$$

are valid for all $z > 0$ *and* $a - b \geq 1/2$.

Proof. Taking into account the integral representation

$$\log(b) - \log(a) = \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt$$

and combining with (2.2), we get

$$G_{a,b}(z) = \int_0^\infty \frac{e^{-at} v_{a,b}(t)}{t(1-e^{-t})} e^{-zt} dt,$$

where $v_{a,b}(t) = e^{(a-b)t} - e^{(a-b-1)t} - t$. Hence,

$$v'_{a,b}(t) = (a-b)e^{(a-b)t} - (a-b-1)e^{(a-b-1)t} - 1,$$
$$v''_{a,b}(t) = (a-b)^2 e^{(a-b)t} w_{a,b}(t),$$

where

$$w_{a,b}(t) = 1 - \frac{(a-b-1)^2}{(a-b)^2} e^{-t}, \quad a \neq b.$$

It is clear that the function $w_{a,b}(t)$ is increasing on $(0,\infty)$. From $w_{a,b}(0) = 2a - 2b - 1 \geq 0$ if and only if $a-b \geq 1/2$, it follows that $v''_{a,b}(t) \geq 0$ and, consequently, the function $v'_{a,b}(t)$ is increasing on $(0,\infty)$. Since $v'_{a,b}(0) = 0$, then $v'_{a,b}(t) \geq 0$ if and only if $a-b \geq 1/2$. This yields that the function $v_{a,b}(t)$ is increasing on $(0,\infty)$ such that $v_{a,b}(0) = 0$ and, consequently, $v_{a,b}(t) \geq 0$ for all $t \in (0,\infty)$ if and only if $a-b \geq 1/2$. Therefore, we deduce that all prerequisites of the Bernstein characterization theorem for the complete monotone functions are fulfilled, that is, the function $G_{a,b}(z)$ is completely monotonic on $(0,\infty)$ if and only if $a-b \geq 1/2$.

Lemma 2.2 is proved.

3. Starlikeness of Le Roy-type Mittag-Leffler function.

Theorem 3.1. *Let the parameters range* $\alpha, \beta, \gamma, \nu, \mu > 0$ *such that* $\alpha\gamma \geq 1$ *and* $\big[\Gamma(\beta)\big]^\gamma (e-1) < \big[\Gamma(\alpha+\beta)\big]^\gamma$. *Also, we suppose that one of the following hypothesis holds true*:

$$(H_1)\colon \begin{cases} \text{(i)} \quad \beta < \nu \leq \alpha^2\mu\gamma, \ \mu \geq 1, \\ \text{(ii)} \quad \alpha\gamma(\log(\alpha+\nu) + \psi(\beta) - \log(\mu)) - \log(1+\mu) - 1 > 0, \end{cases}$$

$$(H_1^1)\colon \begin{cases} \text{(i)} \quad \mu \leq \dfrac{1}{2}, \ \nu \leq \min\left(\beta - \dfrac{1}{2}, \gamma\mu\alpha^2\right), \\ \text{(ii)} \quad \alpha\gamma\log(\alpha+\nu) - \log(1+\mu) + \log(\mu) - \psi(1) - 1 > 0. \end{cases}$$

Then the function $\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)$ *is starlike in* $\mathcal{D}$.

Proof. To prove that the function $\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}$ we just show that

$$\Re\left(\left[z\left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' \Big/ \left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)\right]\right) > 0 \quad \text{for all} \quad z \in \mathcal{D}.$$

For this, it is enough to show that

$$\left|\left[z\left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'\Big/\left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)\right]-1\right|<1$$

for $z\in\mathcal{D}$. After calculation we derive

$$\left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'-\frac{\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)}{z}=\sum_{k=1}^{\infty}\frac{k[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta)]^{\gamma}}z^{k}=\sum_{k=1}^{\infty}\frac{A_k(\alpha,\beta,\gamma)z^{k}}{k!}, \tag{3.1}$$

where $(A_k)_{k\geq1}$ is defined by

$$A_k:=A_k(\alpha,\beta,\gamma)=\frac{k\Gamma(k+1)[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta)]^{\gamma}},\quad k\geq1. \tag{3.2}$$

We define the function $\Phi^{(\gamma)}_{\alpha,\beta}(t)$ by

$$\Phi^{(\gamma)}_{\alpha,\beta}(t)=\frac{t\Gamma(t+1)}{[\Gamma(\alpha t+\beta)]^{\gamma}},\quad t\geq1.$$

Then we have

$$\left(\Phi^{(\gamma)}_{\alpha,\beta}(t)\right)'=\Phi^{(\gamma)}_{\alpha,\beta}(t)\Theta^{(\gamma)}_{\alpha,\beta}(t),$$

where

$$\Theta^{(\gamma)}_{\alpha,\beta}(t)=\frac{1}{t}+\psi(t+1)-\alpha\gamma\psi(\alpha t+\beta),\quad t\geq1.$$

Moreover, by using reverse triangle inequality we obtain

$$\left|\frac{\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)}{z}\right|>1-\sum_{k=1}^{\infty}\frac{B_k(\alpha,\beta,\gamma)}{k!}, \tag{3.3}$$

where $(B_k)_{k\geq1}$ is defined by

$$B_k:=B_k(\alpha,\beta,\gamma)=\frac{\Gamma(k+1)[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta)]^{\gamma}},\quad k\geq1. \tag{3.4}$$

We consider two cases.

Case 1. Assume that the conditions (H_1) are true. From Lemma 2.1, we have

$$\psi(t+1)\leq\log(t+\mu),\quad t>0,\quad \mu\geq1,$$

and

$$\psi(\alpha t+\beta)\geq\psi(\beta)-\log(\nu)+\log(\alpha t+\nu),\quad t>0,\quad \nu\geq\beta.$$

This leads to

$$\Theta^{(\gamma)}_{\alpha,\beta}(t)\leq\tilde{\Theta}^{(\gamma)}_{\alpha,\beta}(t)=\frac{1}{t}+\log(t+\mu)-\alpha\gamma\big[\psi(\beta)-\log(\nu)+\log(\alpha t+\nu)\big],\quad t\geq1.$$

Under the given conditions of (H_1) we deduce that the function $\tilde{\Theta}^{(\gamma)}_{\alpha,\beta}(t)$ is decreasing on $[1,\infty)$ such that $\tilde{\Theta}^{(\gamma)}_{\alpha,\beta}(1)<0$ which implies that $\tilde{\Theta}^{(\gamma)}_{\alpha,\beta}(t)<0$. This in turn implies that the function

$\Theta^{(\gamma)}_{\alpha,\beta}(t) < 0$ for all $t \geq 1$. Hence, $\Phi^{(\gamma)}_{\alpha,\beta}(t)$ is decreasing on $[1, \infty)$. In particular, the sequence $(A_k)_{k\geq 1}$ monotonically decreases. In view of (3.1) we get

$$\left| \left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - \frac{\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)}{z} \right| < \sum_{k=1}^{\infty} \frac{A_1(\alpha, \beta, \gamma)}{k!} = \frac{[\Gamma(\beta)]^\gamma (e-1)}{[\Gamma(\alpha+\beta)]^\gamma}. \tag{3.5}$$

Since the function $\Phi^{(\gamma)}_{\alpha,\beta}(t)$ is decreasing, then the function $\left(\Phi^{(\gamma)}_{\alpha,\beta}(t)\right)/t$ is also decreasing. This implies that the sequence $(B_k)_{k\geq 1}$ is decreasing. Hence, by means of (3.3) we have

$$\left| \frac{\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)}{z} \right| > 1 - \sum_{k=1}^{\infty} \frac{B_1(\alpha, \beta, \gamma)}{k!} = 1 - \frac{[\Gamma(\beta)]^\gamma (e-1)}{[\Gamma(\beta+\alpha)]^\gamma}. \tag{3.6}$$

Bearing in mind (3.5) and (3.6), we obtain

$$\left| \left[z\left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' \Big/ \left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right) \right] - 1 \right| < \frac{[\Gamma(\beta)]^\gamma (e-1)}{[\Gamma(\alpha+\beta)]^\gamma - [\Gamma(\beta)]^\gamma (e-1)}.$$

So, the function $\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}$ if the inequality

$$\frac{[\Gamma(\beta)]^\gamma (e-1)}{[\Gamma(\alpha+\beta)]^\gamma - [\Gamma(\beta)]^\gamma (e-1)} \leq 1$$

holds true.

