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M. Raza[1] (Department of Mathematics, Government College University Faisalabad, Pakistan),
D. K. Thomas (Swansea University, Bay Campus, Swansea, United Kingdom),
A. Riaz (Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Pakistan)

COEFFICIENT ESTIMATES FOR STARLIKE AND CONVEX FUNCTIONS RELATED TO SIGMOID FUNCTIONS

ОЦІНКИ КОЕФІЦІЄНТІВ ДЛЯ ЗІРКОПОДІБНИХ І ОПУКЛИХ ФУНКЦІЙ, ЩО ПОВ'ЯЗАНІ З СИГМОЇДНИМИ ФУНКЦІЯМИ

We give sharp coefficient bounds for starlike and convex functions related to modified sigmoid functions. We also provide some sharp coefficients bounds for the inverse functions and sharp bounds for the initial logarithmic coefficients and some coefficient differences.

Наведено точні межі для коефіцієнтів зіркоподібних і опуклих функцій, що пов'язані з модифікованими сигмоїдними функціями, а також деякі точні коефіцієнтні оцінки для обернених функцій і точні оцінки для початкових логарифмічних коефіцієнтів та деяких різниць коефіцієнтів.

1. Introduction. Denote by $\mathcal{A}$ the class of normalized analytic functions f in the open unit disc $\mathbb{D} := \big\{z : |z| < 1, \quad z \in \mathbb{C}\big\}$ with Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1.1}$$

and let $\mathcal{S}$ denote the subclass of analytic functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$.

An analytic function f is subordinate to a function g, written as $f \prec g$, if there exists an analytic function w with $|w(z)| \le |z|$ and $w(0) = 0$ such that $f(z) = g(w(z))$. If g is univalent and $f(0) = g(0)$, then $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Suppose that the function φ is analytic and univalent in $\mathbb{D}$, is starlike with respect to $\varphi(0) = 1$ with $\varphi'(0) > 0$, and is symmetric about the real axis. Then Ma and Minda [10] generalized the classes of starlike and convex functions as follows:

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} \colon \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{C}(\varphi) := \left\{ f \in \mathcal{A} \colon 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

The class of starlike functions related to a sigmoid function was introduced by Goel and Kumar [8], and is defined as

$$\mathcal{S}^*_{SG} := \left\{ f \in \mathcal{A} \colon \frac{zf'(z)}{f(z)} \prec \frac{2}{1+e^{-z}} \right\}.$$

[1] Corresponding author, e-mail: mohsan976@yahoo.com.

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Here the function $\varphi(z) = 2/(1+e^{-z})$ is a modified sigmoid function which maps $\mathbb{D}$ onto the domain $\Delta_{SG} = \{w \in \mathbb{C} : |\log(w/(2-w))| < 1\}$. The class $\mathcal{S}^*_{SG}$ was further studied in [5, 9].

We will also consider the class of convex functions related to a modified sigmoid functions defined by

$$\mathcal{C}_{SG} = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{2}{1+e^{-z}} \right\}.$$

We now recall some basic definitions of coefficient functionals which will be considered in this paper.

The logarithmic coefficient β_n of $f \in \mathcal{S}$ are defined for $z \in \mathbb{D}$ by

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \beta_n z^n. \tag{1.2}$$

The logarithmic coefficients of f play an important role in the theory of univalent functions. Clearly the Köebe function has logarithmic coefficients $\beta_n = 1/n$, and it is a simple exercise to show that $|\beta_n| \le 1/n$ holds for starlike functions, which is false for the full class $\mathcal{S}$ [7, p. 898]. For some recent work on logarithmic coefficients, see [2, 3].

For any univalent function f, there exists an inverse function f^{-1}, defined on some disc $|w| \le 1/4 \le r(f)$, with Taylor series expansion

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \ldots. \tag{1.3}$$

In the 1960's, L. Zalcman conjectured that if $f \in \mathcal{S}$, then

$$\left|a_n^2 - a_{2n-1}\right| \le (n-1)^2, \quad n \ge 2,$$

which would be sharp for the Koebe function. The Zalcman conjecture implies the famous Bieberbach conjecture $|a_n| \le n$ for $n \ge 2$, see [6, 15].

After the Bieberbach conjecture was settled, it was therefore natural to study the validity of the inequality

$$\big||a_{n+1}| - |a_n|\big| \le 1, \quad n \ge 2.$$

It was shown in [5] that the above inequalty does not hold for $f \in \mathcal{S}$ when $n = 2$, where it was shown that the following sharp bounds hold:

$$-1 \le |a_3| - |a_2| \le \frac{3}{4} + e^{-\alpha_0}(2e^{-\alpha_0} - 1) = 1.029,$$

where α_0 is the unique value of α in $0 < \alpha < 1$, satisfying the equation $4\alpha_0 = e\alpha_0$. For some recent developments concerning coefficient differences, see [13, 14].

2. Lemmas. We will use the following results concerning the functions in the class $\mathcal{P}$.

Let $\mathcal{P}$ denote the class of analytic functions p defined for $z \in \mathbb{D}$ and given by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2.1}$$

with positive real part in $\mathbb{D}$.

