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THE CONCEPT OF TOPOLOGICAL WELL-ORDERED SPACE

ПОНЯТТЯ ТОПОЛОГІЧНОГО ДОБРЕ ВПОРЯДКОВАНОГО ПРОСТОРУ

Since the general definition of topology is based on the characteristics of the standard Euclidean topology, the relationships between the ordering on real numbers and its topology have been generalized over time and studied in numerous aspects. The compatibility of partially ordered sets with the topology on them was studied by many researchers. On the other hand, well-orderedness is an important concept of the set theory. We define the concept of topological well-orderedness, which can be regarded as a topological generalization of well-orderedness in the set theory, and analyze its basic properties. In this way, the relationship between well-orderedness and topology is established from a different point of view. Finally, some basic applications of the concept of topological well-orderedness to the graph theory are investigated.

Оскільки загальне визначення топології ґрунтується на характеристиках стандартної евклідової топології, зв'язки між упорядкуванням дійсних чисел та відповідною топологією були з часом узагальнені та вивчені з багатьох точок зору. Сумісність частково впорядкованих множин з топологією на них вивчалась багатьма дослідниками. З іншого боку, строга впорядкованість є важливим поняттям теорії множин. Визначено поняття топологічної впорядкованості, яку можна розглядати як топологічне узагальнення поняття строгої впорядкованості в теорії множин, та досліджено її основні властивості. Таким чином, зв'язок між строгою впорядкованістю і топологією був встановлений з іншої точки зору. Крім того, досліджено деякі основні застосування поняття топологічної впорядкованості до теорії графів.

1. Introduction and preliminaries. The comparison of two phenomena almost extends to the first people. Its origins are based on logic, laws of thought and mathematics. The concept of order relation (or partial order relation) puts the comparison of two cases or two things in a mathematical framework. An order relation (or a partial order relation) is a relation which satisfies reflexivity, transitivity and antisymmetric properties. Formally, let X be a nonempty set. The Cartesian product of X itself is defined and denoted by the set $X \times X = \{(x,y) \mid x, y \in X\}$. Any subset β of $X \times X$ is called a binary relation on X. If a relation β on X satisfies the following properties, it is called a partial order relation on X:

(i) $(x,x) \in \beta$ for all $x \in X$,

(ii) if $(x,y) \in \beta$ and $(y,z) \in \beta$, then $(x,z) \in \beta$ for each $x,y,z \in X$,

(iii) if $(x,y) \in \beta$ and $(y,x) \in \beta$, then $x = y$ for each $x,y \in X$.

If β is a partial order relation on X, then the pair (X,β) is called a partial ordered set (or poset, briefly). Often, the partial order relation β on X is denoted by the symbol $\leq$, and so (X,β) is denoted by $(X,\leq)$. If $\leq$ is a partial order relation on X and $(x,y) \in \leq$, then it is called that x precedes y or x is less than or equal y and denoted by $x \leq y$. Let $(X,\leq)$ be a poset. If $x \leq y$ or $y \leq x$, then x and y are called comparable elements in X. A poset in which every pair of elements are comparable is called a totally ordered set (or toset, briefly). If any poset $(X,\leq)$ has an element that is smaller than all elements, this element is called the minimum element. Dually, if all elements are less than or equal one element, this element is called maximum element. The smallest among a

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set of comparable elements is called a minimal element in the poset, and the largest among a set of comparable elements is called a maximal element. If every nonempty subset of a poset has a minimum element, then it is called well-ordered set. The following theorem which has a very important place in set theory is given by Zermelo.

Theorem 1.1 (Zermelo's theorem (well-ordering theorem))**.** *Every set can be well-ordered.*

As is well-known, this theorem is equivalent to the axiom of choice, the Hausdorff maximality principle and the Zorn lemma.

The concept of an up-set of a poset $(X, \leq)$ is defined as a subset U of X such that if $x \in U$ and $x \leq y$, then $y \in U$. Dually, a down-set D is defined as a subset of X such that if $x \in D$ and $y \leq x$, then $y \in D$. Let $\varnothing \neq I \subseteq X$. If I is a down-set and, for every $x, y \in I$, there exists some element $z \in I$ such that $x \leq z$ and $y \leq z$, then I is called an ideal of X. The set $\downarrow x = \{y \in X \mid y \leq x\}$ is called a principle ideal of X for each $x \in X$.

For any poset $(X, \leq)$, if for any sequence $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$ in X there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = \ldots$, then X satisfies the ascending chain conditions (ACC) and the dual of the ACC is the descending chain condition (DCC). Let $(X, \leq_1)$ and $(Y, \leq_2)$ be two posets, and $f: X \to Y$ be a function. We say f is monotone (increasing) if $x \leq_1 y$, then $f(x) \leq_2 f(y)$ for each $x, y \in X$. If f is bijective and satisfies the condition; $x \leq_1 y \Leftrightarrow f(x) \leq_2 f(y)$, it is called an order isomorphism from X to Y, and X and Y are called order isomorphic.

We know that topology is one of the most important subfields of mathematics. Formally, a topology $\mathcal{T}$ is a collection of subsets of a nonempty set which satisfies following conditions:

(i) $\varnothing$ and X are in $\mathcal{T}$,

(ii) the intersection of a subfamily of $\mathcal{T}$ containing a finite number of elements is in $\mathcal{T}$,

(iii) the union of a subfamily of $\mathcal{T}$ containing an arbitrary number of elements is in $\mathcal{T}$.

Each element of $\mathcal{T}$ is called an open set. Any set is called closed set if its complement is open. We recommend [5, 12, 14] for all basic information about topology.