Case 2. Assume that the conditions (H_1^1) hold. Then, by Lemma 2.2, we have

$$\psi(t+1) \leq \log(t+\mu) - \log(\mu) + \psi(1), \quad 0 < \mu \leq \frac{1}{2}, \tag{3.7}$$

and

$$\psi(\alpha t + \beta) \geq \log(\alpha t + \nu), \quad 0 < \nu \leq \beta - \frac{1}{2}. \tag{3.8}$$

Having in mind the above inequalities, we obtain

$$\theta^{(\gamma)}_{\alpha,\beta}(t) \leq \breve{\theta}^{(\gamma)}_{\alpha,\beta}(t) = \frac{1}{t} + \log(t+\mu) - \log(\mu) + \psi(1) - \alpha\gamma \log(\alpha t + \nu),$$

$$t \geq 1, \quad 0 < \mu \leq \frac{3}{2}, \quad 0 < \nu \leq \beta - \frac{1}{2}.$$

Obviously, we deduce that the function $\breve{\theta}^{(\gamma)}_{\alpha,\beta}(t)$ is decreasing on $[1, \infty)$ such that $\breve{\theta}^{(\gamma)}_{\alpha,\beta}(1) < 0$. This implies $\theta^{(\gamma)}_{\alpha,\beta}(t) < 0$. Consequently, the function $\Phi^{(\gamma)}_{\alpha,\beta}(t)$ is decreasing on $[1, \infty)$. Then the sequences $(A_k)_{k\geq 1}$ and $(B_k)_{k\geq 1}$ are decreasing. So, the inequalities (3.1) and (3.6) hold true in this case. The rest is obvious.

Theorem 3.1 is proved.

Letting $\alpha = \mu = 1$ and $\gamma = \nu = 2$ in the conditions (H_1) of Theorem 3.1, we get the following result.

***Corollary* 3.1.** *If* $\beta > \sqrt{1-e} \approx 1.310832$, *then the function* $\mathcal{F}^{(2)}_{1,\beta}(z)$ *is starlike in* $\mathcal{D}$.

***Example* 3.1.** The function $\mathcal{F}^{(2)}_{1,\frac{3}{2}}(z)$ is starlike in $\mathcal{D}$ (see Fig. 1).

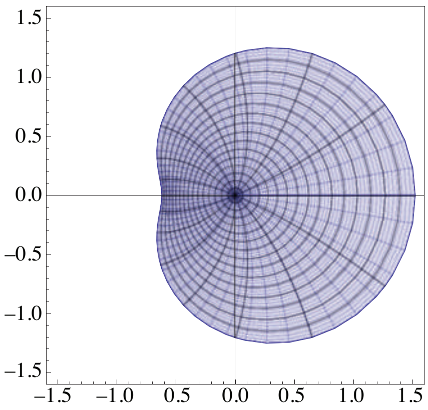

Fig. 1. Mapping of $\mathcal{F}^{(2)}_{1,\frac{3}{2}}(z)$ over $\mathcal{D}$.

Specifying $\gamma = 1$ in Theorem 3.1, we get the following result.

***Corollary* 3.2.** *Let the parameters range $\alpha, \beta, \nu, \mu > 0$ such that $\alpha \geq 1$ and $(e-1)\Gamma(\beta) < \Gamma(\alpha+\beta)$. Also, we suppose that one of the following hypothesis holds true*:

$$(\tilde{H}_1)\colon \begin{cases} \text{(i)} \quad \beta < \nu \leq \alpha^2\mu,\ \mu \geq 1, \\ \text{(ii)} \quad \alpha(\log(\alpha+\nu) + \psi(\beta) - \log(\mu)) - \log(1+\mu) - 1 > 0, \end{cases}$$

$$(\tilde{H}^1_1)\colon \begin{cases} \text{(i)} \quad \mu \leq \dfrac{1}{2},\ \ \nu \leq \min\left(\beta - \dfrac{1}{2}, \mu\alpha^2\right), \\ \text{(ii)} \quad \alpha\log(\alpha+\nu) - \log(1+\mu) + \log(\mu) - \psi(1) - 1 > 0. \end{cases}$$

Then the function $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}$.

***Corollary* 3.3.** *If $\beta \geq (-1+\sqrt{4e-3})/2 \approx 0.9029$, then the function $\mathbb{E}_{2,\beta}(z)$ is starlike in $\mathcal{D}$.*

Proof. Taking $\alpha = 2,\ \mu = \dfrac{1}{2}$ and $\nu = \dfrac{-2+\sqrt{4e-3}}{2}$ in the conditions $(\tilde{H}^1_1)$, we have

$$\alpha\log(\alpha+\nu) - \log(1+\mu) + \log(\mu) - \psi(1) - 1 \approx 1.23 > 0.$$

In addition, the condition $\Gamma(\beta)(e-1) < \Gamma(\alpha+\beta)$ holds true if $\beta \geq \big(-1+\sqrt{4e-3}\big)/2$.

Let us illustrate Corollary 3.3 by an example.

***Example* 3.2.** The function $\mathbb{E}_{2,1}(z)$ is starlike in $\mathcal{D}$ (see Fig. 2).

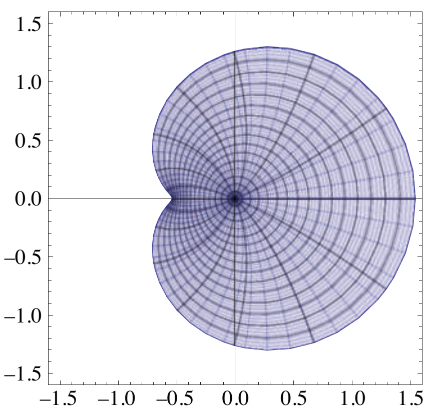

Fig. 2. Mapping of $\mathbb{E}_{2,1}(z)$ over $\mathcal{D}$.

***Corollary* 3.4.** *If $\beta \geq 0.87$, then the function $\mathbb{E}_{3,\beta}(z)$ is starlike in $\mathcal{D}$.*

Proof. Indeed, the constants $\alpha = 3$, $\nu = \mu = 1$ and $\beta \geq 0.87$ satisfy the conditions $(\tilde{H}_1^1)$. In addition, $(e-1)\Gamma(\beta) < \Gamma(3+\beta)$ holds true if $\beta > 0.48$.

***Remark* 3.1.** Recently, in [3], the authors proved that $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}$ if $\alpha \geq 1$ and $\beta \geq (3+\sqrt{17})/2 \approx 3.56155281$. Moreover, Noreen et al. [9] indicates that the function $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}$ if $\alpha \geq 1$ and $\beta \geq 3.214319744$. By using the same technique as in Corollaries 3.3 and 3.4, it can be verified that for each positive integer $\alpha = n \geq 2$, there exists $\beta \in (0,1)$ such that $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}$. Hence, Corollary 3.2 improve the results for $\mathbb{E}_{\alpha,\beta}(z)$ available in [3, 9], for each positive integer $\alpha \geq 2$.

Theorem 3.2. *Let the parameters range $\alpha, \beta, \gamma, \nu, \mu > 0$ such that $\alpha\gamma \geq 1$ and $(e-1)[\Gamma(\beta)]^\gamma < [\Gamma(\alpha+\beta)]^\gamma$. In addition, assume that any one of the following hypothesis holds true*:

$$(H_2)\colon \begin{cases} \text{(i)} & \mu \geq 1,\ \beta \leq \nu \leq \alpha^2\gamma\mu, \\ \text{(ii)} & \alpha\gamma(\log(\alpha+\nu) + \psi(\beta) - \log(\nu)) - \log(1+\mu) > 0, \end{cases}$$

$$(H_2^1)\colon \begin{cases} \text{(i)} & \mu \leq \dfrac{1}{2},\ \nu \leq \min\left(\beta - \dfrac{1}{2}, \alpha^2\gamma\mu\right), \\ \text{(ii)} & \alpha\gamma\log(\alpha+\nu) + \log(\mu) - \log(1+\mu) - \psi(1) > 0. \end{cases}$$

Then the function $\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)$ is univalent and starlike in $\mathcal{D}_{1/2}$.

Proof. According to MacGregor [7], if $f \in \mathcal{A}$ and satisfy $|(f(z)/z) - 1| < 1$ for each $z \in \mathcal{D}$, then f is univalent and starlike in $\mathcal{D}_{\frac{1}{2}}$. A simple computation shows that

$$|(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)/z) - 1| < \sum_{k=1}^{\infty} \frac{B_k(\alpha,\beta,\gamma)}{k!} \qquad \text{for all} \quad z \in \mathcal{D}, \tag{3.9}$$

where $(B_k)_{k\geq 1}$ is defined in (3.4). We define the function $\phi^{(\gamma)}_{\alpha,\beta} : [1,\infty) \to \mathbb{R}$ by

$$\phi^{(\gamma)}_{\alpha,\beta}(t) = \frac{\Gamma(t+1)}{[\Gamma(\alpha t+\beta)]^\gamma}.$$

Differentiating the function $\phi^{(\gamma)}_{\alpha,\beta}(t)$, we obtain

$$\big(\phi^{(\gamma)}_{\alpha,\beta}(t)\big)' = \phi^{(\gamma)}_{\alpha,\beta}(t)\big[\psi(t+1) - \alpha\gamma\psi(\alpha t+\beta)\big] =: \phi^{(\gamma)}_{\alpha,\beta}(t)\tilde{\phi}^{(\gamma)}_{\alpha,\beta}(t).$$

We consider two cases.

