DOI: 10.37863/umzh.v75i6.6958

UDC 517.9

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MULTIPLE SOLUTIONS TO BOUNDARY-VALUE PROBLEMS FOR FOURTH-ORDER ELLIPTIC EQUATIONS

ЧИСЛЕННІ РОЗВ'ЯЗКИ КРАЙОВИХ ЗАДАЧ ДЛЯ ЕЛІПТИЧНИХ РІВНЯНЬ ЧЕТВЕРТОГО ПОРЯДКУ

We study the existence of multiple solutions for the biharmonic problem

$$\Delta^2 u = f(x,u) + g(x,u) \quad \text{in} \quad \Omega,$$

$$u = \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $N > 4$, $f(x,\xi)$ is odd in ξ, and $g(x,\xi)$ is a perturbation term. Under certain growth conditions on f and g, we show that there are infinitely many weak solutions to the problem.

Досліджено існування кількох розв'язків бігармонічної задачі

$$\Delta^2 u = f(x,u) + g(x,u) \quad \text{в} \quad \Omega,$$

$$u = \partial_\nu u = 0 \quad \text{на} \quad \partial\Omega,$$

де Ω — обмежена область із гладкою межею в $\mathbb{R}^N$, $N > 4$, $f(x,\xi)$ непарна по ξ, а $g(x,\xi)$ — член збурення. За деяких умов, накладених на зростання f і g, показано, що існує нескінченна кількість слабких розв'язків задачі.

1. Introduction. In the last decades, the biharmonic elliptic equations

$$\Delta^2 u = f(x,u) \quad \text{in} \quad \Omega,$$
$$u = \Delta u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

has been studied by many authors (see [4–7, 10–12] and the references therein). In [5], there was a survey of results obtained in this direction. In [7], A. M. Micheletti and A. Pistoia showed that (1.1) admits at least two solutions by a variation of linking if $f(x,\xi)$ is sublinear. And in [4], the authors proved that the problem (1.1) has at least three solutions by a variational reduction method and a degree argument. In [10], J. H. Zhang and S. J. Li, showed that (1.1) admits at least two nontrivial solutions by the Morse theory and local linking if $f(x,\xi)$ is superlinear and subcritical on ξ. In [11], J. Zhang and Z. L. Wei obtained the existence of infinitely many solutions for the problem (1.1) where the nonlinearity involves a combination of superlinear and asymptotically linear terms. As far as the problem (1.1) is concerned, existence results of sign-changing solutions were also obtained (see, e.g., [6, 12] and the references therein). Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [25, 26] (see also some recent results in [1, 14–21, 23, 24]).

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In this paper, we study the existence of multiple weak solutions to the following problem:

$$\begin{gathered}\Delta^2 u = f(x,u) + g(x,u) \quad \text{in} \quad \Omega,\\ u = \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega,\end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$, $N > 4$, is a smooth bounded domain, $\nu = (\nu_1, \ldots, \nu_N)$ is the unit outward normal on $\partial\Omega$.

To study the problem (1.2), we make the following assumptions:

We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

(F1) $f(x, -\xi) = -f(x, \xi)$ for all $(x, \xi) \in \Omega \times \mathbb{R}$.

(F2) There exist $2 < p < 2_* := \dfrac{2N}{N-4}$ and $C_1 > 0$ such that for all ξ and almost everywhere in $x \in \Omega$

$$|f(x,\xi)| \le C_1\big(1 + |\xi|^{p-1}\big).$$

(F3) There exist $\mu > 2$ and $R_0 > 0$ such that $0 < \mu F(x,\xi) \le f(x,\xi)\xi$ for $|\xi| \ge R_0$ and almost every $x \in \Omega$, where $F(x,\xi) = \displaystyle\int_0^\xi f(x,\tau)d\tau$.

And $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

(G) $|g(x,\xi)| \le g_1(x) + g_2(x)|\xi|^\theta$ for almost every $(x,\xi) \in \Omega \times \mathbb{R}$, where $g_1(x) \in L^{p_1}(\Omega)$, $g_2(x) \in L^{p_2}(\Omega)$, $p_1/(p_1-1) < \mu$, $(\theta+1)p_2/(p_2-1) < \mu$, $\theta \ge 0$, $p_1 \ge \dfrac{2_* p_2}{p_2\theta + 2_*}$, $p_2 \ge \dfrac{2_*}{2_* - \theta - 1}$.