In general, any topological space $(X, \mathcal{T})$ satisfies the property that a finite intersection of open sets is open, but any arbitrary intersection of open sets need not be open. Topological spaces where arbitrary intersections of open sets are open were studied by Alexandroff for the first time in [1]. The topological spaces that fulfill this mentioned condition are called Alexandroff spaces. In [1], Alexandroff gave examples of Alexandroff spaces on a poset $(X, \leq)$ that accept the families $\mathcal{B} = \{\uparrow x \mid x \in X\}$ or $\mathcal{B}' = \{\downarrow x \mid x \in X\}$ as a subbase. The Alexandroff topology generated by $\mathcal{B}$ is denoted by $\mathcal{T}(\leq^{\uparrow})$ and the Alexandroff topology generated by $\mathcal{B}'$ is denoted by $\mathcal{T}(\leq^{\downarrow})$. A relationship has been established between topological spaces and posets.

Many scientists have been studying the principal relations of interdependence between a topology and an order. Some of these studies are given in [6, 8, 15, 16, 18]. In [7], Engelking et al. defined the concept of topologically well-ordered space given any linearly ordered space which is a linear ordering, equipped with the usual order topology. They investigated some basic properties. In [2, 9, 10], the authors studied relationship between the concept of selection and topologically well-ordered spaces.

In this paper, our focus is on the relationship between the concept of well-ordering and topology, from a different perspective than [7]. In our study, we give the concept of topologically well-orderness, which can be seen as a topological extension of well-order, using it together with the topology on a poset. We define this by the fact that the open sets of the topology have minimums with respect to the partial order relation. By our definition, this concept will be slightly different from the well-ordered set concept. Therefore, we regard that topological well-ordered spaces different from well-ordered sets, or even as a special extension. After making some basic definitions, we examine the basic properties of topological well-ordered spaces. Then we examine the relationships between well-orderness and topological well-orderness. We give two basic ways of making a poset topologically well-ordered on which an arbitrary topology is defined. We examine the relationships between topological well-orderness and some special topological spaces such as Alexandroff, Fort and Fortissimo spaces. Finally, we define the concept of basic well-orderedness using bases of topological space and give some results.

As it is known that there is a direct relationship between posets and graphs, and we can make each poset correspond to a graph. Therefore, in the last part of the study, some basic applications of topological well-ordered spaces to graph theory are given.

2. Topological well-ordered spaces.

Definition 2.1. *Let $(X,\mathcal{T})$ be a topological space and $\leq$ be a partial order relation on X. The triplet $(X,\mathcal{T},\leq)$ is called a quasitopological well-ordered space or briefly q-TWO space (or just q-TWO) if $U\in\mathcal{T}$ and $\varnothing\neq U\neq X$. Then U has a minimum element.*

***Example* 2.1.** The Sierpinski space with any partial order relation on it is a q-TWO.

***Example* 2.2.** Let $\mathbb{R}$ be a set of reals with the topology $\mathcal{T}=\{\varnothing,\mathbb{R},\{0\}\}$ on it. The topological space is a q-TWO with respect to usual order relation on $\mathbb{R}$.

***Example* 2.3.** Let $(\mathbb{R},\mathcal{U})$ be a usual space with usual ordering on it. Then it is not a q-TWO.

Let $(X,\leq)$ be a poset. Suppose that $\mathcal{I}$ is the indiscrete topology on X. Then X is q-TWO, obviously. Then we can say that every partially ordered set has a topology on it which makes it q-TWO.

Definition 2.2. *Let $(X,\mathcal{T})$ be a topological space and $\leq$ be a partial order relation on X. The triplet $(X,\mathcal{T},\leq)$ is called a topological well-ordered space or briefly TWO space (or just TWO) if each nonempty open set in a topological space $(X,\mathcal{T})$ has a minimum element.*

***Example* 2.4.** Let $X=\{a,b,c,d\}$, $\mathcal{T}=\big\{\varnothing,X,\{a,b\},\{a,c\},\{a\},\{a,b,c\}\big\}$, and $\leq=\big\{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(b,c),(a,d)\big\}$. Then X is a topological well-ordered space.

***Example* 2.5.** Let $\mathbb{N}$ be the set of natural numbers, and define the topology $\mathcal{T}=\{\{1,\ldots,n\}\mid n\in\mathbb{N}\}\cup\{\varnothing\}$ on $\mathbb{N}$. $(\mathbb{N},\mathcal{T})$ is TWO.

***Example* 2.6.** One of the special topologies defined on an arbitrary set X is

$$\mathcal{T}_a=\big\{U\subseteq X\mid a\in U\ \text{ or }\ U=\varnothing\big\}$$

called the particular point topology. It is clearly obtained that $(X,\mathcal{T}_a,\leq)$ is TWO if there exist a minimum element of X and $\min X=a$ for this topological space and a partial order relation $\leq$ defined on X.

Recall from the set theory, in fact, a well-ordered set is a totally ordered set that has a minimum element of each subset. Note that, as can be seen from the examples above, an underlying set of a TWO space does not have to be a totally ordered set.

Munkres stated in [14] that a standard topology can be obtained by using the partial order relation on a partially ordered set. Let X be a poset which has more than one element, $(a,b)=\{x\mid a<$

$x < b\}$ open interval, $[a, b] = \{x \mid a \leq x \leq b\}$ closed interval and $[a, b) = \{x \mid a \leq x < b\}$ and $(a, b] = \{x \mid a < x \leq b\}$ half-open intervals determined by $a, b \in X$. Suppose that $\mathcal{B}$ is the family of all sets following types:

(1) all open intervals (a, b) in X,

(2) all intervals of the form $[a_0, b)$, where a_0 is the minimum element (if any) of X,

(3) all intervals of the form $(a, b_0]$, where b_0 is the maximum element (if any) of X.