Case 1. Assume that the conditions (H_2) are valid. By (2.1) we get

$$\psi(t+1) \leq \log(t+\mu), \quad t > 0, \quad \mu \geq 1,$$

and

$$\psi(\beta) - \log(\nu) + \log(\alpha t+\nu) \leq \psi(\alpha t+\beta), \quad t > 0, \quad \nu \geq \beta.$$

It follows that

$$\tilde{\phi}^{(\gamma)}_{\alpha,\beta}(t) \leq \breve{\phi}^{(\gamma)}_{\alpha,\beta}(t) := \log(t+\mu) - \alpha\gamma(\psi(\beta) - \log(\nu) + \log(\alpha t+\nu)), \quad t \geq 1.$$

Differentiating the function $\breve{\phi}^{(\gamma)}_{\alpha,\beta}(t)$, we have

$$\left(\breve{\phi}_{\alpha,\beta}^{(\gamma)}(t)\right)' = \frac{\alpha(1-\alpha\gamma)t + \nu - \alpha^2\gamma\mu}{(t+\mu)(\alpha t+\nu)}.$$

This implies $\left(\breve{\phi}_{\alpha,\beta}^{(\gamma)}(t)\right)' < 0$. Thus the function $\breve{\phi}_{\alpha,\beta}^{(\gamma)}(t)$ is decreasing on $[1,\infty)$. Since $\breve{\phi}_{\alpha,\beta}^{(\gamma)}(1) < 0$, we obtain that the function $\breve{\phi}_{\alpha,\beta}^{(\gamma)}(t)$ is negative for all $t \geq 1$, and, consequently, the function $\phi_{\alpha,\beta}^{(\gamma)}(t)$ is decreasing on $[1,\infty)$. Thus in turn implies that the sequence $(B_k)_{k\geq 1}$ is decreasing. Therefore, by means of (3.9) we get

$$\left|\left(\mathcal{F}_{\alpha,\beta}^{(\gamma)}(z)/z\right) - 1\right| < B_1(e-1) \quad \text{for all} \quad z \in \mathcal{D}. \tag{3.10}$$

Under the given hypothesis we conclude that $B_1(e-1) < 1$, which show that the function $\mathcal{F}_{\alpha,\beta}^{(\gamma)}(z)$ is starlike in $\mathcal{D}_{1/2}$.

Case 2. We suppose that the conditions (H_2^1) holds true. In view of the inequalities (3.7) and (3.8), we obtain

$$\tilde{\phi}_{\alpha,\beta}^{(\gamma)}(t) \leq \breve{\phi}_{\alpha,\beta}^{(\gamma)}(t) := \log(t+\mu) - \alpha\gamma\log(\alpha t+\nu) - \log(\mu) + \psi(1),$$

$$0 < \mu \leq \frac{1}{2}, \quad 0 < \nu \leq \beta - \frac{1}{2}.$$

It is easy to verify that the function $\breve{\phi}_{\alpha,\beta}^{(\gamma)}(t)$ is decreasing on $[1,\infty)$ under the conditions (H_2^1). Since $\breve{\phi}_{\alpha,\beta}^{(\gamma)}(1) < 0$ for all $t \geq 1$, we deduce that the function $\phi_{\alpha,\beta}^{(\gamma)}(t)$ is decreasing on $[1,\infty)$ and, consequently, $(B_k)_{k\geq 1}$ is also decreasing. So, the inequality (3.10) is true in the second case.

Theorem 3.2 is proved.

Specifying $\alpha = 1$, $\nu = \mu = \dfrac{1}{2}$ and $\gamma = 2$ in the second assumptions of Theorem 3.2, we have the following result.

***Corollary* 3.5.** *If $\beta > \sqrt{e-1}$, then the function $\mathcal{F}_{1,\beta}^{(2)}(z)$ is univalent and starlike in $\mathcal{D}_{1/2}$.*

***Example* 3.3.** The function $\mathcal{F}_{1,\frac{4}{3}}^{(2)}(z)$ is univalent and starlike in $\mathcal{D}_{1/2}$ (see Fig. 3).

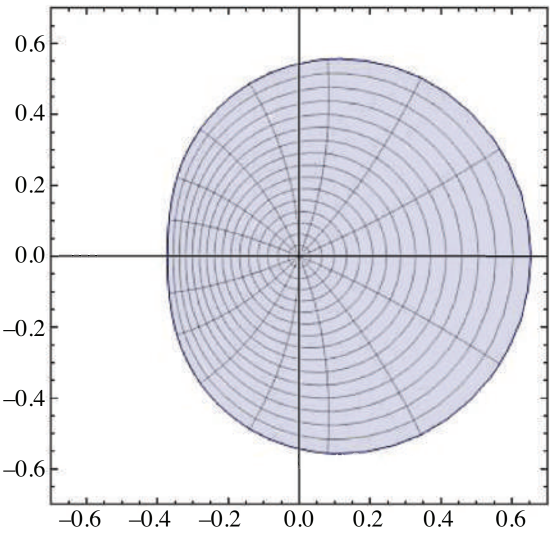

Fig. 3. Mapping of $\mathcal{F}_{1,\frac{4}{3}}^{(2)}(z)$ over $\mathcal{D}$.

Letting $\gamma = 1$ in Theorem 3.2, we get the following results.

***Corollary* 3.6.** *Let the parameters range $\alpha, \beta, \nu, \mu > 0$ such that $\alpha \geq 1$ and $(e-1)\Gamma(\beta) < \Gamma(\alpha+\beta)$. Also, assume that any one of the following hypothesis holds true*:

$$(\tilde{H}_2)\colon \begin{cases} \text{(i)} & \mu \geq 1,\, \beta \leq \nu \leq \alpha^2\mu, \\ \text{(ii)} & \alpha(\log(\alpha+\nu) + \psi(\beta) - \log(\nu)) - \log(1+\mu) > 0, \end{cases}$$

$$(\tilde{H}_2^1)\colon \begin{cases} \text{(i)} & \mu \leq \dfrac{1}{2},\, \nu \leq \min\left(\beta - \dfrac{1}{2}, \alpha^2\mu\right), \\ \text{(ii)} & \alpha\log(\alpha+\nu) + \log(\mu) - \log(1+\mu) - \psi(1) > 0. \end{cases}$$

Then the function $\mathbb{E}_{\alpha,\beta}(z)$ is univalent and starlike in $\mathcal{D}_{1/2}$.

Let us illustrate Theorem 3.2 and Corollary 3.6 by the following consequences and examples.

***Corollary* 3.7.** *If $\beta > \big(-1+\sqrt{4e-3}\big)/2 \approx 0.9029\ldots$, then the function $\mathbb{E}_{2,\beta}(z)$ is univalent and starlike in $\mathcal{D}_{1/2}$.*

Proof. Specifying $\alpha = 2$, $\mu = \dfrac{1}{2}$ and $\nu = \big(-2+\sqrt{4e-3}\big)/2$ in the conditions $(\tilde{H}_2^1)$ of Corollary 3.6, we derive the desired result.

***Example* 3.4.** The function $\mathbb{E}_{2,\frac{10}{11}}(z)$ is univalent and starlike in $\mathcal{D}_{1/2}$ (see Fig. 4).

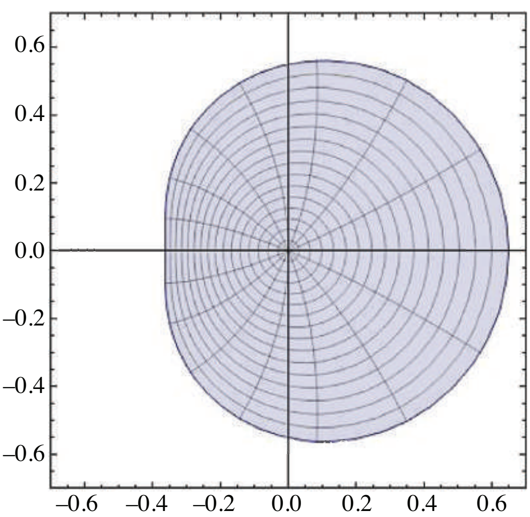

Fig. 4. Mapping of $\mathbb{E}_{2,\frac{10}{11}}(z)$ over $\mathcal{D}$.

***Corollary* 3.8.** *The function $\mathbb{E}_{3,\beta}(z)$ is univalent and starlike in $\mathcal{D}_{1/2}$, if $\beta \geq \dfrac{2}{3}$.*

Proof. The constants $\alpha = 3$, $\mu = \dfrac{1}{2}$, $\nu = \dfrac{1}{6}$ and $\beta = \dfrac{2}{3}$ satisfy the conditions $(\tilde{H}_2^1)$ of Corollary 3.6. In addition, the inequality $(e-1)\Gamma(\beta) < \Gamma(\beta+3)$ is true if $\beta \geq \dfrac{2}{3}$.