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that f and g satisfy* (F1)–(F3), (G) *and*

$$\frac{4p}{N(p-2)} - 1 > \frac{\mu}{\mu - \theta - 1}. \tag{1.3}$$

Then the problem (1.2) *has an unbounded sequence of solutions in $H_0^2(\Omega)$.*

2. Proof of Theorem 1.1. Define the Euler–Lagrange functional associated with the problem (1.2) as follows:

$$\Phi(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \int_\Omega F(x,u)\,dx - \int_\Omega G(x,u)\,dx.$$

Lemma 2.1. *Assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $|g(x,\xi)| \le g_1(x) + g_2(x)|\xi|^\theta$ for almost every $(x,\xi) \in \Omega \times \mathbb{R}$, where $g_1(x) \in L^{p_1}(\Omega)$, $g_2(x) \in L^{p_2}(\Omega)$, $p_1/(p_1-1) \le 2_*$, $(\theta+1)p_2/(p_2-1) \le 2_*$, $\theta \ge 0$, $p_1 \ge \dfrac{2_* p_2}{p_2\theta + 2_*}$, $p_2 \ge \dfrac{2_*}{2_* - \theta - 1}$. Then $\Phi_1(u) \in C^1(H_0^2(\Omega), \mathbb{R})$ and*

$$\langle \Phi_1'(u), v\rangle = \int_\Omega g(x,u)v\,dx$$

for all $v \in H_0^2(\Omega)$, where

$$\Phi_1(u) = \int_\Omega G(x,u)\,dx$$

and $G(x,u) = \displaystyle\int_0^u g(x,\xi)\,d\xi$.

Proof. With slight modification, the proof of this lemma is similar to Lemma 2.3 in [22]. We omit the details.

Definition 2.1. *We say that* $u \in H_0^2(\Omega)$ *is a weak solution of the problem* (1.2) *if*

$$\int\limits_{\Omega} \Delta u \Delta v \, dx - \int\limits_{\Omega} f(x,u) v \, dx - \int\limits_{\Omega} g(x,u) v \, dx = 0$$

for all $v \in H_0^2(\Omega)$.

From Lemma 2.1 and f satisfies (F2), g satisfies (G), we obtained that Φ is well defined on $H_0^2(\Omega)$ and $\Phi \in C^1(H_0^2(\Omega), \mathbb{R})$ with

$$\langle \Phi'(u), v \rangle = \int\limits_{\Omega} \Delta u \Delta v \, dx - \int\limits_{\Omega} f(x,u) v \, dx - \int\limits_{\Omega} g(x,u) v \, dx$$

for all $v \in H_0^2(\Omega)$. Thus, we will seek weak solutions of the problem (1.2) as the critical points of the functional Φ.

For future reference we note that (F3) implies there are constants $C_2, C_3, C_4 > 0$ such that

$$\frac{1}{\mu}(\xi f(x,\xi) + C_2) \geq F(x,\xi) + C_3 \geq C_4 |\xi|^{\mu} \quad \text{for all} \quad \xi \in \mathbb{R}. \tag{2.1}$$

Lemma 2.2. *Assume that* f *and* g *satisfy* (F1)–(F3), (G) *and* u *is a critical point of* Φ. *Then there exists a constant* C_5 *such that*

$$\int\limits_{\Omega} (F(x,u) + C_3) \, dx \leq \frac{1}{\mu} \int\limits_{\Omega} (u f(x,u) + C_2) \, dx \leq C_5 \big((\Phi(u))^2 + 1 \big)^{\frac{1}{2}}. \tag{2.2}$$