Then $\mathcal{B}$ is a basis for a topology on X and it is called *order topology*.

We know that the order topology obtained according to the usual order relation on $\mathbb{R}$ is the known usual topology. From Example 2.3 we stated that this space is neither q-TWO nor TWO. An ordered set with the order topology may or may not be TWO. However, the example below shows that an ordered set with the order topology may be TWO.

***Example* 2.7** [14]. The positive integers $\mathbb{Z}^+$ form an ordered set with a minimum element with respect to usual order on it. The order topology on $\mathbb{Z}^+$ is the discrete topology. So, this space is a TWO, clearly.

Suppose that $(X, \leq)$ is a poset. The partial order relation defined in the form

$$x \leq_d y \Leftrightarrow y \leq x \quad \forall x, y \in X,$$

is called the dual of $\leq$. Clearly, $(X, \leq_d)$ is a poset [3, 11]. Therefore, for a poset $(X, \leq)$ with a topology $\mathcal{T}$ on it; we can clearly express that $(X, \mathcal{T}, \leq_d)$ is a TWO if each nonempty open subset of X has a maximum element.

We clearly get that every TWO is a q-TWO from Definitions 2.1 and 2.2.

The reverse of this inference is not true.

***Example* 2.8.** Let $\mathbb{R}$ be a set of reals with usual partial order relation on it. Consider the topology $\mathcal{T} = \{\varnothing, \mathbb{R}, \{0\}\}$. So, $(\mathbb{R}, \mathcal{T}, \leq)$ is a q-TWO but not TWO.

***Remark* 2.1.** A subspace of a space which is not q-TWO or TWO space can be q-TWO or TWO.

***Example* 2.9.** Let $X = \{a, b, c, d, e\}$ be a poset with Hasse diagram is as Fig. 1.

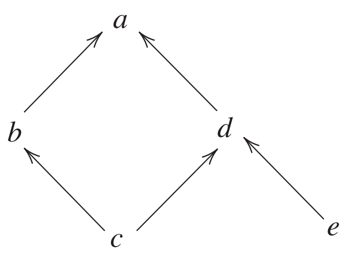

Fig. 1. Hasse diagram of partial order relation on X.

Define the topology $\mathcal{T} = \{\varnothing, X, \{c, e\}, \{c, d\}, \{c\}, \{c, d, e\}\}$. $(X, \mathcal{T})$ is not q-TWO but if we take the subset $A = \{c, d\}$, we obtain the subspace topology $\mathcal{T}_A = \{\varnothing, A, \{c\}\}$. Hence, $(A, \mathcal{T}_A, \leq_A)$ is a TWO. Moreover, consider the subset $B = \{c, e\}$ and its subspace topology $\mathcal{T}_B = \{\varnothing, B, \{c\}\}$. Then $(B, \mathcal{T}_B, \leq_B)$ is a q-TWO but not TWO.

We can give the following example that the space is TWO but its subspace is not.

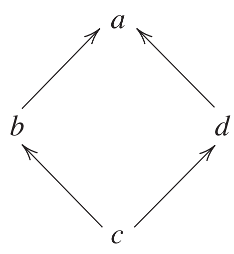

Fig. 2. Hasse diagram of partial order relation on X.

***Example* 2.10.** Let $X = \{a, b, c, d\}$ be a poset with Hasse diagram as in Fig. 2.

Define the topology $\mathcal{T} = \{\varnothing, X, \{b, c, d\}\}$. The space $(X, \mathcal{T}, \leq)$ is obviously TWO. If we take the subset $A = \{b, d\}$, then we have the subspace topology $\mathcal{T}_A = \{\varnothing, A\}$. However, $(A, \mathcal{T}_A, \leq_A)$ is not a TWO.

Theorem 2.1. *If $(X, \mathcal{T}, \leq)$ is a TWO and $A \subseteq X$ is open, then $(A, \mathcal{T}_A, \leq_A)$ is a TWO.*

Proof. Suppose that $U \in \mathcal{T}_A$. Then there exists $V \in \mathcal{T}$ such that $U = A \cap V$. Since $A \in \mathcal{T}$, then we obtain that $A \cap V = U \in \mathcal{T}$. By the hypothesis $(X, \mathcal{T})$ is a TWO, then $U \in \mathcal{T}$ has a minimum element. Hence, $(A, \mathcal{T}_A)$ is a TWO.

Theorem 2.2. *Let $(X, \mathcal{T}, \leq)$ be a TWO and A be a nonempty subset of X. If A is a down-set, then $(A, \mathcal{T}_A, \leq_A)$ is a TWO.*

Proof. Suppose that $U \in \mathcal{T}_A$ and $a \in U$. Since $U \in \mathcal{T}_A$, there exists $V \in \mathcal{T}$ such that $U = A \cap V$. So, $a \in A$ and $a \in V$. Since $(X, \mathcal{T})$ is TWO, V has minimum element m such that $m \leq a$. Since A is a down-set, then $m \in A$. Thus, $m \in A \cap V = U$. $U \subseteq V$, then m is the minimum element of U. Hence, we obtain that $(A, \mathcal{T}_A, \leq_A)$ is a TWO.

From Theorem 2.2, obviously, we obtain following result.

***Corollary* 2.1.** *Let $(X, \mathcal{T}, \leq)$ be a TWO and A be a nonempty subset of X. If A is an order ideal, then $(A, \mathcal{T}_A, \leq_A)$ is a TWO.*

Theorem 2.3. *Let $(X, \mathcal{T})$ be a topological space and $\leq$ be a partial order relation on X. If X has a minimum and each nonempty open set is a down-set, then $(X, \mathcal{T}, \leq)$ is a TWO.*

Proof. Consider arbitrary nonempty open set U in X. From hypothesis, since U is a down-set and X has the minimum element x_0, then we obtain that $x_0 \in U$. Thus, $(X, \mathcal{T}, \leq)$ is a TWO.