***Example* 3.5.** The function $\mathbb{E}_{3,\frac{2}{3}}(z)$ is univalent and starlike in $\mathcal{D}_{1/2}$ (see Fig. 5).

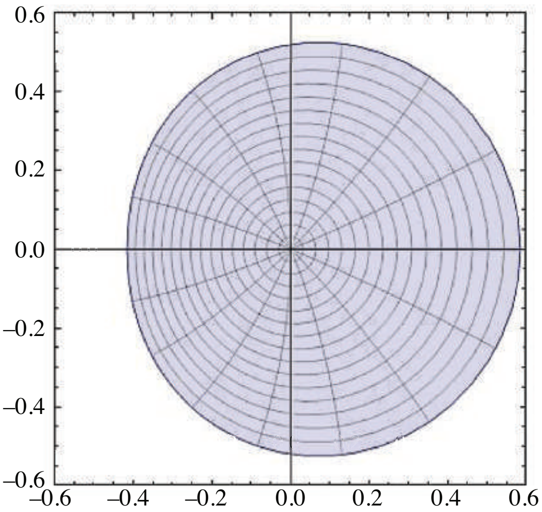

Fig. 5. Mapping of $\mathbb{E}_{3,\frac{2}{3}}(z)$ over $\mathcal{D}$.

***Remark* 3.2.** It is important to note that Corollary 3.6 discusses the cases when $0 < \beta \leq 1$. Upon setting $\nu = \mu = 1$ in the hypothesis $(\tilde{H}_2)$ of Corollary 3.6, we deduce that the function $\mathbb{E}_{\alpha,\beta}(z)$ is univalent and starlike in $\mathcal{D}_{1/2}$, if $\beta \leq 1$ and $\alpha \geq 1$ satisfies the conditions $(e-1)\Gamma(\beta) < \Gamma(\alpha+\beta)$ and $\alpha(\log(\alpha+1) + \psi(\beta)) - \log(2) > 0$. For example, if we set $\mu = \nu = 1$ and $\alpha = 2$ in the hypothesis $(\tilde{H}_2)$ of Corollary 3.6, we derive that the function $\mathbb{E}_{2,\beta}(z)$ is univalent and starlike in $\mathcal{D}_{1/2}$ if $\left(-1 + \sqrt{4e-3}\right)/2 < \beta \leq 1$.

***Remark* 3.3.** It can be noted from [3, Theorem 2.4] that the function $\mathbb{E}_{\alpha,\beta}(z)$ is univalent and starlike in $\mathcal{D}_{1/2}$ if $\beta \geq \left(1 + \sqrt{5}\right)/2 \approx 1.6180339$ and $\alpha \geq 1$. Hence, the above result improves the result for $\mathbb{E}_{\alpha,\beta}(z)$ with $\alpha = 2, 3$, proved in [3].

4. Close-to-convexity of Le Roy-type Mittag-Leffler function with respect to certain starlike functions.

Theorem 4.1. *Let* α, β, γ, μ, ν *be positive real numbers and* $x^* \approx 1.461632144\ldots$ *is the abscissa of the minimum of the Gamma function. Assume that any one of the following hypothesis* (H_3), (H_3^1) *or* (H_3^2) *holds true*:

$$(H_3)\colon \begin{cases} \text{(i)} & \beta - \alpha \leq \mu, \\ \text{(ii)} & \alpha\gamma(\psi(\beta-\alpha) - \log(\mu) + \log(\alpha+\mu)) - 1 > 0, \end{cases}$$

$$(H_3^1)\colon \begin{cases} \text{(i)} & \beta - \alpha - \nu \geq \dfrac{1}{2},\ \alpha + \nu > 1, \\ \text{(ii)} & \alpha\gamma\log(\alpha+\nu) - 1 > 0, \end{cases}$$

$$(H_3^2)\colon \begin{cases} \text{(i)} & \beta > x^*, \\ \text{(ii)} & \alpha\gamma\psi(\beta) - 1 > 0. \end{cases}$$

Then the function $\mathcal{F}_{\alpha,\beta}^{(\gamma)}(z)$ *is close-to-convex with respect to starlike function* $-\log(1-z)$ *in* $\mathcal{D}$ *and, consequently, is univalent in* $\mathcal{D}$.

Proof. To prove that $\mathcal{F}_{\alpha,\beta}^{(\gamma)}(z)$ is close-to-convex with respect to starlike function $-\log(1-z)$ in $\mathcal{D}$, it is sufficient to prove, that the sequence $\{ka_k\}_{k\geq 1}$ is decreasing (cf. [4, Corollary 7]), where $(a_k)_{k\geq 1}$ is defined by

$$a_k = \frac{[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha k + \beta - \alpha)]^\gamma}, \quad k \geq 1.$$

We consider the function $\Omega^{(\gamma)}_{\alpha,\beta}(t)$ defined by

$$\Omega^{(\gamma)}_{\alpha,\beta}(t) = \frac{t}{[\Gamma(\alpha t + \beta - \alpha)]^\gamma}, \quad t > 0.$$

Therefore, we get

$$(\Omega^{(\gamma)}_{\alpha,\beta}(t))' = \frac{\Omega^{(\gamma)}_{\alpha,\beta}(t)[1 - \alpha\gamma t\psi(\alpha t + \beta - \alpha)]}{t}. \tag{4.1}$$

We consider three cases.

Case 1. Assume that the hypothesis (H_3) holds true. By (2.1) we have

$$\psi(\alpha t + \beta - \alpha) \geq \psi(\beta - \alpha) - \log(\mu) + \log(\alpha t + \mu), \quad t > 0, \quad 0 < \beta - \alpha \leq \mu. \tag{4.2}$$

In view of (4.1) and (4.2) we get

$$(\Omega^{(\gamma)}_{\alpha,\beta}(t))' \leq \frac{\Omega^{(\gamma)}_{\alpha,\beta}(t)[1 - \alpha\gamma t[\psi(\beta - \alpha) - \log(\mu) + \log(\alpha t + \mu)]]}{t}$$
$$=: \frac{\Omega^{(\gamma)}_{\alpha,\beta}(t)\tilde{\Omega}^{(\gamma)}_{\alpha,\beta}(t)}{t}, \quad t > 0, \quad 0 < \beta - \alpha \leq \mu.$$

Then we have

$$(\tilde{\Omega}^{(\gamma)}_{\alpha,\beta}(t))' = -\alpha\gamma[\psi(\beta - \alpha) - \log(\mu) + \log(\alpha t + \mu)] - \frac{\alpha^2\gamma t}{\alpha t + \mu}$$
$$=: -\alpha\gamma\breve{\Omega}^{(\gamma)}_{\alpha,\beta}(t) - \frac{\alpha^2\gamma t}{\alpha t + \beta - \alpha}.$$

We observe that the function $\breve{\Omega}^{(\gamma)}_{\alpha,\beta}(t)$ is increasing on $[1, \infty)$. On the other hand,

$$\breve{\Omega}^{(\gamma)}_{\alpha,\beta}(t) = \psi(\beta - \alpha) - \log(\mu) + \log(\alpha + \mu) > 0.$$

Therefore, $\breve{\Omega}^{(\gamma)}_{\alpha,\beta}(t) > 0$ for all $t \geq 1$ under the given hypothesis and, consequently, the function $\tilde{\Omega}^{(\gamma)}_{\alpha,\beta}(t)$ is decreasing on $[1, \infty)$. Moreover, $\tilde{\Omega}^{(\gamma)}_{\alpha,\beta}(t)(1) < 0$. Hence, $\tilde{\Omega}^{(\gamma)}_{\alpha,\beta}(t) < 0$ for all $t \geq 1$ under the given conditions. Consequently, the function $\Omega^{(\gamma)}_{\alpha,\beta}(t)$ is decreasing on $[1, \infty)$ under the hypothesis (H_3). Therefore, we conclude that the sequence $(ka_k)_{k\geq 1}$ is decreasing. So, the stated result asserted by Theorem 4.1 is true under the conditions (H_3).

Case 2. Assume that the conditions of hypothesis (H^1_3) hold. Bearing in mind the left-hand side of inequalities (2.3) and (4.1), we get

$$(\Omega^{(\gamma)}_{\alpha,\beta}(t))' \leq \Omega^{(\gamma)}_{\alpha,\beta}(t)[1 - \alpha\gamma t \log(\alpha t + \nu)]$$
$$=: \Omega^{(\gamma)}_{\alpha,\beta}(t)\Xi^{(\gamma)}_{\alpha,\beta}(t), \qquad t > 0, \quad \beta - \alpha - \nu \geq \frac{1}{2}.$$

Clearly, we get

$$(\Xi_{\alpha,\beta}^{(\gamma)}(t))' = -\frac{\alpha\gamma[\alpha t + (\alpha t + \nu)\log(\alpha t + \nu)]}{\alpha t + \nu} \leq 0$$

for all $t \geq 1$ and $\alpha + \nu > 1$. Hence, the function $\Xi_{\alpha,\beta}^{(\gamma)}(t)$ is decreasing on $[1, \infty)$ and $\Xi_{\alpha,\beta}^{(\gamma,\tau)}(1) = 1 - \alpha\gamma\log(\alpha + \nu) < 0$ under the given conditions of (H_3^1) and, consequently, the function $\Omega_{\alpha,\beta}^{(\gamma)}(t)$ is decreasing on $[1, \infty)$ under the hypothesis (H_3^1). It follows that the sequence $\{ka_k\}_{k\geq 1}$ is decreasing. So, the function $\mathcal{F}_{\alpha,\beta}^{(\gamma)}(z)$ is close-to-convex with respect to starlike function $-\log(1-z)$ in $\mathcal{D}$ under the second hypothesis of Theorem 4.1.