Proof. The left-hand side inequality (2.2) can easily be obtained by integrating the left-hand side inequality (2.1) over in Ω. At the critical point u of Φ, by (2.1), applying Hölder's and Young's inequalities, we get

$$\begin{aligned}
\Phi(u) &= \Phi(u) - \frac{1}{2} \langle \Phi'(u), u \rangle \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \int\limits_{\Omega} (f(x,u)u + C_2) \, dx + \frac{1}{2} \int\limits_{\Omega} g(x,u) u \, dx - \int\limits_{\Omega} G(x,u) \, dx - C_6 \\
&\geq C_6 \int\limits_{\Omega} (f(x,u)u + C_2) \, dx - \overline{C}_1(\epsilon) - \epsilon \Big\| u \Big\|_{L^{\mu}(\Omega)}^{\mu}
\end{aligned} \tag{2.3}$$

for any $\epsilon > 0$. Choosing $\epsilon = \mu C_6 C_4 / 2$, from (2.1), (2.3) and applying Cauchy's inequalities, we have

$$\frac{1}{\mu} \int\limits_{\Omega} \big(u f(x,u) + C_2 \big) \, dx \leq C_5 \big((\Phi(u))^2 + 1 \big)^{\frac{1}{2}}.$$

Lemma 2.2 is proved.

Next, define a modified functional $\overline{\Phi}(u)$. Let $\chi \in C^{\infty}(\mathbb{R},\mathbb{R})$ such that $\chi(t) = 1$ for $t \le 1$, $\chi(t) = 0$ for $t > 2$ and $-2 < \chi' < 0$ for $t \in (1,2)$. For $u \in H_0^2(\Omega)$, we put

$$\kappa(u) = 2\Theta\big((\Phi(u))^2 + 1\big)^{\frac{1}{2}}, \qquad \psi(u) = \chi\left(\kappa(u)^{-1}\int\limits_{\Omega}(F(x,u) + C_3)\,dx\right)$$

and

$$\overline{\Phi}(u) = \int\limits_{\Omega}\left(\frac{1}{2}|\Delta u|^2 - F(x,u) - \psi(u)G(x,u)\right)dx, \tag{2.4}$$

where Θ is positive constant, which will be chosen later in Lemma 2.7.

***Remark* 2.1.** From the definition of χ, we have that if

$$u \in H_0^2(\Omega), \kappa(u)^{-1}\int\limits_{\Omega}(F(x,u) + C_3)\,dx \le 1,$$

then $\Phi(u) = \overline{\Phi}(u),\ \Phi'(u) = \overline{\Phi}'(u)$.

Let $\operatorname{supp}(\psi)$ denote the support of ψ.

Lemma 2.3. *Assume that f and g satisfy* (F1)–(F3), (G) *and* $u \in \operatorname{supp}(\psi)$. *Then*

$$\left|\int\limits_{\Omega} G(x,u)\,dx\right| \le C_8\Big(|\Phi(u)|^{\frac{\theta+1}{\mu}} + 1\Big). \tag{2.5}$$

Proof. From (G) and (2.1), applying embedding inequalities combined with Hölder's inequality, we have

$$\left|\int\limits_{\Omega} G(x,u)\,dx\right| \le C_9\left[\left(\int\limits_{\Omega}(F(x,u) + C_3)\,dx\right)^{\frac{1}{\mu}} + \left(\int\limits_{\Omega}(F(x,u) + C_3)\,dx\right)^{\frac{\theta+1}{\mu}}\right]. \tag{2.6}$$

On the other hand, since $u \in \operatorname{supp}(\psi)$, we get

$$\int\limits_{\Omega}(F(x,u) + C_3)\,dx \le 4\Theta\big((\Phi(u))^2 + 1\big)^{\frac{1}{2}} \le C_{10}(|\Phi(u)| + 1), \tag{2.7}$$

so (2.5) follows from (2.6) and (2.7).

Lemma 2.3 is proved.

Lemma 2.4. *Assume that f and g satisfy* (F1)–(F3), (G). *Then there exists a constant C_{11}, such that, for any $u \in H_0^2(\Omega)$,*

$$\big|\overline{\Phi}(u) - \overline{\Phi}(-u)\big| \le C_{11}\left(\big|\overline{\Phi}(u)\big|^{\frac{\theta+1}{\mu}} + 1\right).$$

Proof. By (F1) and (2.4), we get

$$\big|\overline{\Phi}(u) - \overline{\Phi}(-u)\big| \le |\psi(u)|\left|\int\limits_{\Omega} G(x,u)\,dx\right| + |\psi(-u)|\left|\int\limits_{\Omega} f(x,-u)\,dx\right|. \tag{2.8}$$

Consider four cases.