Theorem 2.4. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. If $(X, \mathcal{T}_2, \leq)$ is a TWO and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $(X, \mathcal{T}_1, \leq)$ is also TWO.*

Proof. It is straightforward.

Theorem 2.5. *Let $(X, \leq)$ and $(Y, \leq')$ be two posets, $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces $f : X \to Y$ be a monotone and continuous function. If $(X, \mathcal{T})$ is a TWO, then $\left(f[X], \mathcal{T}'_{f[X]}, \leq_{f[X]}\right)$ is a TWO.*

Proof. Take any nonempty open set U in the space $\left(f[X], \mathcal{T}'_{f[X]}\right)$. Since f is continuous and $(X, \mathcal{T})$ is TWO, then $f^{-1}[U] \in \mathcal{T}$ and there exists the minimum element m of $f^{-1}[U]$, i.e., $m \leq x$ for each $x \in f^{-1}[U]$. Since f is monotone, then we have that $f(m) \leq' f(x)$ and $f(m), f(x) \in U$. Since $f(x)$ is arbitrary, $f(m)$ is the minimum element of U. Since $U \in \mathcal{T}'_{f[X]}$ is arbitrary and it has a minimum element, finally $\left(f[X], \mathcal{T}'_{f[X]}, \leq_{f[X]}\right)$ is a TWO.

***Corollary* 2.2.** *$(X,\leq)$ and $(Y,\leq')$ be two posets, $(X,\mathcal{T})$ and $(Y,\mathcal{T}')$ be topological spaces $f: X\to Y$ be a monotone and continuous surjection. If $(X,\mathcal{T},\leq)$ is a TWO, then $(Y,\mathcal{T}',\leq')$ is a TWO.*

Theorem 2.6. *Let $(X,\leq)$ and $(Y,\leq')$ be two posets, $(X,\mathcal{T})$ and $(Y,\mathcal{T}')$ be topological spaces $f: X\to Y$ be an order isomorphism and open function. If $(Y,\mathcal{T}',\leq')$ is a TWO, then $(X,\mathcal{T},\leq)$ is a TWO.*

Proof. Consider arbitrary nonempty open set U in the space $(X,\mathcal{T})$. Since f is open, then $f[U]$ is open in Y. From hypothesis $(Y,\mathcal{T}',\leq')$ is TWO. So there exists $m\in f[U]$ such that $m\leq' y$ for each $y\in f[U]$. Since f is an order isomorphism, there exist $m_0, x\in U$ such that $m_0\leq x$, $f(m_0)=m,\ f(x)=y$, and m_0 is the minimum element in U. Hence, $(X,\mathcal{T},\leq)$ is a TWO.

Let X and Y be topological spaces and $f: X\to Y$ be surjective function. f is called a quotient map if it satisfies the following condition:

$$U\subseteq Y \text{ is open } \Leftrightarrow f^{-1}[U]\subseteq X \text{ is open.}$$

Assume that $(X,\mathcal{T})$ is a topological space, Y is a nonempty set and $f: X\to Y$ is a surjective function. Thereby, there exists a topology σ on Y which f is a quotient map, and this topology is called a quotient topology [14]. We can give the following result using these arguments.

Theorem 2.7. *Let $f: X\to Y$ be a quotient map and monotone, $\leq$ be a partial order relation on X and $\leq'$ be a partial order relation on Y. If $(X,\mathcal{T},\leq)$ is a TWO, then the quotient space $(Y,\sigma,\leq')$ is a TWO.*

Theorem 2.8. *Every finite totally ordered set with a topology is TWO.*

Proof. It is obvious.

Theorem 2.9. *Let X be a poset. The discrete space $(X,\mathcal{P}(X),\leq)$ is TWO iff X is well-ordered set.*

Proof. It is obvious.

We know that every set can be well ordered from *Well-Ordering Principle*. Now, from this point of view, let us show that each set with a topology can be topologically well ordered. For this, we can give the following theorem whose proof is obvious.

Theorem 2.10. *Let X be a poset and $\mathcal{T}$ be a topology on X. If X is well ordered, then $(X,\mathcal{T},\leq)$ is a TWO.*

Because of Theorem 2.10, every well-ordered sets with a topology is a TWO. As a result of Theorem 2.10 and Well-Ordering Principle we have following corollary.

***Corollary* 2.3.** *All sets with a topology can be topologically well-ordered.*

***Example* 2.11.** Let $\mathcal{T}$ be an arbitrary topology on $\mathbb{Z}$. Since there exists no minimum element of $\mathbb{Z}$, then $(\mathbb{Z},\mathcal{T},\leq)$ is not TWO. However, if we define the order relation

$$x\leq' y \iff (|x|<|y|) \qquad \text{or} \qquad (|x|=|y| \text{ and } x\leq y),$$

then $(\mathbb{Z},\leq')$ is well-ordered. Hence, from Theorem 2.10, $(\mathbb{Z},\mathcal{T},\leq')$ is TWO.

Obtained TWOs from Theorem 2.10, Corollary 2.3, Example 2.11 is called topological well-ordering (or TWOing, briefly) of first space with a partial order.

Moreover, we obtain following remark.

***Remark* 2.2.** Let $(X, \leq)$ be a poset and $(X, \mathcal{T})$ be a topological space. If $m \in X$ is the minimum element, then we can derive a

$$\mathcal{T}^m = \{U \cup \{m\} \mid U \in \mathcal{T}\} \cup \{\varnothing\}$$

from $\mathcal{T}$ so that $(X, \mathcal{T}^m, \leq)$ is a TWO.