Case 3. Suppose that the conditions (H_3^2) hold. We define the function $\varphi_{\alpha,\beta}^{(\gamma)}(t)$ by

$$\varphi_{\alpha,\beta}^{(\gamma)}(t) = 1 - \alpha\gamma t\psi(\alpha t + \beta - \alpha), \quad t \geq 1.$$

Since the digamma function $\psi(t)$ is increasing on $(0, \infty)$, then the function $t\psi(\alpha t + \beta - \alpha)$ is increasing on $[1, \infty)$ if $\beta > x^*$. This implies that the function $\varphi_{\alpha,\beta}^{(\gamma)}(t)$ is decreasing on $[1, \infty)$. Since $\varphi_{\alpha,\beta}^{(\gamma)}(1) < 0$ we obtain $(\Omega_{\alpha,\beta}^{(\gamma)}(t))' < 0$. This implies that the sequence $(ka_k)_{k\geq 1}$ is decreasing.

Theorem 4.1 is proved.

Setting $\alpha = 1$ and $\gamma = 2$ in the hypothesis (H_3^2) of Theorem 4.1, we obtain the following result.

***Corollary* 4.1.** *If $\beta > x^*$ such that $\psi(\beta) > \dfrac{1}{2}$, then the function $\mathcal{F}_{1,\beta}^{(2)}(z)$ is close-to-convex with respect to starlike function $-\log(1-z)$ in $\mathcal{D}$.*

***Example* 4.1.** The function $\mathcal{F}_{1,\frac{7}{3}}^{(2)}(z)$ is close-to-convex with respect to starlike function $-\log(1-z)$ in $\mathcal{D}$.

Now, putting $\gamma = 1$ in Theorem 4.1, we compute the following result.

***Corollary* 4.2.** *Let α, β, γ, μ, ν be positive real numbers and $x^* \approx 1.461632144\ldots$ is the abscissa of the minimum of the Gamma function. Assume that any one of the following hypothesis (H_3), (H_3^1) or (H_3^2) holds true*:

$$(\tilde{H}_3)\colon \begin{cases} \text{(i)} & 0 < \beta - \alpha \leq \mu, \\ \text{(ii)} & \alpha(\psi(\beta - \alpha) - \log(\mu) + \log(\alpha + \mu)) - 1 > 0, \end{cases}$$

$$(\tilde{H}_3^1)\colon \begin{cases} \text{(i)} & \beta - \alpha - \nu \geq \dfrac{1}{2}, \quad \alpha + \nu > 1, \\ \text{(ii)} & \alpha\log(\alpha + \nu) - 1 > 0, \end{cases}$$

$$(\tilde{H}_3^2)\colon \begin{cases} \text{(i)} & \beta > x^*, \\ \text{(ii)} & \alpha\psi(\beta) - 1 > 0. \end{cases}$$

Then the function $\mathbb{E}_{\alpha,\beta}(z)$ is close-to-convex with respect to starlike function $-\log(1-z)$ in $\mathcal{D}$.

***Example* 4.2.** Let $\alpha = \dfrac{3}{4}$, $\beta = 5$ and $\alpha = \dfrac{2}{3}$, $\beta = 6$ in $\left(\tilde{H}_3^2\right)$, respectively. We deduce that the functions $\mathbb{E}_{\frac{3}{4},5}(z)$ and $\mathbb{E}_{\frac{2}{3},6}(z)$ are close-to-convex with respect to starlike function $-\log(1-z)$ in $\mathcal{D}$.

***Example* 4.3.** We set $\alpha = \dfrac{1}{2}$ in the conditions $\left(\tilde{H}_3^1\right)$ of Corollary 4.2. Then we obtain that the function $\mathbb{E}_{\frac{1}{2},\beta}(z)$ is close-to-convex with respect to starlike function $-\log(1-z)$ in $\mathcal{D}$, if $\beta \geq 8$.

***Remark* 4.1.** By Examples 4.2 and 4.3, we note that Corollary 4.2 is useful to discus the close-to-convexity of $\mathbb{E}_{\alpha,\beta}(z)$ in $\mathcal{D}$ when $0 < \alpha < 1$. In [3, Theorem 2.5], sufficient condition for close-to-convexity of $\mathbb{E}_{\alpha,\beta}(z)$ is given as $\alpha \geq 1$ and $\beta \geq 1$. Therefore, Corollary 4.2 improves the result in [3].

Theorem 4.2. *Keeping the hypothesis of Theorem* 3.1, *so that* $\beta^\gamma > 2(e-1)$. *Then the function* $\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)$ *is close-to-convex with respect to starlike function* $\mathcal{F}^{(\gamma)}_{1,\beta}(z)$ *in* $\mathcal{D}$ *and, consequently, is univalent in* $\mathcal{D}$.

Proof. From Theorem 3.1, the function $\mathcal{F}^{(\gamma)}_{1,\beta}(z)$ is starlike in $\mathcal{D}$ under the hypothesis (H_1) and (H^1_1). Then, from the definition, we need to show that

$$\Re\left(\left[z\left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'\right]\Big/\left(\mathcal{F}^{(\gamma)}_{1,\beta}(z)\right)\right) > 0 \quad \text{for all} \quad z \in \mathcal{D},$$

which is equivalent to

$$\left|\left[z\left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'\right]\Big/\left(\mathcal{F}^{(\gamma)}_{1,\beta}(z)\right) - 1\right| < 1 \quad \text{for all} \quad z \in \mathcal{D}.$$

From (3.6) we get

$$\left|\frac{\mathcal{F}^{(\gamma)}_{1,\beta}(z)}{z}\right| > \frac{\beta^\gamma - (e-1)}{\beta^\gamma}, \quad z \in \mathcal{D}. \tag{4.3}$$

By a short computation we obtain

$$\left|(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z))' - \frac{\mathcal{F}^{(\gamma)}_{1,\beta}(z)}{z}\right| < \sum_{k=1}^{\infty}[\Gamma(\beta)]^\gamma\left|\frac{k+1}{[\Gamma(\alpha k+\beta)]^\gamma} - \frac{1}{[\Gamma(k+\beta)]^\gamma}\right|$$

$$\leq \sum_{k=1}^{\infty}\frac{k[\Gamma(\beta)]^\gamma}{[\Gamma(k+\beta)]^\gamma} = \sum_{k=1}^{\infty}\frac{[\Gamma(\beta)]^\gamma A_k(1,\beta,\gamma)}{k!},$$

where the sequence $(A_k)_{k\geq 1}$ is defined in (3.2). As $(A_k)_{k\geq 1}$ monotonically decreases under the given hypothesis, we get

$$\left|(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z))' - \frac{\mathcal{F}^{(\gamma)}_{1,\beta}(z)}{z}\right| < \frac{e-1}{\beta^\gamma}, \quad \text{for all} \quad z \in \mathcal{D}.$$

Combining the above inequality with (4.3) we obtain the following bound:

$$\left|\left[z\left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'\right]\Big/\left(\mathcal{F}^{(\gamma)}_{1,\beta}(z)\right) - 1\right| < \frac{e-1}{\beta^\gamma - e + 1} < 1 \quad \text{for all} \quad z \in \mathcal{D}.$$

Theorem 4.2 is proved.

Taking in hypothesis (H^1_1) of Theorem 3.1, the values $\alpha = \gamma = 2$ and $\nu = \mu = 1$, we get the following result.

***Corollary* 4.3.** *If* $\beta > \sqrt{2(e-1)}$, *then the function* $\mathcal{F}^{(2)}_{2,\beta}(z)$ *is close-to-convex with respect to starlike function* $\mathcal{F}^{(2)}_{1,\beta}(z)$ *in* $\mathcal{D}$ *and*, *consequently*, *is univalent in* $\mathcal{D}$.

Specifying $\gamma = 1$ in Theorem 4.2, we immediately obtain the following result.

***Corollary* 4.4.** *Under the hypothesis* $(\tilde{H}_1)$ *and* $(\tilde{H}_1^1$ *of Corollary* 3.2 *such that* $\beta > 2(e-1)$, *the function* $\mathbb{E}_{\alpha,\beta}(z)$ *is close-to-convex with respect to starlike function* $\mathbb{E}_{1,\beta}(z)$.