Case 1: $u \in \operatorname{supp}(\psi)$ and $-u \in \operatorname{supp}(\psi)$. From Lemma 2.3, since (2.4), we have

$$|\Phi(u)| \le \left|\overline{\Phi}(u)\right| + 2\left|\int_\Omega F(x,u)\,dx\right|. \tag{2.9}$$

From (2.5) and (2.9), we obtain

$$\left|\int_\Omega G(x,u)\,dx\right| \le C_{13}\left(\left|\overline{\Phi}(u)\right|^{\frac{\theta+1}{\mu}} + \left|\int_\Omega G(x,u)\,dx\right|^{\frac{\theta+1}{\mu}} + 1\right).$$

Applying Young's inequality and the definition of ψ, we get the conclusion of the lemma.

Case 2: $u \in \operatorname{supp}(\psi)$ and $-u \notin \operatorname{supp}(\psi)$. From (2.8), we have

$$\left|\overline{\Phi}(u) - \overline{\Phi}(-u)\right| \le |\psi(u)|\left|\int_\Omega G(x,u)\,dx\right|.$$

By using the same argument as in case 1, the statement is proved.

Case 3: $u \notin \operatorname{supp}(\psi)$ and $-u \notin \operatorname{supp}(\psi)$, the proof is trivial.

Case 4: $u \notin \operatorname{supp}(\psi)$ and $-u \in \operatorname{supp}(\psi)$. From (2.8), we get

$$\left|\overline{\Phi}(u) - \overline{\Phi}(-u)\right| \le 2\left|\int_\Omega G(x,-u)\,dx\right|.$$

From (2.5), we obtain

$$\left|\int_\Omega G(x,-u)\,dx\right| \le C_{12}\left(\left|\overline{\Phi}(u)\right|^{\frac{\theta+1}{\mu}} + \left|\int_\Omega G(x,-u)\,dx\right|^{\frac{\theta+1}{\mu}} + 1\right).$$

Applying Young's inequality, we get the conclusion of the lemma.

Lemma 2.4 is proved.

Lemma 2.5. *Assume that f and g satisfy* (F1)–(F3), (G) *and there exist constants M_0 and $C_{13} > 0$ such that whenever $M \ge M_0$, $\overline{\Phi}(u) \ge M, u \in \operatorname{supp}(\psi)$. Then $\Phi(u) \ge C_{13}M$.*

Proof. From (2.4), we deduce that

$$\Phi(u) \ge \overline{\Phi}(u) - \left|\int_\Omega G(x,u)\,dx\right|. \tag{2.10}$$

If $u \in \operatorname{supp}(\psi)$, by (2.10) and (2.5), we have

$$\Phi(u) + C_8|\Phi(u)|^{\frac{\theta+1}{\mu}} \ge \overline{\Phi}(u) - C_8 \ge \frac{M}{2}$$

for large enough M_0. Therefore, $\Phi(u) > 0$ and $\Phi(u) > \dfrac{M}{2(2C_8+1)}$.

Lemma 2.5 is proved.

From (2.4), we see that

$$\langle \overline{\Phi}'(u), u\rangle = (1 + T_1(u)) \int\limits_\Omega |\Delta u|^2\, dx$$

$$- (1 + T_2(u)) \int\limits_\Omega f(x,u)u\, dx - (\psi(u) + T_1(u)) \int\limits_\Omega g(x,u)u\, dx, \tag{2.11}$$

where

$$T_1(u) = \chi'\Big(\kappa(u)^{-1}\int_\Omega (F(x,u) + C_3)\, dx\Big)\kappa(u)^{-3}(2\Theta)^2\Phi(u)$$

$$\times \int_\Omega (F(x,u) + C_3)\, dx \int\limits_\Omega G(x,u)\, dx,$$

$$T_2(u) = \chi'\Big(\kappa(u)^{-1}\int_\Omega (F(x,u) + C_3)\, dx\Big)\kappa(u)^{-1}\int_\Omega G(x,u)\, dx + T_1(u).$$

Lemma 2.6. *Assume that f and g satisfy* (F1) – (F3), (G). *Then, for every small enough $\delta > 0$, there exists large enough $M > 0$ such that, for all $u \in H_0^2(\Omega), \overline{\Phi}(u) \ge M$, we have $|T_1(u)| \le \delta$, $|T_2(u)| \le \delta$.*

Proof. Consider two cases.