Moreover, suppose that X does not have a minimum element. Define a new set $X^* = X \cup \{*\}$ such that $* \leq^* x$ for each $x \in X$ and $x \leq^* y \Leftrightarrow x \leq y$ for each $x, y \in X$. So, we can achieve

$$\mathcal{T}^* = \{U \cup \{*\} \mid U \in \mathcal{T}\} \cup \{\varnothing\}$$

which is called closed extention topology as in [17]. Thus, $(X^*, \mathcal{T}^*, \leq^*)$ is a TWO derived from $(X, \mathcal{T}, \leq)$.

Definition 2.3. *Let $(X, \mathcal{T})$ be a topological space, $(X, \leq)$ be a poset. The topological well ordered space $(X^*, \mathcal{T}^*, \leq^*)$ obtained as in Remark 2.2 is called the one point topological well-ordering (or op-TWOing, briefly) of X.*

***Example* 2.12.** Let $(\mathbb{R}, \mathcal{U})$ be the usual topology, and $\leq$ be a usual ordering on it. Define the set $\mathbb{R}^{-\infty} = \mathbb{R} \cup \{-\infty\}$. So, $\mathcal{U}^{-\infty}$ is obtained as a union of all elements of $\mathcal{U}$ with $-\infty$, i.e.,

$$\mathcal{U}^{-\infty} = \{U \cup \{-\infty\} \mid U \in \mathcal{U}\} \cup \{\varnothing, \{-\infty\}\}.$$

Thus, we obtain that $(\mathbb{R}^{-\infty}, \mathcal{U}^{-\infty}, \leq^{-\infty})$ is TWO, and it is a one point topological well ordering (op-TWOing) of $\mathbb{R}$.

Note that, from Example 2.12, we conclude that the op-TWOing of a topological space cannot be T_0, because we can not separate $-\infty$ from any element.

We know that all finite T_1 spaces are discrete space. Using this argument we can obviously say that if $(X, \mathcal{T}, \leq)$ is a finite T_1 TWO space, then X is well-ordered set.

Theorem 2.11. *Let X be a poset, $x_0 \in X$ and define the topology*

$$\mathcal{T} = \{U \subseteq X \mid x_0 \notin U\} \cup \{X\}$$

called excluded point topology. If $(X, \mathcal{T})$ is a TWO and $x \leq x_0$ or $x_0 \leq x$ for each $x \in X$, then X is a well-ordered set.

Proof. Suppose that A is arbitrary nonempty subset of X. If $x_0 \notin A$, then $A \in \mathcal{T}$. Since X is TWO, then A has a minimum element.

Now, suppose that $x_0 \in A$. Then $A - \{x_0\} \in \mathcal{T}$. Since X is TWO, then $A - \{x_0\}$ has a minimum element and say $\min A - \{x_0\} = a$. From hypothesis, since $x \leq x_0$ or $x_0 \leq x$ for each $x \in X$, we have $a \leq x_0$ or $x_0 \leq a$. Thus, we obtain either $\min A = a$ or $\min A = x_0$. Hence, A has a minimum element.

As a result, X is well-ordered because an arbitrary subset of X has the minimum element.

The concept of Fort space and Fortissimo spaces are given in [17]. If X is any infinite set and $x_0 \in X$ is a particular point, then the topology can be defined on X by defining the open sets to be those whose complement either is finite or includes x_0. If X is countable infinite, this space is called countable Fort space. If X is uncountable, then it called uncountable Fort space. Moreover, let X be an uncountable set and x_0 be a particular point in X. We can define a topology on X by defining the open sets to be those whose complement either is countable or includes x_0. This space is called the Fortissimo space. Formally, let X be any infinite set, $x_0 \in X$. The Fort space topology is

$$\mathcal{T}_F = \big\{U \subseteq X \mid x_0 \notin U\big\} \cup \big\{U \subseteq X \mid X - U \text{ is finite}\big\},$$

and let X be any uncountable set, and $x_0 \in X$. The Fortissimo space topology is

$$\mathcal{T}_{Ft} = \big\{U \subseteq X \mid x_0 \notin U\big\} \cup \big\{U \subseteq X \mid X - U \text{ is countable}\big\}.$$

Clearly, part of the Fort and Fortissimo space topologies is the topology given in Theorem 2.11. Using the arguments in Theorem 2.11 and the definition of Fort and Fortissimo spaces we can give the following result.

***Corollary* 2.4.** *Let X be a poset, $\mathcal{T}_F$ and $\mathcal{T}_{Ft}$ be the Fort space topology and the Fortissimo space topology on X, respectively.*

(i) *If $(X, \mathcal{T}_F, \leq)$ is TWO and either $x \leq x_0$ or $x_0 \leq x$ for each X, then X is well-ordered.*

(ii) *If $(X, \mathcal{T}_{Ft}, \leq)$ is TWO and either $x \leq x_0$ or $x_0 \leq x$ for each X, then X is well-ordered.*

Lemma 2.1 [1]. *If a topological space X is both T_1 and Alexandroff, then the topology on it is discrete.*

Theorem 2.12. *If a T_1-Alexandroff space X is a TWO space, then X is well-ordered.*

Proof. From Theorem 2.9 and Lemma 2.1, it is obvious.

***Remark* 2.3.** In $(X, \mathcal{T}(\leq^{\uparrow}))$, the open sets are the increasing sets and the closed sets are the decreasing sets, so D is dense in X if and only if D contains the set of maximal elements of X.