***Corollary* 4.5.** *If* $\beta > 2(e-1)$, *then the function* $\mathbb{E}_{2,\beta}(z)$ *is close-to-convex with respect to starlike function* $\mathbb{E}_{1,\beta}(z)$ *in* $\mathcal{D}$.

Proof. Taking $\alpha = \nu = 2$ and $\mu = \dfrac{1}{2}$ in Corollary 4.4, with the second hypothesis of Theorem 3.1, we deduce the desired result.

***Remark* 4.2.** In [3, Theorem 2.6], the authors proved that the function $\mathbb{E}_{\alpha,\beta}(z)$ is close-to-convex with respect to the starlike function $\mathbb{E}_{1,\beta}(z)$ if $\alpha \geq 1$ and $\beta \geq \big(3+\sqrt{17}\big)/2 \approx 3.56155281281$. Hence, Corollary 4.4 proved result for $\mathbb{E}_{2,\beta}(z)$ better than the result available in [3].

5. Convexity of Le Roy-type Mittag-Leffler function.

Theorem 5.1. *Let the parameters range* $\alpha,\ \beta,\ \gamma,\ \tau,\ \sigma > 0$ *such that* $\alpha\gamma \geq 1$ *and* $4(e-1)[\Gamma(\beta)]^\gamma < \Gamma(\alpha+\beta)]^\gamma$. *In addition, assume that one of the following hypothesis holds true*:

$$(H_4)\colon \begin{cases} \text{(i)} & \mu \geq 2, \beta \leq \nu \leq \mu\alpha^2\gamma, \\ \\ \text{(ii)} & \alpha\gamma(\psi(\beta) - \log(\nu) + \log(\alpha+\nu)) - \log(1+\mu) - 1 > 0, \end{cases}$$

$$\big(H_4^1\big)\colon \begin{cases} \text{(i)} & \mu\alpha^2\gamma \geq \nu,\ \mu \leq \dfrac{3}{2},\ \beta - \nu \geq \dfrac{1}{2}, \\ \\ \text{(ii)} & \alpha\gamma\log(\alpha+\nu) - \log(1+\mu) + \log(\mu) - \psi(2) - 1 > 0. \end{cases}$$

Then the function $\mathcal{F}_{\alpha,\beta}^{(\gamma)}(z)$ *is convex on* $\mathcal{D}$.

Proof. It is known that $f(z)$ is convex if and only if $zf'(z)$ is starlike. So in order to prove $\mathcal{F}_{\alpha,\beta}^{(\gamma)}(z)$ is convex it is sufficient to prove that the function

$$\mathcal{G}_{\alpha,\beta}^{(\gamma)}(z) := z\Big(\mathcal{F}_{\alpha,\beta}^{(\gamma)}(z)\Big)'$$

is starlike in $\mathcal{D}$. A calculation gives

$$\left|\frac{\mathcal{G}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right| = |(\mathcal{F}_{\alpha,\beta}^{(\gamma)}(z))'| > 1 - \sum_{k=1}^{\infty} \frac{C_k(\alpha,\beta,\gamma)}{k!}, \tag{5.1}$$

where $(C_k)_{k\geq 1}$ is defined by

$$C_k := C_k(\alpha,\beta,\gamma) = \frac{[\Gamma(\beta)]^\gamma\Gamma(k+2)}{[\Gamma(\alpha k+\beta)]^\gamma}, \quad k \geq 1. \tag{5.2}$$

Straightforward calculation would yield

$$\big(\mathcal{G}_{\alpha,\beta}^{(\gamma)}(z)\big)' - \frac{\mathcal{G}_{\alpha,\beta}^{(\gamma)}(z)}{z} = \sum_{k=1}^{\infty} \frac{D_k(\alpha,\beta,\gamma)z^k}{k!}, \tag{5.3}$$

where $(D_k)_{k\geq 1}$ is defined by

$$D_k := D_k(\alpha,\beta,\gamma) = \frac{k[\Gamma(\beta)]^\gamma\Gamma(k+2)}{[\Gamma(\alpha k+\beta)]^\gamma}, \quad k \geq 1.$$

For convenience, we denote

$$\Delta^{(\gamma)}_{\alpha,\beta}(t)=\frac{t\Gamma(t+2)}{[\Gamma(\alpha t+\beta)]^{\gamma}},\quad t\geq 1.$$

Hence,

$$\big(\Delta^{(\gamma)}_{\alpha,\beta}(t)\big)'=\Delta^{(\gamma)}_{\alpha,\beta}(t)\left[\frac{1}{t}+\psi(t+2)-\alpha\gamma\psi(\alpha t+\beta)\right]=:\Delta^{(\gamma)}_{\alpha,\beta}(t)\tilde{\Delta}^{(\gamma)}_{\alpha,\beta}(t).\tag{5.4}$$

We consider two cases.

Case 1. Assume that the conditions (H_4) are valid. By using (2.1), we get

$$\psi(t+2)\leq\log(t+\mu),\quad t>0,\quad \mu\geq 2,$$

and

$$\psi(\beta)-\log(\nu)+\log(\alpha t+\nu)\leq\psi(\alpha t+\beta),\quad t>0,\quad \nu\geq\beta.$$

Therefore,

$$\tilde{\Delta}^{(\gamma)}_{\alpha,\beta}(t)\leq\breve{\Delta}^{(\gamma)}_{\alpha,\beta}(t):=\frac{1}{t}+\log(t+\mu)-\gamma\alpha(\psi(\beta)-\log(\nu)+\log(\alpha t+\nu)),$$

$$t>0,\quad \mu\geq 2,\quad \nu\geq\beta.$$

Differentiating the function $\breve{\Delta}^{(\gamma)}_{\alpha,\beta}(t)$, we obtain

$$\big(\breve{\Delta}^{(\gamma)}_{\alpha,\beta}(t)\big)'=-\frac{1}{t^2}+\frac{\alpha(1-\alpha\gamma)t+\nu-\mu\alpha^2\gamma}{(t+\mu)(\alpha t+\nu)},\quad t\geq 1.$$

This equality implies $\big(\breve{\Delta}^{(\gamma)}_{\alpha,\beta}(t)\big)'\leq 0$ for all $t\geq 1,\alpha\gamma\geq 1$ and $\mu\alpha^2\gamma\geq\nu$. Hence, the function $\breve{\Delta}^{(\gamma)}_{\alpha,\beta}(t)$ is decreasing on $[1,\infty)$. Since $\breve{\Delta}^{(\gamma)}_{\alpha,\beta}(1)<0$, it follows that the function $\breve{\Delta}^{(\gamma)}_{\alpha,\beta}(t)<0$ for all $t\geq 1$ and, consequently, $\tilde{\Delta}^{(\gamma)}_{\alpha,\beta}(t)<0$ for $t\geq 1$. From (5.4) it follows that the function $\Delta^{(\gamma)}_{\alpha,\beta}(t)$ is decreasing on $[1,\infty)$. This in turn implies that $(D_k)_{k\geq 1}$ monotonically decreases. By (5.3) we find that

$$\left|\big(\mathcal{G}^{(\gamma)}_{\alpha,\beta}(z)\big)'-\frac{\mathcal{G}^{(\gamma)}_{\alpha,\beta}(z)}{z}\right|<\sum_{k=1}^{\infty}\frac{D_1(\alpha,\beta,\gamma)}{k!}=D_1(\alpha,\beta,\gamma)(e-1)\quad\text{for all}\quad z\in\mathcal{D}.\tag{5.5}$$

We observe that if $(D_k)_{k\geq 1}$ is decreasing, then the sequence $(C_k)_{k\geq 1}$ is decreasing. Hence, (5.1) yields

$$\left|\frac{\mathcal{G}^{(\gamma)}_{\alpha,\beta}(z)}{z}\right|>1-C_1(\alpha,\beta,\gamma)(e-1)=\frac{[\Gamma(\alpha+\beta)]^{\gamma}-2(e-1)[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha+\beta)]^{\gamma}}\quad\text{for all}\quad z\in\mathcal{D}.\tag{5.6}$$

Now, collecting (5.5) and (5.6) we obtain

$$\left|\Big[z\big(\mathcal{G}^{(\gamma)}_{\alpha,\beta}(z)\big)'\Big]\Big/\Big[\mathcal{G}^{(\gamma)}_{\alpha,\beta}(z)\Big]-1\right|<\frac{2(e-1)[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha+\beta)]^{\gamma}-2(e-1)[\Gamma(\beta)]^{\gamma}}<1\quad\text{for all}\quad z\in\mathcal{D}.\tag{5.7}$$

Therefore, the function $\mathcal{G}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}$.