Case 1: If $u \notin \mathrm{supp}(\psi)$, then the proof is trivial.

Case 2: If $u \in \mathrm{supp}(\psi)$, then let M_0 be as in Lemma 2.5. Let $u \in H_0^2(\Omega)$ be such that $\overline{\Phi}(u) \ge M$ and $M \ge M_0$. Then Lemmas 2.3 and 2.5 imply

$$|T_1(u)| \le C_{14}\Big(\Phi(u)^{\frac{\theta+1}{\mu}-1} + \Phi(u)^{-1}\Big) \le C_{15}\Big(M^{\frac{\theta+1}{\mu}-1} + M^{-1}\Big) \to 0 \quad \text{as} \quad M \to \infty,$$

$$|T_2(u)| \le |T_1(u)| + C_{16}\Big(M^{\frac{\theta+1}{\mu}-1} + M^{-1}\Big) \to 0 \quad \text{as} \quad M \to \infty.$$

Lemma 2.6 is proved.

We shall show that large critical values of $\overline{\Phi}$ are critical values of Φ.

Lemma 2.7. *Assume that f and g satisfy* (F1) – (F3), (G) *and constant Θ is large enough. Then there exists $M_1 > 0$ such that if $u \in H_0^2(\Omega)$ is critical point of $\overline{\Phi}$ and $\overline{\Phi}(u) \ge M_1$, then u is a critical point of Φ and $\overline{\Phi}(u) = \Phi(u)$.*

Proof. Let $u \in H_0^2(\Omega)$ be such that $\overline{\Phi}'(u) = 0$. By (2.11), we have

$$\Phi(u) = \Phi(u) - \frac{\langle \overline{\Phi}'(u), u\rangle}{2(1 + T_1(u))}$$

$$\ge \left(\frac{1 + T_2(u)}{2(1 + T_1(u))} - \frac{1}{\mu}\right) \int\limits_\Omega (f(x,u)u + C_2)\, dx$$

$$+ \frac{\psi(u) + T_1(u)}{2(1 + T_1(u))} \int\limits_\Omega g(x,u)u\, dx - \int\limits_\Omega G(x,u)\, dx - C_{17}$$

$$\geq C_{18} \int_\Omega (f(x,u)u + C_2)\,dx - \overline{C}_2(\epsilon) - \epsilon \|u\|^\mu_{L^\mu(\Omega)}.$$

For sufficiently large M_1 such that $M_1 > M_0$ and sufficiently small T_1, T_2, if we choose large enough Θ, then

$$\kappa(u)^{-1} \int_\Omega (F(x,u) + C_3)\,dx \leq 1.$$

Hence it follows that $\psi(u) = 1$ and $\psi'(u) = 0$.

Lemma 2.7 is proved.

Lemma 2.8. *Assume that f and g satisfy* (F1) – (F3), (G). *Then $\overline{\Phi} \in C^1(H^2_0(\Omega), \mathbb{R})$ and there exists a constant $M_2 > 0$ such that $\overline{\Phi}$ satisfies the Palais – Smale condition on $\widehat{A}_{M_2} = \{u \in H^2_0(\Omega) : \overline{\Phi}(u) \geq M_2\}$.*

Proof. Since f and g satisfy (F1) – (F3), (G) and $\chi \in C^\infty(\mathbb{R},\mathbb{R})$, then $\overline{\Phi} \in C^1(H^2_0(\Omega),\mathbb{R})$. Let M_0 be as in Lemma 2.5 and take $M_2 \geq M_0$. Let $\{u_m\}^\infty_{m=1}$ be a sequence in $\widehat{A}_{M_2}$ such that

$$\overline{\Phi}(u_m) \leq K \quad \text{for every} \quad m \in \mathbb{N}, \qquad \lim_{m\to\infty} \overline{\Phi}'(u_m) = 0$$

for some $K \geq M_2$. Then, for all small enough $\rho_2 > 0$, large enough m and $\rho_1 > 0$, by (2.1) and Young's inequality, we deduce that

$$\begin{aligned}\rho_1 K + \rho_2 \|u_m\|_{H^2_0(\Omega)} &\geq \rho_1 \overline{\Phi}(u_m) - \langle \overline{\Phi}'(u_m), u_m\rangle \\ &\geq \left(\frac{\rho_1}{2} - (1 + T_1(u_m))\right) \|u_m\|^2_{H^2_0(\Omega)} \\ &\quad + \left(1 + T_2(u_m) - \frac{\rho_1}{\mu}\right) C_4 \mu \|u\|^\mu_{L^\mu(\Omega)} \overline{C}_3(\epsilon) - \epsilon \|u\|^\mu_{L^\mu(\Omega)}.\end{aligned}$$