Lemma 2.1 [11]. *Let w be a Schwarz function given by $w(z)=\sum_{n=0}^{\infty}w_nz^n$ and*

$$\psi(u,v)=\left|w_3+\mu w_1w_2+\nu w_1^3\right|.$$

Then $\psi(u,v)\le|\nu|$ if $(u,v)\in D_6$, where

$$D_6=\left\{(u,v)\colon 2\le|\mu|\le 4,\ \nu\ge\frac{1}{12}(\mu^2+8)\right\}.$$

Lemma 2.2 [10]. *Let $p\in\mathcal{P}$ and be given by* (2.1). *Then*

$$\left|c_2-vc_1^2\right|\le\begin{cases}-4v+2, & v<0,\\ 2, & 0\le v\le 1,\\ 4v-2, & v>1.\end{cases}$$

When $v<0$ or $v>1$, equality holds if and only if $h(z)=\dfrac{1+z}{1-z}$ or one of its rotations. If $0<v<1$, then equality holds if and only if $h(z)=\dfrac{1+z^2}{1-z^2}$ or one of its rotations.

Lemma 2.3 [1]. *Let $p\in\mathcal{P}$ and be given by* (2.1) *with $0\le B\le 1$ and $B(2B-1)\le D\le B$. Then*

$$\left|c_3-2B\,c_1c_2+Dc_1^3\right|\le 2.$$

Lemma 2.4 [12]. *Let $p\in\mathcal{P}$ and be given by* (2.1). *If $0<a<1,\ 0<b<1$ and*

$$8a(1-a)\left\{(b\beta-2\lambda)^2+(b(a+b)-\beta)^2\right\}+b(1-b)(\beta-2ab)^2\le 4b^2a(1-b)^2(1-a),$$

then

$$\left|\lambda c_1^4+ac_2^2+2bc_1c_3-\frac{3}{2}\beta c_1^2c_2-c_4\right|\le 2.$$

Lemma 2.5 [14]. *Let B_1, B_2, and B_3 be numbers such that $B_1\ge 0$, $B_2\in\mathbb{C}$, and $B_3\in\mathbb{R}$. Let $p\in\mathcal{P}$ and be given by* (2.1). *Define $\psi_+(c_1,c_2)$ and $\psi_-(c_1,c_2)$ by*

$$\psi_+(c_1,c_2)=|B_2c_1^2+B_3c_2|-|B_1c_1|$$

and $\psi_-(c_1,c_2)=-\psi_+(c_1,c_2)$. Then

$$\psi_+(c_1,c_2)\le\begin{cases}|4B_2+2B_3|-2B_1, & \text{when}\quad |2B_2+B_3|\ge|B_3|+B_1,\\ 2|B_3|, & \text{otherwise},\end{cases}$$

and

$$\psi_-(c_1,c_2)\le\begin{cases}2B_1-B_4, & \text{when}\quad B_1\ge 2|B_3|+B_4,\\ 2B_1\sqrt{\dfrac{2|B_3|}{2|B_3|+B_4}}, & \text{when}\quad B_1^2\le 2|B_3|\big(2|B_3|+B_4\big),\\ 2|B_3|+\dfrac{B_1^2}{2|B_3|+B_4}, & \text{otherwise},\end{cases}$$

where $B_4=|4B_2+2B_3|$. All inequalities are sharp.

3. Some coefficient inequalities for the classes $\mathcal{S}^*_{SG}$ and $\mathcal{C}_{SG}$.

Theorem 3.1 [8]. *Let $f \in \mathcal{S}^*_{SG}$ and be given by* (1.1). *Then*

$$|a_2| \le \frac{1}{2}, \qquad |a_3| \le \frac{1}{4}, \qquad |a_4| \le \frac{1}{6}, \qquad |a_5| \le \frac{1}{8}.$$

All inequalities are sharp.

Proof. Let $f \in \mathcal{S}^*_{SG}$. Then there exists the Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1 + e^{-w(z)}}. \tag{3.1}$$

Let $p \in \mathcal{P}$, then using the definition of subordination we can write

$$w(z) = \frac{p(z) - 1}{p(z) + 1}. \tag{3.2}$$

Let p be of the form (2.1). Using (3.1) and (3.2) and equating coefficients, we obtain

$$a_2 = \frac{1}{4}c_1, \tag{3.3}$$

$$a_3 = \frac{1}{8}c_2 - \frac{1}{32}c_1^2, \tag{3.4}$$

$$a_4 = \frac{7}{1152}c_1^3 - \frac{5}{96}c_1c_2 + \frac{1}{12}c_3, \tag{3.5}$$

$$a_5 = \frac{-17}{18432}c_1^4 + \frac{7}{384}c_1^2c_2 - \frac{1}{24}c_1c_3 - \frac{3}{128}c_2^2 + \frac{1}{16}c_4. \tag{3.6}$$

(3.3) follows at once from the classical inequality $|c_n| \le 2$ for $n \ge 1$.

For (3.4) we apply Lemma 2.2 with $\nu = \dfrac{1}{4}$.

For (3.5) we apply Lemma 2.3 with $B = \dfrac{5}{16}$, and $D = \dfrac{7}{96}$.