Case 2. Suppose that the conditions of $\left(H_4^1\right)$ hold true. In virtue of (2.3) we have

$$\psi(t+2) \le \log(t+\mu) - \log(\mu) + \psi(2), \quad t>0, \quad 0<\mu \le \frac{3}{2},$$

and

$$\log(\alpha t+\nu) \le \psi(\alpha t+\beta), \quad t>0, \quad \alpha>0, \quad \beta-\nu \ge \frac{1}{2}.$$

Bearing in mind the above formulas, we find that

$$\tilde{\Delta}_{\alpha,\beta}^{(\gamma)}(t) \le \Psi_{\alpha,\beta}^{(\gamma)}(t) = \frac{1}{t} + \log(t+\mu) - \log(\mu) + \psi(2) - \alpha\gamma\log(\alpha t+\nu),$$

$$t,\alpha,\mu,\nu>0, \quad \beta-\nu \ge \frac{1}{2}, \quad \mu \le \frac{3}{2}.$$

Since

$$\left(\Psi_{\alpha,\beta}^{(\gamma)}(t)\right)' = -\frac{1}{t^2} + \frac{\alpha(1-\alpha\gamma)t+\nu-\mu\alpha^2\gamma}{(t+2)(\alpha t+\sigma)} < 0$$

and

$$\Psi_{\alpha,\beta}^{(\gamma)}(1) = 1 + \log(1+\mu) - \log(\mu) + \psi(2) - \alpha\gamma\log(\alpha+\nu) < 0,$$

we have $\tilde{\Delta}_{\alpha,\beta}^{(\gamma)}(t)<0$ under the given conditions of $\left(H_4^1\right)$. This implies that the function $\Delta_{\alpha,\beta}^{(\gamma)}(t)$ is decreasing on $[1,\infty)$. This in turn implies that the sequences $(D_k)_{k\ge1}$ and $(C_k)_{k\ge1}$ are decreasing. Therefore, the inequalities (5.5), (5.6) and (5.7) hold true under the conditions $\left(H_4^1\right)$.

Theorem 5.1 is proved.

If we set $\alpha=\mu=\nu=1$ and $\gamma=2$ in the hypothesis $\left(H_4^1\right)$ of Theorem 5.1, we get the following result.

***Corollary* 5.1.** *If* $\beta > \sqrt{4(e-1)} \approx 2.62166498886$, *then the function* $\mathcal{F}_{1,\beta}^{(2)}(z)$ *is convex on* $\mathcal{D}$.

***Example* 5.1.** The function $\mathcal{F}_{1,\frac{11}{4}}^{(2)}(z)$ is convex on $\mathcal{D}$ (see Fig. 6).

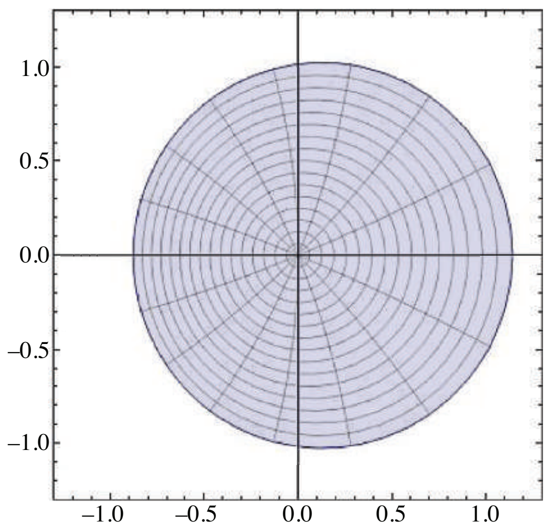

Fig. 6. Mapping of $\mathcal{F}_{1,\frac{11}{4}}^{(2)}(z)$ over $\mathcal{D}$.

As an immediate consequence of Theorem 5.1 we have the following result.

***Corollary* 5.2.** *Let the parameters range $\alpha, \beta, \tau, \sigma > 0$ such that $\alpha \geq 1$ and $4(e-1)\Gamma(\beta) < \Gamma(\alpha+\beta)$. Also, assume that any one of the following hypothesis holds true*:

$$(\tilde{H}_4): \begin{cases} \text{(i)} & \mu \geq 2,\ \beta \leq \nu \leq \mu\alpha^2\gamma, \\ \text{(ii)} & \alpha(\psi(\beta) - \log(\nu) + \log(\alpha+\nu)) - \log(1+\mu) - 1 > 0, \end{cases}$$

$$(\tilde{H}_4^1): \begin{cases} \text{(i)} & \mu\alpha^2 \geq \nu,\ \mu \leq \dfrac{3}{2},\ \beta - \nu \geq \dfrac{1}{2}, \\ \text{(ii)} & \alpha\log(\alpha+\nu) - \log(1+\mu) + \log(\mu) - \psi(2) - 1 > 0. \end{cases}$$

Then the function $\mathbb{E}_{\alpha,\beta}(z)$ is convex on $\mathcal{D}$.

Specifying $\alpha = 2$, $\mu = \dfrac{3}{2}$ and $\nu = 1$ in the hypothesis $(\tilde{H}_4^1)$ of Corollary 5.2, we get the following result.

***Corollary* 5.3.** *If $\beta > \left(-1+\sqrt{16e-15}\right)/2 \approx 2.1689187537$, then the function $\mathbb{E}_{2,\beta}(z)$ is convex on $\mathcal{D}$.*

***Example* 5.2.** The function $\mathbb{E}_{2,\frac{7}{3}}(z)$ is convex on $\mathcal{D}$ (see Fig. 7).

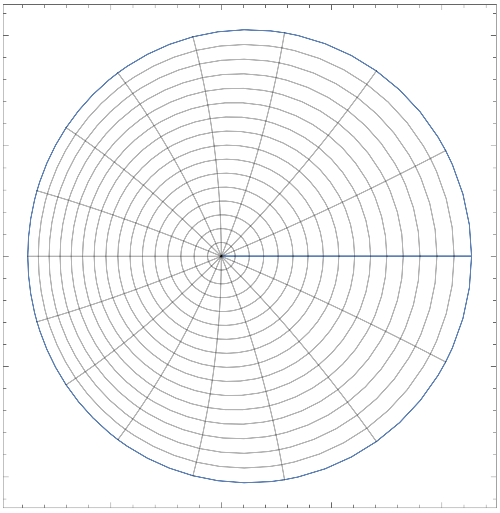

Fig. 7. Mapping of $\mathbb{E}_{2,\frac{7}{3}}(z)$ over $\mathcal{D}$.

***Remark* 5.1.** In [9, Theorem 7], the function $\mathbb{E}_{2,\beta}(z)$ is convex on $\mathcal{D}$ if $\beta > 3.56155281$. Hence, Corollary 5.3 provide result for $\mathbb{E}_{2,\beta}(z)$ is better than the result available in [9, Theorem 7].

Theorem 5.2. *Let the parameters range $\alpha, \beta, \gamma, \nu, \mu > 0$ such that $\alpha\gamma \geq 1$ and $2(e-1)[\Gamma(\beta)]^\gamma < [\Gamma(\alpha+\beta)]^\gamma$, together with the constraints (H_5) and $\left(H_5^1\right)$ as follows*:

$$(H_5): \begin{cases} \text{(i)} & \mu \geq 2,\ \beta \leq \nu \leq \mu\gamma\alpha^2, \\ \text{(ii)} & \alpha\gamma(\psi(\beta) - \log(\nu) + \log(\alpha+\nu)) - \log(1+\mu) > 0, \end{cases}$$

$$(H_5^1): \begin{cases} \text{(i)} & \mu\gamma\alpha^2 \geq \nu,\ \mu \leq \dfrac{3}{2},\ \beta - \nu \geq \dfrac{1}{2}, \\ \text{(ii)} & \alpha\gamma\log(\alpha+\nu) - \log(1+\mu) + \log(\mu) - \psi(2) > 0. \end{cases}$$

Then the function $\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)$ is convex in $\mathcal{D}_{1/2}$.