For sufficiently large M_2 and sufficiently small T_1, T_2 , we can choose ρ_1, ρ_2 such that

$$\frac{\rho_1}{2} - (1 + T_1(u_m) > 0, 1 + T_2(u_m) - \frac{\rho_1}{\mu} > 0, \quad \rho_2 > 0$$

and $\epsilon = \left(1 + T_2(u_m) - \dfrac{\rho_1}{\mu}\right) C_4 \mu$. Hence $\{u_m\}^\infty_{m=1}$ is bounded in $H^2_0(\Omega)$.

Therefore, we can (by passing to a subsequence if necessary) suppose that

$$\begin{aligned} &u_m \rightharpoonup u \quad \text{in} \quad H^2_0(\Omega) \quad \text{as} \quad m \to \infty, \\ &u_m \to u \quad \text{a.e. in} \quad \Omega \quad \text{as} \quad m \to \infty, \\ &u_m \to u \quad \text{in} \quad L^q(\Omega), \quad 1 \leq q < 2_* \quad \text{as} \quad m \to \infty. \end{aligned} \tag{2.12}$$

Thus by (F2), (G), applying Hölder's inequality and (2.12), we obtain

$$\int_\Omega (f(x,u_m) - f(x,u))(u_m - u)\,dx \to 0 \quad \text{as} \quad m \to \infty, \tag{2.13}$$

$$\int\limits_{\Omega}(g(x,u_m)-g(x,u))(u_m-u)\,dx\to 0 \quad \text{as} \quad m\to\infty. \tag{2.14}$$

It follows from $\lim_{m\to\infty}\Phi'(u_m)=0$ and (2.12) that

$$\Big\langle(1+T_1(u))\overline{\Phi}'(u_m)-(1+T_1(u_m))\overline{\Phi}'(u),u_m-u\Big\rangle\to 0 \quad \text{as} \quad m\to\infty. \tag{2.15}$$

By (2.13), (2.14) and (2.15), we have

$$\int\limits_{\Omega}|\Delta u_m-\Delta u|^2\,dx\to 0 \quad \text{as} \quad m\to\infty.$$

Therefore, we conclude that $u_m\to u$ strongly in $H_0^2(\Omega)$.

Lemma 2.8 is proved.

Now we can show that $\overline{\Phi}$ has an unbounded sequence of critical values. Let $0<\lambda_1<\lambda_2\le\lambda_3\le\ldots\le\lambda_k\le\ldots$ denote the eigenvalues of the problems

$$\Delta^2u=\lambda u \quad \text{in} \quad \Omega,$$

$$u=\partial_\nu u=0 \quad \text{on} \quad \partial\Omega,$$

and $e_1,e_2,\ldots$ denote the corresponding eigenfunctions normalized such that $\|e_j\|_{H_0^2(\Omega)}=1$ for all $j=1,2,\ldots$. For any $k>0$, we put $\mathbb{V}_k=\operatorname{span}\{e_j;j\le k\}$, and $\mathbb{V}_k^\perp$ its orthogonal complement.