Theorem 2.13. *Let $(X, \mathcal{T}(\leq^{\uparrow}), \leq)$ be an Alexandroff space. Define the topology*

$$\mathcal{DBT} = \big\{U \subseteq X \mid U \textit{ is a dense subset in } (X, \mathcal{T}(\leq^{\uparrow}), \leq)\big\} \cup \{\varnothing\}.$$

If $(X, \mathcal{DBT}, \leq)$ is TWO, then:

(i) *the cardinality of $M(X)$ is 1 where $M(X)$ is the set of all maximal elements in X,*

(ii) *X is well-ordered.*

Proof. (i) From Remark 2.3, we can write the $\mathcal{DBT}$ as

$$\mathcal{DBT} = \{U \subseteq X \mid M(X) \subseteq U\} \cup \{\varnothing\}.$$

Since $M(X) \subseteq M(X)$, then $M(X) \in \mathcal{DBT}$. From hypothesis, $(X, \mathcal{DBT}, \leq)$ is TWO, so there exists $M \in M(X)$ such that $\min(M(X)) = M$. Suppose that $M' \in M(X)$. Since $\min(M(X)) = M$, $M \leq M'$, then we obtain $M = M'$. Since M' is arbitrarily selected and $M = M'$, $M(X)$ is a singleton set and $M(X) = \{M\}$.

(ii) From (i), we know that $M(X) = \{M\}$. Let us rewrite $\mathcal{DBT}$ based on this information:

$$\begin{aligned}\mathcal{DBT} &= \{U \subseteq X \mid M(X) \subseteq U\} \cup \{\varnothing\} \\ &= \{U \subseteq X \mid \{M\} \subseteq U\} \cup \{\varnothing\} \\ &= \{U = P \cup \{M\} \mid P \in \mathcal{P}(X - \{M\})\} \cup \{\varnothing\}.\end{aligned}$$

Now, take a nonempty subset $U = P \cup \{M\}$ of X. If P is empty, $\min U = M$. Since $(X, \mathcal{DBT}, \leq)$ is TWO, then U has the minimum element. Let us say $\min U = m$. Since $m \in U = P \cup \{M\}$, $m = \min P$, obviously. Therefore, we conclude that arbitrary subset of $X - \{M\}$ has the minimum element. So, $X - \{M\}$ is well-ordered. Since $X - \{M\} \cup M(X) = X$, X is well-ordered.

Theorem 2.14. *Let $(X, \mathcal{T}(\leq^{\uparrow}), \leq)$ be an Alexandroff space. If it is TWO, then every closed subset of X has a minimum element, in fact all closed sets have the minimum element of X.*

Proof. The closed sets are the decreasing sets. Since X is TWO, $\min X$ exists and the result follows.

Theorem 2.15. *Let $(X, \leq)$ be a poset and $(X, \mathcal{T}(\leq^{\downarrow}))$ be the Alexandroff space generated by $(X, \leq)$. X has a minimum element if and only if $(X, \mathcal{T}(\leq^{\downarrow}), \leq)$ is TWO.*

The result is obvious since the $\mathcal{T}(\leq^{\downarrow})$-open sets are the $\leq$-decreasing sets.

Connectedness is a distinctive property for topological spaces. A topological space that cannot be written as a union of two disjoint nonempty open subsets is called *connected space*.

Let $(X, \mathcal{T})$ be a topological space and $\leq$ be a partial order relation on X. Suppose that $(X^*, \mathcal{T}^*, \leq^*)$ is an op-TWOing of this space. Since $* \in U$ for each nonempty open set $U \in \mathcal{T}^*$, $* \notin X - U$. Therefore, the space X cannot have both open and closed sets other than the empty set and itself. Hence, X is connected. As a result, an op-TWOing of a topological space with a partial order relation is connected.

As can be seen from the example below, a TWO space may not be connected.

***Example* 2.13.** Consider the set $X = \{a, b, c, d\}$ and the topology $\mathcal{T} = \{\varnothing, X, \{a, b, c\}, \{d\}\}$ on X. Let the diagram of the partial order relation $\leq$ on X be given as Fig. 3.

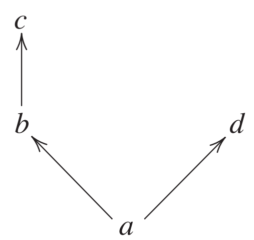

Fig. 3. Hasse diagram of partial order relation on X.

Obviously, $(X, \mathcal{T}, \leq)$ is a TWO and $(X, \mathcal{T})$ is not connected.

In order for any TWO to be connected, we can give following obvious result.

Theorem 2.16. *If $(X, \mathcal{T}, \leq)$ is a TWO and all nonempty open subsets contain the minimum of X, then $(X, \mathcal{T})$ is connected.*

The reverse of the expression given in the Theorem 2.16 is not true.

***Example* 2.14.** Consider the set $X = \{a, b, c, d, e\}$ and given the topology $\mathcal{T} = \{\varnothing, X, \{b, d\}, \{b, e\}\}$ on it. The Hasse diagram of the relation $\leq$ is as follows (Fig. 4).

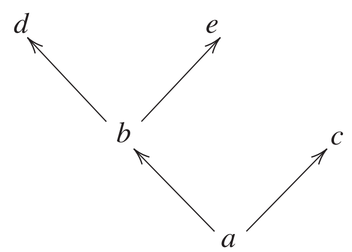

Fig. 4. Hasse diagram of partial order relation on X.

Clearly, $(X, \mathcal{T}, \leq)$ is TWO and connected. But, all open sets of the space do not have same minimum element.

In topology, the notion of path-connectedness is a stronger concept of connectedness.