Proof. Due to T. H. MacGregor [6], it is known that if $f \in \mathcal{A}$ and satisfies $|f'(z) - 1| < 1$ for each $z \in \mathcal{D}$, then f is convex on $\mathcal{D}_{\frac{1}{2}} = \left\{ z \in C,\, |z| < \dfrac{1}{2} \right\}$. Straightforward calculation would yield

$$\left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1 = \sum_{k=1}^{\infty} \frac{C_k(\alpha,\beta,\gamma) z^k}{k!},$$

where $(C_k)_k$ is defined in (5.2). Now, we prove that the sequence $(C_k)_{k\geq 1}$ monotonically decreases. For this we consider the function $\chi^{(\gamma)}_{\alpha,\beta}(t)$ defined by

$$\chi^{(\gamma)}_{\alpha,\beta}(t) = \frac{\Gamma(t+2)}{[\Gamma(\alpha t + \beta)]^{\gamma}}, \quad t \geq 1.$$

Differentiation gives

$$(\chi^{(\gamma)}_{\alpha,\beta}(t))' = \chi^{(\gamma)}_{\alpha,\beta}(t)[\psi(t+2) - \gamma\alpha\psi(\alpha t + \beta)] =: \chi^{(\gamma)}_{\alpha,\beta}(t)\tilde{\chi}^{(\gamma)}_{\alpha,\beta}(t). \tag{5.8}$$

Case 1. We suppose that the conditions (H_5) are valid. From the right-hand side of inequalities (2.1), we have

$$\psi(t+2) \leq \log(t+\mu), \quad t > 0, \quad \mu \geq 2. \tag{5.9}$$

By left-hand side of inequalities (2.1) we get

$$\psi(\alpha t + \beta) \geq \psi(\beta) - \log(\nu) + \log(\alpha t + \nu), \quad t > 0, \quad \nu \geq \beta. \tag{5.10}$$

Combining (5.9), (5.10) with (5.8), we obtain

$$(\chi^{(\gamma)}_{\alpha,\beta}(t))' \leq \chi^{(\gamma)}_{\alpha,\beta}(t)\check{\chi}^{(\gamma)}_{\alpha,\beta}(t),$$

where

$$\check{\chi}^{(\gamma)}_{\alpha,\beta}(t) = \log(t+\mu) - \gamma\alpha(\log(\alpha t + \nu) + \psi(\beta) - \log(\nu)), \quad t \geq 1, \quad \mu \geq 2, \quad \nu \geq \beta.$$

We see that the function $\check{\chi}^{(\gamma)}_{\alpha,\beta}(t)$ is decreasing on $[1,\infty)$ such that $\check{\chi}^{(\gamma)}_{\alpha,\beta}(1) < 0$ under the given hypothesis (H_5). This in turn implies that the function $\chi^{(\gamma)}_{\alpha,\beta}(t)$ is decreasing in $[1,\infty)$ and consequently the sequence $(C_k)_{k\geq 1}$ monotonically decreases. Thus, we get

$$\left|\left(\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1\right| \leq \sum_{k=1}^{\infty} \frac{C_1(\alpha,\beta,\gamma) z^k}{k!} = C_1(\alpha,\beta,\gamma)(e-1) \quad \text{for all} \quad z \in \mathcal{D}. \tag{5.11}$$

Hence, if $C_1(\alpha,\beta,\gamma)(e-1) < 1$, then the function $\mathcal{F}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}$. This complete the proof of Theorem 5.2 under the conditions (H_5).

Case 2. Assume that the conditions $\left(H_5^1\right)$ are satisfied. From inequalities (2.3), we get

$$\psi(t+2) \leq \log(t+\mu) + \psi(2) - \log(\mu), \quad t > 0, \quad 0 < \mu \leq \frac{3}{2},$$

and

$$\psi(\alpha t+\beta)\geq \log(\alpha t+\nu), \quad t>0, \quad \nu>0, \quad \beta-\nu\geq \frac{1}{2}.$$

Bearing in mind the above inequalities and (5.8) we find that

$$(\chi_{\alpha,\beta}^{(\gamma)}(t))'\leq \chi_{\alpha,\beta}^{(\gamma)}(t)\big[\log(t+\mu)+\psi(2)-\log(\mu)-\alpha\gamma\log(\alpha t+\nu)\big]=:\chi_{\alpha,\beta}^{(\gamma)}(t)K_{\alpha,\beta}^{(\gamma)}(t). \quad (5.12)$$

We observe that the function $K_{\alpha,\beta}^{(\gamma)}(z)$ is decreasing on $[1,\infty)$ for all $\alpha\gamma\geq 1$ and $\nu\leq\alpha^2\gamma\mu$ such that $K_{\alpha,\beta}^{(\gamma)}(1)<0$. Applying (5.12) yields that the function $\chi_{\alpha,\beta}^{(\gamma)}(t)$ is decreasing on $[1,\infty)$ and, consequently, the sequence $(C_k)_{k\geq 1}$ is decreasing. So the inequality (5.11) holds true in this case. Then $\mathcal{F}_{\alpha,\beta}^{(\gamma)}(z)$ is starlike in $\mathcal{D}$ under the conditions of $\big(H_5^1\big)$.

Theorem 5.2 is proved.

Taking $\alpha=\nu=1$, $\mu=\dfrac{3}{2}$ and $\gamma=2$ in the hypothesis $\big(H_5^1\big)$ of Theorem 5.2, we obtain the following result.

***Corollary* 5.4.** *If $\beta>\sqrt{2(e-1)}$, then the function $\mathcal{F}_{1,\beta}^{(2)}(z)$ is convex in $\mathcal{D}_{1/2}$.*

***Example* 5.3.** The function $\mathcal{F}_{1,\frac{19}{10}}^{(2)}(z)$ is convex in $\mathcal{D}_{1/2}$ (see Fig. 8).

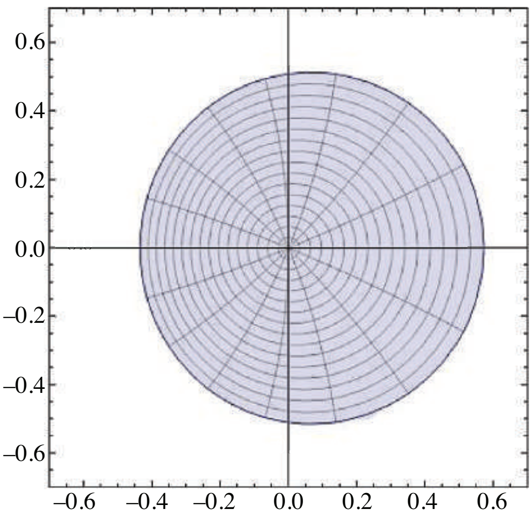

Fig. 8. Mapping of $\mathcal{F}_{1,\frac{19}{10}}^{(2)}(z)$ over $\mathcal{D}$.

Upon setting $\gamma=1$ in Theorem 5.2, we get the following result.

***Corollary* 5.5.** *Assume that the parameters $\alpha,\beta,\nu,\mu,\delta>0$ such that $\alpha\geq 1$ and $2(e-1)\Gamma(\beta)<\Gamma(\alpha+\beta)$, together with the constraints $\big(\tilde{H}_5\big)$ and $\big(\tilde{H}_5^1\big)$ as follows*:

$$\big(\tilde{H}_5\big)\colon \begin{cases} \text{(i)} & \mu\geq 2,\ \beta\leq\nu\leq\mu\alpha^2,\\ \text{(ii)} & \alpha(\psi(\beta)-\log(\nu)+\log(\alpha+\nu))-\log(1+\mu)>0,\end{cases}$$

$$\big(\tilde{H}_5^1\big)\colon \begin{cases} \text{(i)} & \mu\alpha^2\geq\nu,\ \mu\leq\dfrac{3}{2},\ \beta-\nu\geq\dfrac{1}{2},\\ \text{(ii)} & \alpha\log(\alpha+\nu)-\log(1+\mu)+\log(\mu)-\psi(2)>0.\end{cases}$$

Then the function $\mathbb{E}_{\alpha,\beta}(z)$ is convex in $\mathcal{D}_{1/2}$.

If we set $\alpha = 2$, $\mu = \dfrac{3}{2}$ and $\nu = 1$ in the conditions $\left(\tilde{H}_5^1\right)$ of Corollary 5.5, we obtain the following result.

***Corollary* 5.6.** *If* $\beta > \left(-1 + \sqrt{8e-7}\right)/2 \approx 1.420042618$, *then the function* $\mathbb{E}_{2,\beta}(z)$ *is convex in* $\mathcal{D}_{1/2}$.

***Example* 5.4.** The function $\mathbb{E}_{2,\frac{3}{2}}(z)$ is convex in $\mathcal{D}_{1/2}$ (see Fig. 9).

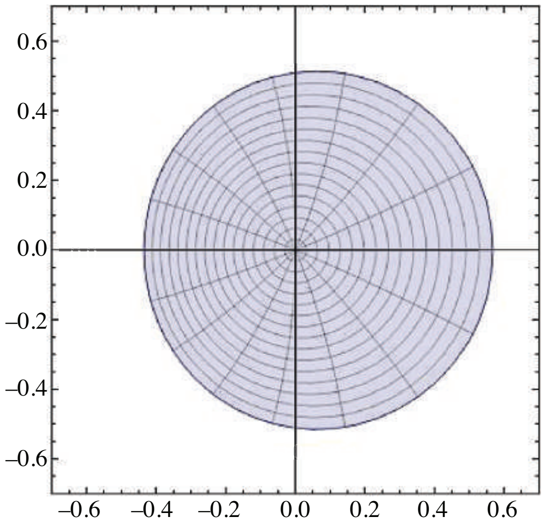

Fig. 9. Mapping of $\mathbb{E}_{2,\frac{3}{2}}(z)$ over $\mathcal{D}_{1/2}$.

***Remark* 5.2.** It can be noted from [3, Theorem 2.4], that $\mathbb{E}_{2,\beta}(z)$ is convex in $\mathcal{D}_{1/2}$ if $\beta \geq \left(3 + \sqrt{17}\right)/2 \approx 3.561552813$. Hence, the above result improves the result for $\mathbb{E}_{2,\beta}(z)$, available in [3].

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