The inequalities are sharp for the functions $f_n \in \mathcal{S}^*_{SG}$ given by formula

$$f_n(z) = z \exp\left(\int_0^z \frac{\frac{2}{1 + e^{-t^n}} - 1}{t} dt \right), \quad n = 1, 2, 3, 4. \tag{3.7}$$

Theorem 3.1 is proved.

Theorem 3.2. *Let $f \in \mathcal{S}^*_{SG}$ and be given by* (1.1). *Then*

$$-\frac{1}{\sqrt{6}} \le |a_3| - |a_2| \le \frac{1}{4}.$$

Both inequalities are sharp.

Proof. From (3.3) and (3.4), we have

$$\psi_+(c_1, c_2) = |a_3| - |a_2| = \left|\frac{1}{8}c_2 - \frac{1}{32}c_1^2\right| - \left|\frac{1}{4}c_1\right|$$

$$= |B_2 c_1^2 + B_3 c_2| - |B_1 c_1|,$$

where $B_1 = \dfrac{1}{4}$, $B_2 = -\dfrac{1}{32}$ and $B_3 = \dfrac{1}{8}$. Now $|2B_2 + B_3| = \dfrac{1}{16}$ and $|B_3| + B_1 = \dfrac{3}{8}$, so that $|2B_2 + B_3| \ngeq |B_3| + B_1$. Hence, from Lemma 2.5, we obtain

$$\psi_+(c_1, c_2) = |a_3| - |a_2| \le 2|B_3| = \frac{1}{4}.$$

For the lower bound

$$\psi_-(c_1, c_2) = |a_2| - |a_3|,$$

and since $2|B_3|\big(2|B_3| + B_4\big) - B_1^2 = 3/8 > 0$, Lemma 2.5 gives

$$\psi_-(c_1, c_2) \le 2B_1\sqrt{\frac{2|B_3|}{2|B_3| + B_4}} = \frac{1}{\sqrt{6}},$$

as required.

To see that the upper bound is sharp, consider the function $f_2 \in \mathcal{S}^*_{SG}$, defined by formula

$$f_2(z) = z\exp\left(\int_0^z \frac{\dfrac{2}{1+e^{-t^2}} - 1}{t}\,dt\right) = z + \frac{z^3}{4} + \ldots. \tag{3.8}$$

For the lower consider the function

$$p_0(z) = \frac{1 - z^2}{1 - 2t_0 z + z^2},$$

where $t_0 = \sqrt{\dfrac{2}{3}}$. Let

$$w_0(z) = \frac{p_0(z) - 1}{p_0(z) + 1} = \frac{z\big(3z - \sqrt{6}\big)}{-3 + \sqrt{6}z} = \frac{\sqrt{6}}{3}z - \frac{1}{3}z^2 - \frac{\sqrt{6}}{9}z^3 - \ldots$$

and

$$q_0(z) = \frac{2}{1 + e^{-w_0(z)}}.$$

It is easy to see that $w_0(0) = 0$ and $|w_0(z)| < 1$ for $z \in \mathbb{D}$. Now let the function $f_* \in \mathcal{S}^*_{SG}$ be defined by

$$f_*(z) = z\exp\int_0^z \frac{q_0(t) - 1}{t}\,dt = z + \frac{1}{\sqrt{6}}z^2 - \frac{5\sqrt{6}}{162}z^4 + \ldots.$$

Then the lower bound is sharp for the function f_*.

Theorem 3.2 is proved.

Theorem 3.3. *Let* $f \in \mathcal{S}^*_{SG}$ *and be given by* (1.1). *Then*

$$|A_2| \le \frac{1}{2}, \qquad |A_3| \le \frac{3}{8}, \qquad |A_4| \le \frac{23}{72}.$$

All inequalities are sharp.

Proof. Since $f\big(f^{-1}(w)\big) = w$, using (1.3) it is easy to see that

$$A_2 = -a_2,$$

$$A_3 = 2a_2^2 - a_3,$$

$$A_4 = -5a_2^3 + 5a_2a_3 - a_4.$$

Using (3.3) – (3.5) in the above and equating coefficients, we obtain

$$A_2 = \frac{-1}{4}c_1, \qquad A_3 = \frac{5}{32}c_1^2 - \frac{1}{8}c_2, \qquad A_4 = \frac{-71}{576}c_1^3 + \frac{5}{24}c_1c_2 - \frac{1}{12}c_3.$$

The first bound follows at once from the inequality $|c_1| \le 2$. For $|A_3|$, we have

$$|A_3| = \frac{1}{8}\Big|c_2 - \frac{5}{4}, c_1^2\Big|,$$

and using Lemma 2.2 for $v = \dfrac{5}{4} > 1$, we obtain the required result.