Theorem 2.17. *An op-TWOing of a topological space with a partial order relation is path-connected.*

Proof. Let $(X, \mathcal{T})$ be a topological space and $\leq$ be a partial order relation on X. Suppose that $(X^*, \mathcal{T}^*, \leq^*)$ is an op-TWOing. For any $x, y \in X$ and $* \in X^* - X$, define the function $\alpha_{x,y} : [0,1] \to X^*$ such that

$$\alpha_{x,y}(t) = \begin{cases} x, & t = 0, \\ *, & t \in (0,1), \\ y, & t = 1. \end{cases}$$

Obviously, $\alpha_{x,y}$ is a path from x to y in X^*. Thus, $(X^*, \mathcal{T}^*, \leq^*)$ is path-connected.

Definition 2.4. *Let X be a nonempty set with a partial order relation $\leq$ and a topology $\mathcal{T}$. $(X, \mathcal{T}, \leq)$ is called basic well-ordered space (or briefly BWO space) (or just BWO) if there exists a base $\mathcal{B}$ such that each basic element has a minimum element.*

***Example* 2.15.** Consider the real numbers set $\mathbb{R}$, and its lower limit topology $\mathcal{U}_l$. We know that $\mathcal{B}_l = \{[a, b) \mid a, b \in \mathbb{R}\}$ is a basis for $\mathcal{U}_l$. Since every basic element of $\mathcal{B}_l$ has a minimum element, then the space $(\mathbb{R}, \mathcal{U}_l)$ is BWO.

***Example* 2.16.** Let X be a poset. Obviously, the discrete space $(X, \mathcal{P}(X))$ is BWO, since its base is $\mathcal{B} = \{\{x\} \mid x \in X\}$ and $\min\{x\} = x$.

Theorem 2.18. *If X is TWO, then it is BWO.*

Proof. Suppose that $B \in \mathcal{B}$. Then it is in $\mathcal{T}$, and since $(X, \mathcal{T})$ is TWO, then B has a minimum element. Thus, $(X, \mathcal{T})$ is BWO.

Note that the converse of theorem is not true. For example, we know that $(\mathbb{R}, \mathcal{U}_l)$ is BWO and $(a, b) \in \mathcal{U}_l$ for each $a, b \in \mathbb{R}$, and there is no minimum element of (a, b). Hence, $(\mathbb{R}, \mathcal{U}_l)$ is not TWO.

***Remark* 2.4.** Since the Alexandroff space $(X, \mathcal{T}(\leq^{\uparrow}), \leq)$ generated by $\mathcal{B} = \{\uparrow x \mid x \in X\}$, then it is BWO, obviously.

Theorem 2.19. *Let $(X, \leq)$ be a poset. If X has a minimum element, then $(X, \mathcal{T}(\leq^{\downarrow}), \leq)$ is BWO.*

***Remark* 2.5.** Let $(X_i, \mathcal{T}_i)$ be TWO space for each $i \in I$. Let $\mathcal{T}_B$ be the box topology and $\mathcal{T}_P$ be the product topology on $\prod_{i \in I} X_i$ and consider the lexiographic order or product order on $\prod_{i \in I} X_i$. Then $(\prod_{i \in I} X_i, \mathcal{T}_B)$ and $(\prod_{i \in I} X_i, \mathcal{T}_B)$ are BWOs according to the relevant relations, obviously.

3. Some applications of TWOs to graph theory. A graph is a mathematical structure that models binary relationships between objects. Graphs have applications in many fields such as physical, biological and information system sciences. A graph has a structure consisting of vertices and edges connecting vertices. Formally, $G = (V, E)$ is called a graph where V is the set of vertices, and $E \subseteq \{\{x, y\} \mid x, y \in V, \ x \neq y\}$ is the set of edges. Symbolically, the edge $\{x, y\} \in E$ is denoted by xy or yx. Let $G = (V, E)$ be a graph. If $xy \in E$, then x and y are called adjacent edges. Vertices that are not adjacent to any vertex are called isolated vertices. Let $x \in V$. The total number of vertices adjacent to x is called the degree of x and is denoted by $d(x)$. If the degree of all the vertices of a graph is k, this graph is called k-regular graph. Graphs in which every pair of vertices are adjacent are called complete graphs. All the vertices of a complete graph with n vertices have a degree of $n - 1$. In a graph with n-vertices, if only one vertex has a degree of $n - 1$ and the remaining vertices have a degree of 1, this graph is called an n-star graph and it is usually denoted by S_n. Let $G = (V, E)$ be a graph and $x_0, \ldots, x_i \in V$. If there exists a subset $P = \{x_0x_1, x_1x_2, \ldots, x_{i-1}x_i\} \subseteq E$, then P is called a path from x_0 to x_i in G. If there exists at least one path between any two vertices in a graph, then it is called a connected graph. Let $G = (V, E)$ be a graph and $S \subseteq V$. If every vertex in $V - S$ is adjacent to at least one vertex in S, then S is called a dominant set in G. The number of elements of the dominant set with the least number of elements is called the dominance number of the graph G and it is denoted by $\gamma(G)$.

Let $(X, \leq)$ be a poset and $x, y \in X$. In the case where y covers x, we will call the graph $G = (X, E)$ obtained by constituting an edge between x and y, as the graph corresponding to the poset $(X, \leq)$. As can be easily understood, for $x, y \in X$, $xy \in E$ if y covers x. If the graph G corresponding to the poset $(X, \leq)$ is a connected graph, then the poset $(X, \leq)$ is called a path-connected poset.

Let $(X, \leq)$ be a poset and $G = (X, E)$ be the graph corresponding to this poset. If G satisfies the condition

$$x, y \in X \quad \Leftrightarrow \quad \text{there exists } z \in X \text{ such that } x, y \leq z,$$

then G is called an upper bound graph, and denoted by $UB(X)$ [4, 11].

In [13], Kılıcman and Abdulkalek defined a topological space that is called incidence topology associated with simple graphs. Let $G = (V, E)$ be a graph without an isolated vertex and I_e be the set of endpoints of e for any $e \in E$. The $\mathcal{T}_G$ topology, which considers the family $\mathcal{S}_G = \{I_e \mid e \in E\}$ as a subbase on the set V, is called incidence topology. Suppose that $\mathcal{T}_G$ is the incidence topology of the graph $G = (V, E)$. If $d(x) \geq 2$, then $\{x\} \in \mathcal{T}_G$ [13].

We can now discuss some relations between the concept of topological well-orderness and graphs.

***Remark* 3.1.** Let $(X, \mathcal{T}, \leq)$ be a TWO. Then each nonempty open set $U \subseteq X$ has the minimum element m. So, for all $x, y \in U$, $P = \{mx, my\}$ is a path between x and y. Thus, G is a connected graph. As a consequence, U is a path-connected poset.

Note that $UB(U)$ will denote that the upper bound graph.

Theorem 3.1. *Let $(X, \mathcal{T}, \leq)$ be a finite TWO space. For any nonempty open set U, if $\min U = m_U$, then $d(m_U) = |U| - 1$ in the graph $UB(U)$.*

Proof. Suppose that $U \in \mathcal{T}$ is a nonempty open set and $\min U = m_U$. For each $x \in U$ which satisfies the condition $x \neq m_U$, there is an edge between x and m_U. Thus, we obtain that $d(m_U) = |U| - 1$.

Theorem 3.2. *Let $(X, \mathcal{T})$ be a topological space with a partial order relation $\leq$. If $UB(U)$ is a complete graph for each nonempty open set U, then $(X, \mathcal{T}, \leq)$ is a TWO space.*

Proof. Since $UB(U)$ is a complete graph, $x \leq y$ or $y \leq x$ for each $x, y \in U$. So, U is a chain. Therefore, there is the minimum element in finite chain U. Consequently, we have that $(X, \mathcal{T}, \leq)$ is TWO.

Note that if any graph with n-vertices is $n-1$-regular, then it is complete. Using this argument, we get the following result from Theorem 3.1, directly.

***Corollary* 3.1.** *Let $(X, \mathcal{T})$ be a topological space with a partial order relation $\leq$. If $UB(U)$ is $(|U| - 1)$-regular for each nonempty open set U, then $(X, \mathcal{T}, \leq)$ is a TWO space.*

Theorem 3.3. *Let X be finite, $(X, \mathcal{T}, \leq)$ be a TWO space and U be a nonempty open set in X. For each $x, y \in U$ satisfying $x \neq y$, the length of the shortest path between the vertices x and y in the graph $UB(U)$ is either 1 or 2.*

Proof. If $x \leq y$ or $y \leq x$ for each $x, y \in U$ that satisfies $x \neq y$, the graph $UB(U)$ has an edge between x and y. So, the length of the path xy is 1.

Suppose that x and y are incomparable. Since $(X, \mathcal{T}, \leq)$ is TWO, U has the minimum element $\min U = m$. So, we have that $m \leq x$ and $m \leq y$. Therefore, there is an edge between m and x and m and y in the graph $UB(U)$. Thus, $P = \{xm, my\}$ is a path from x to y and the length of P is 2.

Theorem 3.4. *Let X be finite and $(X, \mathcal{T}, \leq)$ be a TWO space. For each nonempty open set U in X, The dominance number of the graph $UB(U)$ is* 1.

Proof. Since $(X, \mathcal{T}, \leq)$ is TWO, any nonempty open set U has a minimum element m. Since $x \leq x$ and $m \leq x$ for each $x \in U$ that satisfies $x \neq m$, there is an edge between x and m. Therefore, the subset $S = \{m\}$ of U is the dominant set of the graph $UB(U)$. Thus, the dominance number of S is 1.

Theorem 3.5. *Let S_n be a star graph and $(X, \leq)$ be the partially ordered set that accepts this graph as the upper bound graph. In the circumstances, $(X, \mathcal{T}_G, \leq)$ is TWO where $\mathcal{T}_G$ is the incidence topology of the S_n.*

Proof. From definition of star graph, there exists $x_0 \in X$ such that $d(x_0) = n - 1$. In the meantime, the family $\mathcal{B}_G = \{x_0\} \cup \{\{x_0, x\} \mid x_0 \neq x,\ x \in X\}$ is a basis for the incidence topology $\mathcal{T}_G$. Since $\min B = \min X = x_0$ for each $B \in \mathcal{B}_G$, we have that $\min U = x_0$ for all nonempty open set U. Thus, $(X, \mathcal{T}_G, \leq)$ is TWO.

Example **3.1.** Let $(X, \leq)$ be a poset that accepts the star graph S_4 given in Fig. 5 as the upper bound graph.

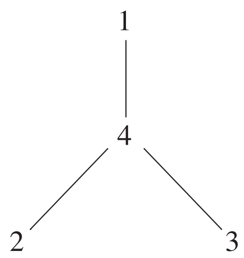

Fig. 5. The star graph S_4.

We obtain that

$$\leq = \{(1,1), (2,2), (3,3), (4,4), (4,1), (4,2), (4,3)\}$$

on the set $X = \{1, 2, 3, 4\}$. Since $d(4) = 3$ and $d(x) = 1$ for each $x \in X - \{4\}$, we have

$$\mathcal{B}_G = \{\{4\}, \{1,4\}, \{2,4\}, \{3,4\}\}$$

and, so,

$$\mathcal{T}_G = \{\varnothing, X, \{4\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}.$$

Hence, $(X, \mathcal{T}_G, \leq)$ is a TWO space.

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