For $|A_4|$ consider a function $p \in \mathcal{P}$ given by (2.1), and the Schwarz function $w(z) = \sum_{n=0}^{\infty} w_n z^n$, so we can write

$$p(z) = \frac{1 + w(z)}{1 - w(z)}.$$

Equating coefficients gives

$$c_1 = 2w_1, \qquad c_2 = 2w_2 + 2w_1^2, \qquad c_3 = 2w_3 + 4w_1w_2 + 2w_1^3,$$

and so

$$A_4 = -\frac{1}{6}\left(w_3 - 3w_2w_1 + \frac{23}{12}w_1^3\right) = -\frac{1}{6}\big(w_3 + \mu w_2w_1 + \nu w_1^3\big),$$

where $\mu = -3$ and $\nu = \dfrac{23}{12}$. Since $2 < |\mu| = 3 < 4$ and the relation $\nu \ge \dfrac{1}{12}[\mu^2 + 8]$ implies that $\dfrac{23}{12} > \dfrac{17}{12}$, all conditions of Lemma 2.1 are satisfied, and the required inequality follows.

The bounds are sharp for the function f_1 given by (3.7).

Theorem 3.3 is proved.

Theorem 3.4. *Let* $f \in \mathcal{S}^*_{SG}$ *and be given by* (1.1). *Then*

$$-\frac{1}{\sqrt{10}} \le |A_3| - |A_2| \le \frac{1}{4}.$$

Proof. From (3.3) and (3.4) we have

$$\psi_+(c_1, c_2) = |A_3| - |A_2| = \left|\frac{5}{32}c_1^2 - \frac{1}{8}c_2\right| - \left|\frac{1}{4}c_1\right|$$

$$= |B_2 c_1^2 + B_3 c_2| - |B_1 c_1|,$$

where $B_1 = \frac{1}{4}$, $B_2 = \frac{5}{32}$ and $B_3 = -\frac{1}{8}$. Since $|2B_2 + B_3| = 3/16$ and $|B_3| + B_1 = \frac{3}{8}$, it follows that $|2B_2 + B_3| \ngeq |B_3| + B_1$. Hence, using Lemma 2.5, we obtain

$$\psi_+(c_1, c_2) = |A_3| - |A_2| \leq 2|B_3| = \frac{1}{4}.$$

For the lower bound we get

$$\psi_-(c_1, c_2) = -\psi_+(c_1, c_2) = |A_2| - |A_3|.$$

Since $2|B_3|(2|B_3| + B_4) - B_1^2 = \frac{3}{32} > 0$, again using Lemma 2.5, we have

$$\psi_-(c_1, c_2) \leq 2B_1\sqrt{\frac{2|B_3|}{2|B_3| + B_4}} = \frac{1}{\sqrt{10}},$$

as required.

The upper bound is sharp for f_2, defined in (3.8), and for the lower bound, consider the function

$$p_0(z) = \frac{1 + 2t_0 z + z^2}{1 - z^2},$$

where $t_0 = \sqrt{\frac{2}{5}}$. Let

$$w_0(z) = \frac{p_0(z) - 1}{p_0(z) + 1} = \frac{z\big(5z + \sqrt{10}\big)}{5 + \sqrt{10}z} = \frac{1}{5}\sqrt{10}z + \frac{3}{5}z^2 - \frac{3\sqrt{10}}{25}z^3 - \ldots$$

and

$$q_0(z) = \frac{2}{1 + e^{-w_0(z)}}.$$

It is easy to see that $w_0(0) = 0$ and $|w_0(z)| < 1$ for $z \in \mathbb{D}$. Then the function $f_{**} \in \mathcal{S}^*_{SG}$, where

$$f_{**}(z) = z\exp\int_0^z \frac{q_0(t) - 1}{t}dt = z + \frac{1}{\sqrt{10}}z^2 + \frac{1}{5}z^3 - \frac{\sqrt{10}}{225}z^4 + \ldots,$$

so that

$$|A_2| - |A_3| = |-a_2| - |2a_2^2 - a_3| = \frac{1}{\sqrt{10}},$$

which shows that the lower bound is sharp for the function f_{**}.

Theorem 3.4 is proved.

We next give some coefficient bounds for functions in $\mathcal{C}_{SG}$.

Theorem 3.5. *Let $f \in \mathcal{C}_{SG}$ and be given by* (1.1). *Then*

$$|a_2| \le \frac{1}{4}, \qquad |a_3| \le \frac{1}{12}, \qquad |a_4| \le \frac{1}{24}, \qquad |a_5| \le \frac{1}{40}. \tag{3.9}$$

All inequalities are sharp.

Proof. Let $f \in \mathcal{C}_{SG}$. From the definition of subordination, we have

$$1+\frac{zf''(z)}{f'(z)} = \frac{2}{1+e^{-w(z)}}, \tag{3.10}$$

where w is analytic with $w(0)=0$ and $|w(z)|<1$ in $\mathbb{D}$. Thus, for $p \in \mathcal{P}$, we can write

$$w(z) = \frac{p(z)-1}{p(z)+1}.$$

Let p be given by (2.1). Then after some simple calculations

$$\begin{aligned}\frac{2}{1+e^{-w(z)}} &= 1+\frac{1}{4}c_1 z + \left(\frac{1}{4}c_2 - \frac{1}{8}c_1^2\right)z^2 + \left(\frac{1}{4}c_3 - \frac{1}{4}c_1c_2 + \frac{11}{192}c_1^3\right)z^3 \\ &\quad + \left(\frac{-3}{128}c_1^4 + \frac{11}{64}c_1^2c_2 - \frac{1}{4}c_1c_3 + \frac{1}{4}c_4 - \frac{1}{8}c_2^2\right)z^4 + \ldots .\end{aligned}$$

Also

$$\begin{aligned}1+\frac{zf''(z)}{f'(z)} &= 1+2a_2z+\left(6a_3-4a_2^2\right)z^2+\left(12a_4-18a_2a_3+8a_2^3\right)z^3 \\ &\quad + \left(20a_5-32a_2a_4-18a_3^2+48a_3a_2^2-16a_2^4\right)z^4+\ldots .\end{aligned}$$

Substituting the above into (3.10) and equating coefficients, we obtain

$$a_2 = \frac{1}{8}c_1, \tag{3.11}$$

$$a_3 = \frac{1}{24}c_2 - \frac{1}{96}c_1^2, \tag{3.12}$$

$$a_4 = \frac{7}{4608}c_1^3 - \frac{5}{384}c_1c_2 + \frac{1}{48}c_3, \tag{3.13}$$

$$a_5 = -\frac{17}{92160}c_1^4 + \frac{7}{1920}c_2c_1^2 - \frac{1}{120}c_1c_3 - \frac{3}{640}c_2^2 + \frac{1}{80}c_4. \tag{3.14}$$

The bound for $|a_2|$ follows at once using the well-known coefficient bound $|c_1| \le 2$ for class $\mathcal{P}$, and the bound for $|a_3|$ is obtained using Lemma 2.2 with $v = \dfrac{1}{4}$.

For $|a_4|$, consider

$$|a_4| = \frac{1}{48}\left|c_3 - \frac{5}{8}c_1c_2 + \frac{7}{96}c_1^3\right| = \frac{1}{48}\left|c_3 - 2Bc_1c_2 + Dc_1^3\right|,$$

so that in Lemma 2.3, $B = \dfrac{5}{16}$ and $D = \dfrac{7}{96}$. It is clear that $0 \le B \le 1$ and $B(2B-1) \le D \le B$ implies that $-\dfrac{15}{128} \le \dfrac{7}{96} \le \dfrac{5}{16}$. Thus, by Lemma 2.3 we obtain the required result.

For $|a_5|$ we use Lemma 2.4 and rewrite (3.14) as

$$|a_5| = \frac{1}{80}\left|\frac{17}{1152}c_1^4 + \frac{3}{8}c_2^2 + \frac{2}{3}c_1c_3 - \frac{7}{24}c_2c_1^2 - c_4\right|$$

$$= \frac{1}{80}\left|\gamma c_1^4 + ac_2^2 + 2bc_1c_3 - \frac{3}{2}\beta c_2c_1^2 - c_4\right|.$$

Here, $\gamma = \dfrac{17}{1152}$, $a = \dfrac{3}{8}$, $b = \dfrac{1}{3}$ and $\beta = \dfrac{7}{36}$, and so simple calculations give

$$8a(1-a)\big\{(b\beta - 2\lambda)^2 + (b(a+b) - \beta)^2\big\}$$

$$+ b(1-b)(\beta - 2ab)^2 - 4b^2a(1-b)^2(1-a)$$

$$= -955961/23887872 < 0.$$

Since all conditions of the Lemma 2.4 are satisfied, it follows that $|a_5| \le \dfrac{1}{40}$.

To show that the inequalities are sharp, consider the function $f_n : \mathbb{D} \to \mathbb{C}$ defined by

$$f_n(z) = \int_0^z \left(\exp\left(\int_0^x \frac{\frac{2}{1+e^{-t^n}} - 1}{t}\,dt\right)\right)dx, \quad n = 1,2,3,4. \tag{3.15}$$

Then clearly $f_n \in \mathcal{C}_{SG}$, and simple calculations show that the inequalities are sharp by taking $n = 1,2,3,4$, which completes the proof of the theorem.

Theorem 3.6. *Let $f \in \mathcal{C}_{SG}$ and be given by* (1.1). *Then*

$$-\frac{5}{24} \le |a_3| - |a_2| \le \frac{1}{12}.$$

Both inequalities are sharp.

Proof. We use Lemma 2.5, so that, from (3.11) and (3.12), we have

$$\psi_+(c_1,c_2) = |a_3| - |a_2| = \left|\frac{1}{24}c_2 - \frac{1}{96}c_1^2\right| - \left|\frac{1}{8}c_1\right| = |B_2c_1^2 + B_3c_2| - |B_1c_1|,$$

where $B_1 = \dfrac{1}{8}$, $B_2 = -\dfrac{1}{96}$ and $B_3 = \dfrac{1}{24}$. Now $|2B_2 + B_3| = \dfrac{1}{48}$, and $|B_3| + B_1 = \dfrac{1}{6}$. This shows that $|2B_2 + B_3| \ngeq |B_3| + B_1$. Hence, from Lemma 2.5, we have

$$\psi_+(c_1,c_2) = |a_3| - |a_2| \le 2|B_3| = \frac{1}{12}.$$

For the lower bound, we see that

$$\psi_-(c_1,c_2) = -\psi_+(c_1,c_2) = |a_2| - |a_3|.$$

Since $B_1 - 2|B_3| - B_4 = 0$, Lemma 2.5 gives

$$\psi_-(c_1, c_2) \le 2B_1 - B_4 = \frac{5}{24},$$

as required.

The upper bound is sharp for f_2 defined in (3.15), and the lower bound is sharp for f_1 defined in (3.15).

Theorem 3.6 is proved.

Theorem 3.7. *Let* $f \in \mathcal{C}_{SG}$ *and be given by* (1.1). *Then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{24}\begin{cases} \dfrac{1}{2}(2-3\mu), & \mu < -\dfrac{2}{3}, \\ 2, & -\dfrac{2}{3} \le \mu \le 2, \\ \dfrac{1}{2}(3\mu - 2), & \mu > 2. \end{cases}$$

All inequalities are sharp.

Proof. Using (3.11) and (3.12), we have

$$\left|a_3 - \mu a_2^2\right| = \frac{1}{24}\left|c_2 - \frac{1}{8}(3\mu + 2)c_1^2\right|,$$

and the result follows from Lemma 2.2. The result is sharp for the functions f_1 when $\mu < -\dfrac{2}{3}$ or $\mu > 2$ and f_2 when $-\dfrac{2}{3} \le \mu \le 2$, defined in (3.15) for $n = 1, 2$.

Note that when $\mu = 1$, we have $H_{2,1}(f)$ and so we deduce the following corollary.

***Corollary* 3.1.** *Let* $f \in \mathcal{C}_{SG}$ *and be given by* (1.1). *Then*

$$|a_3 - a_2^2| \le \frac{1}{12}.$$

The inequality is sharp for the function f_2 *defined by* (3.15).

Theorem 3.8. *Let* $f \in \mathcal{C}_{SG}$ *and be given by* (1.1). *Then*

$$|a_2 a_3 - a_4| \le \frac{1}{24}.$$

The inequality is sharp.

Proof. We use Lemma 2.3 and, from (3.11), (3.12) and (3.13), obtain

$$|a_2 a_3 - a_4| = \frac{1}{48}\left|c_3 - \frac{7}{8}c_1 c_2 + \frac{13}{96}c_1^3\right| = \frac{1}{48}\left|c_3 - 2Bc_1 c_2 + Dc_1^3\right|,$$

where $B = \dfrac{7}{16}$ and $D = \dfrac{13}{96}$. It is clear that $0 \le B \le 1$ and the relation $B(2B-1) \le D \le B$ implies that $-\dfrac{7}{128} \le \dfrac{13}{96} \le \dfrac{7}{16}$. Then, from Lemma 2.3, we obtain the required result. The inequality is sharp for the function f_3 defined by (3.15).

Theorem 3.8 is proved.

We end this section, by finding a sharp bound for the Zalcman functional when $n = 3$.

Theorem 3.9. *Let* $f \in \mathcal{C}_{SG}$ *and be given by* (1.1). *Then*

$$|a_3^2 - a_5| \leq \frac{1}{40}.$$

The inequality is sharp.

Proof. We use Lemma 2.4. By (3.12) and (3.14) we have

$$|a_3^2 - a_5| = \frac{1}{80}\left|\frac{3}{128}c_1^4 + \frac{37}{72}c_2^2 + \frac{2}{3}c_1c_3 - \frac{13}{36}c_2c_1^2 - c_4\right|$$

$$= \frac{1}{80}\left|\gamma c_1^4 + ac_2^2 + 2bc_1c_3 - \frac{3}{2}\beta c_2c_1^2 - c_4\right|.$$

Here, $\gamma = \dfrac{3}{128}$, $a = \dfrac{37}{72}$, $b = \dfrac{1}{3}$ and $\beta = \dfrac{13}{54}$, and simple computations give

$$8a(1-a)\big\{(b\beta - 2\lambda)^2 + (b(a+b) - \beta)^2\big\}$$

$$+\, b(1-b)(\beta - 2ab)^2 - 4b^2a(1-b)^2(1-a)$$

$$= -719977369/17414258688 < 0.$$

Since all conditions of the Lemma 2.4 are satisfied, it follows that $|a_3^2 - a_5| \leq \dfrac{1}{40}$. The inequality is sharp for f_4 given in (3.15), which is equivalent to choosing $a_3 = 0$ and $a_5 = \dfrac{1}{40}$, which completes the proof.

Theorem 3.10. *Let* $f \in \mathcal{C}_{SG}$ *and be given by* (1.1). *Then*

$$|A_2| \leq \frac{1}{4}, \qquad |A_3| \leq \frac{1}{12}, \qquad |A_4| \leq \frac{1}{24}.$$

All inequalities are sharp.

Proof. Since $f\big(f^{-1}(w)\big) = w$, using (1.3) it is easy to see that

$$A_2 = -a_2,$$

$$A_3 = 2a_2^2 - a_3,$$

$$A_4 = -5a_2^3 + 5a_2a_3 - a_4.$$

From (3.11) – (3.13), we obtain

$$A_2 = \frac{-1}{8}c_1, \qquad A_3 = \frac{1}{48}c_1^2 - \frac{1}{24}c_2, \qquad A_4 = -\frac{41}{2304}c_1^3 + \frac{5}{128}c_2c_1 - \frac{1}{48}c_3.$$

The first bound follows at once from the inequality $|c_1| \leq 2$. For $|A_3|$, we have

$$|A_3| = \frac{1}{24}\left|c_2 - \frac{1}{2}c_1^2\right|,$$

and, using Lemma 2.2 with $v = \dfrac{1}{2}$, we obtain the required result.

For $|A_4|$, we use Lemma 2.3 so that

$$|A_4| = \frac{1}{48}\big|c_3 - 2Bc_1c_2 + Dc_1^3\big|,$$

where $B = \dfrac{15}{16}$ and $D = \dfrac{41}{48}$. It is easy to see that $0 \le B \le 1$ and $B(2B-1) \le D \le B$ are satisfied and so, from Lemma 2.3, we have $|A_4| \le \dfrac{1}{24}$. The inequalities are sharp for the function $f_n,\ n = 1, 2, 3$ defined in (3.15).

Theorem 3.10 is proved.

Theorem 3.11. *Let* $f \in \mathcal{C}_{SG}$ *and be given by* (1.1). *Then*

$$-\frac{1}{4} \le |A_3| - |A_2| \le \frac{1}{12}.$$

Proof. We again use Lemma 2.5, so that, from (3.11) and (3.12), we have

$$\psi_+(c_1, c_2) = |A_3| - |A_2| = \left|\frac{1}{48}c_1^2 - \frac{1}{24}c_2\right| - \left|\frac{1}{8}c_1\right| = \big|B_2c_1^2 + B_3c_2\big| - |B_1c_1|,$$

where $B_1 = \dfrac{1}{8},\ B_2 = \dfrac{1}{48}$ and $B_3 = -\dfrac{1}{24}$. Now $|2B_2 + B_3| = 0$ and $|B_3| + B_1 = \dfrac{1}{6}$ and so $|2B_2 + B_3| \not\ge |B_3| + B_1$. Hence, from Lemma 2.5, we obtain

$$\psi_+(c_1, c_2) = |A_3| - |A_2| \le 2|B_3| = \frac{1}{12}.$$

For the lower bound we see that

$$\psi_-(c_1, c_2) = -\psi_+(c_1, c_2) = |A_2| - |A_3|.$$

Since $B_1 - 2|B_3| - B_4 = \dfrac{1}{24} > 0$, Lemma 2.5, gives

$$\psi_-(c_1, c_2) \le 2B_1 - B_4 = \frac{1}{4},$$

as required.

The upper bound is sharp for the function f_2 defined in (3.15). The lower bound is sharp for the function f_1 defined in (3.15).

Theorem 3.11 is proved.

4. Logarithmic coefficients for the classes $\mathcal{S}_{SG}^*$ and $\mathcal{C}_{SG}$.

Theorem 4.1. *Let* $f \in \mathcal{S}_{SG}^*$ *and be given by* (1.1). *Then*

$$|\beta_n| \le \frac{1}{4n}, \quad n = 1, 2, 3, 4.$$

These bounds are sharp.

Proof. Differentiating (1.2) and comparing the coefficients gives

$$\beta_1 = \frac{a_2}{2}, \qquad \beta_2 = \frac{1}{4}(2a_3 - a_2^2), \qquad \beta_3 = \frac{1}{6}(3a_4 - 3a_2a_3 + a_2^3), \tag{4.1}$$

$$\beta_4 = \frac{1}{8}(4a_5 - 4a_2a_4 - 2a_3^2 + 4a_2^2a_3 - a_2^4). \tag{4.2}$$

Substituting (3.3) – (3.6) into (4.1) and (4.2), we obtain

$$\beta_1 = \frac{1}{8}c_1, \qquad \beta_2 = \frac{-1}{32}c_1^2 + \frac{1}{16}c_2, \tag{4.3}$$

$$\beta_3 = \frac{11}{1152}c_1^3 - \frac{1}{24}c_1c_2 + \frac{1}{24}c_3, \tag{4.4}$$

$$\beta_4 = d\frac{-3}{1024}c_1^4 + \frac{11}{512}c_2c_1^2 - \frac{1}{32}c_1c_3 - \frac{1}{64}c_2^2 + \frac{1}{32}c_4. \tag{4.5}$$

The bound for $|\beta_1|$ follows at once from the well-known coefficient estimate $|c_1| \leq 2$ for the class $\mathcal{P}$. From (4.3), we can write

$$\beta_2 = \frac{1}{16}\big(c_2 - vc_1^2\big),$$

where $v = \dfrac{1}{2}$, and applying Lemma 2.2 we obtain the required bound for $|\beta_2|$. Next using (4.4), we have

$$|\beta_3| = \frac{1}{24}|c_3 - 2Bc_1c_2 + Dc_1^3|,$$

where $B = \dfrac{1}{2}$ and $D = \dfrac{11}{48}$. It is easy to see that $0 \leq B \leq 1$ and $B(2B-1) \leq D \leq B$. Thus, from Lemma 2.3, we get $|\beta_3| \leq \dfrac{1}{12}$. For $|\beta_4|$, from (4.5) we can write

$$|\beta_4| = \frac{1}{32}\left|\frac{3}{32}c_1^4 + \frac{1}{2}c_2^2 + c_1c_3 - \frac{11}{16}c_1^2c_2 - c_4\right| = \frac{1}{32}\left|\lambda c_1^4 + ac_2^2 + 2bc_1c_3 - \frac{3}{2}\beta c_1^2c_2 - c_4\right|.$$

Here, $\lambda = \dfrac{3}{32}$, $a = \dfrac{1}{2}$, $b = \dfrac{1}{2}$, $\beta = \dfrac{11}{24}$ and simple computations give

$$8a(1-a)\big\{(b\beta-2\lambda)^2+(b(a+b)-\beta)^2\big\}+b(1-b)(\beta-2ab)^2-4b^2a(1-b)^2(1-a) = -127/2304 < 0.$$

Since all conditions of the Lemma 2.4 are satisfied, it follows that $|\beta_4| \leq \dfrac{1}{8}$.

To see that these inequalities are sharp, consider the function $f_n : \mathbb{D} \to \mathbb{C}$ defined by

$$f_n(z) = z\exp\int\limits_0^z \frac{1-e^{-t^n}}{t(1+e^{-t^n})}dt, \quad n = 1,2,3,4.$$

Then $f_n \in \mathcal{S}_{SG}^*$ and simple calculations show that the inequalities are sharp taking $n = 1,2,3,4$, which completes the proof.

Theorem 4.2. *Let* $f \in \mathcal{C}_{SG}$ *and be given by* (1.1). *Then*

$$|\beta_1| \le \frac{1}{8}, \qquad |\beta_2| \le \frac{1}{24}, \qquad |\beta_3| \le \frac{1}{48}, \qquad |\beta_4| \le \frac{1}{80}.$$

All bounds are sharp.

Proof. Differentiating (1.2) and comparing the coefficients gives

$$\beta_1 = \frac{a_2}{2}, \qquad \beta_2 = \frac{1}{4}(2a_3 - a_2^2), \qquad \beta_3 = \frac{1}{6}(3a_4 - 3a_2a_3 + a_2^3), \tag{4.6}$$

$$\beta_4 = \frac{1}{8}(4a_5 - 4a_2a_4 - 2a_3^2 + 4a_2^2a_3 - a_2^4). \tag{4.7}$$

Substituting (3.11)–(3.14) into (4.6) and (4.7), we obtain

$$\beta_1 = \frac{1}{16}c_1, \qquad \beta_2 = \frac{1}{48}c_2 - \frac{7}{768}c_1^2, \tag{4.8}$$

$$\beta_3 = \frac{1}{576}c_1^3 - \frac{7}{768}c_1c_2 + \frac{1}{96}c_3, \tag{4.9}$$

$$\beta_4 = \frac{-209}{1474560}c_1^4 + \frac{293}{92160}c_2c_1^2 - \frac{7}{1280}c_1c_3 - \frac{1}{360}c_2^2 + \frac{1}{160}c_4. \tag{4.10}$$

The bound for $|\beta_1|$ follows at once from the well-known coefficient estimate $|c_1| \le 2$ for the class $\mathcal{P}$. From (4.8), we can write

$$\beta_2 = \frac{1}{48}\big(c_2 - vc_1^2\big),$$

where $v = \dfrac{7}{16}$, and applying Lemma 2.2, we obtain the required bound for $|\beta_2|$. Next using (4.9), we obtain

$$|\beta_3| = \frac{1}{96}|c_3 - 2Bc_1c_2 + Dc_1^3|,$$

where $B = \dfrac{7}{16}$ and $D = \dfrac{1}{6}$. It is easy to see that $0 \le B \le 1$ and $B(2B-1) \le D \le B$. Thus, from Lemma 2.3, we have $|\beta_3| \le \dfrac{1}{48}$. For $|\beta_4|$, from (4.10) we can write

$$|\beta_4| = \frac{1}{160}\left|\frac{209}{9216}c_1^4 + \frac{4}{9}c_2^2 + \frac{7}{8}c_1c_3 - \frac{293}{576}c_1^2c_2 - c_4\right|$$

$$= \frac{1}{160}|\lambda c_1^4 + ac_2^2 + 2bc_1c_3 - \frac{3}{2}\beta c_1^2c_2 - c_4|.$$

Here, $\lambda = \dfrac{209}{9216}$, $a = \dfrac{4}{9}$, $b = \dfrac{7}{16}$ $\beta = \dfrac{293}{864}$ and simple computations give

$$8a(1-a)\big\{(b\beta - 2\lambda)^2 + (b(a+b) - \beta)^2\big\}$$

$$+ b(1-b)(\beta - 2ab)^2 - 4b^2a(1-b)^2(1-a)$$

$$= -113732233/15479341056 < 0.$$

Since all conditions of the Lemma 2.3 are satisfied, it follows that $|\beta_4| \le \dfrac{1}{80}$.

Simple calculations show that inequalities are sharp for the function f_n defined by (3.15) taking $n = 1, 2, 3, 4,$ which completes the proof.

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