Lemma 2.9. *Assume that f and g satisfy* (F1)–(F3), (G). *Then, for any $k>0$, there exists $R_k>0$ such that, for any $u\in\mathbb{V}_k$ with $\|u\|_{H_0^2(\Omega)}\ge R_k$, we have $\overline{\Phi}(u)\le 0$.*

Proof. Let $u\in\mathbb{V}_k$. From (2.1), (2.4) and condition (G), by Young's inequality, we get

$$\overline{\Phi}(u)\le\frac{1}{2}\|u\|^2_{H_0^2(\Omega)}-C_{19}\|u\|^\mu_{L^\mu\Omega}+C_{20}, \qquad C_{19}>0, \quad C_{20}>0.$$

Since in $\mathbb{V}_k$ there exists $d=d_k>0$ such that $\|u\|_{L^\mu(\Omega)}\ge d\|u\|_{H_0^2(\Omega)}$ for all $u\in\mathbb{V}_k$, we have

$$\overline{\Phi}(u)\le\frac{1}{2}\|u\|^2_{H_0^2(\Omega)}-C_{19}d^\mu\|u\|^\mu_{H_0^2(\Omega)}+C_{20}$$

which implies that $\overline{\Phi}(u)\to-\infty$ as $u\in\mathbb{V}_k,\ \|u\|_{H_0^2(\Omega)}\to+\infty$.

Lemma 2.9 is proved.

Choose an increasing sequence R_k such that $\overline{\Phi}(u)\le 0$ if $u\in\mathbb{V}_k,\ \|u\|_{H_0^2(\Omega)}\ge R_k$. Let B_{R_k} denote the closed ball of radius R_k in $H_0^2(\Omega)$, $\mathbb{W}_k\equiv B_{R_k}\bigcap\mathbb{V}_k$, and

$$\Gamma_k=\Big\{\gamma\in C(\mathbb{W}_k,H_0^2(\Omega))\colon\gamma \text{ is odd and } \gamma(u)=u \text{ if } \|u\|_{H_0^2(\Omega)}=R_k\Big\},$$

$$\mathbb{U}_k=\Big\{u=te_{k+1}+w\colon t\in[0,R_{k+1}],\ w\in B_{R_{k+1}}\bigcap\mathbb{V}_k,\ \|u\|_{H_0^2(\Omega)}\le R_{k+1}\Big\},$$

$$\Lambda_k=\Big\{\Psi\in C(\mathbb{U}_k,H_0^2(\Omega))\colon\Psi|_{\mathbb{W}_k}\in\Gamma_k \text{ and } \Psi(u)=u$$

$$\text{if } \|u\|_{H_0^2(\Omega)}=R_{k+1} \text{ or } u\in(B_{R_{k+1}}\backslash B_{R_k})\bigcap\mathbb{V}_k\Big\}.$$

With the help of these continuous maps, we define two sequences of minimax values

$$\alpha_k = \inf_{\gamma\in\Gamma_k} \max_{u\in\mathbb{W}_k} \overline{\Phi}(\gamma(u)), \quad k \in \mathbb{N}. \tag{2.16}$$

$$\beta_k = \inf_{\Psi\in\Lambda_k} \max_{u\in\mathbb{U}_k} \overline{\Phi}(\Psi(u)), \quad k \in \mathbb{N}. \tag{2.17}$$

It is obvious that $\beta_k \geq \alpha_k$. For the sake of getting the lower bound of the above minimax values, we give an intersection property which has been proved in Lemma 1.44 of [9] by Rabinowitz.

Lemma 2.10. *Let $\rho > 0$. For any $k \in \mathbb{N}$, $R_k > \rho$ and $\gamma \in \Gamma_k$, we have*

$$\gamma(\mathbb{W}_k)\bigcap \partial B_\rho \bigcap \mathbb{V}_{k-1}^\perp \neq \varnothing.$$

We give the lower bounds for α_k in the next lemma.

Lemma 2.11. *Assume that f and g satisfy* (F1)–(F3), (G). *Then there exist constants $C_{21} > 0$ and $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$,*

$$\alpha_k \geq C_{21} k^{\frac{4p}{N(p-2)}-1}.$$

Proof. By (F2), (G), Sobolev's embedding $H_0^2(\Omega) \hookrightarrow L^{2_*}(\Omega)$, and using the interpolation inequality, for any $u \in H_0^2(\Omega)$, we obtain

$$\begin{aligned}
\overline{\Phi}(u) &\geq \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - C_{28}\int_\Omega |u|^p\,dx - C_{29} \\
&\geq \frac{1}{2}\|u\|^2_{H_0^2(\Omega)} - C_{28}\|u\|^r_{L^2(\Omega)}\|u\|^{p-r}_{L^{2_*}(\Omega)} - C_{29} \\
&\geq \frac{1}{2}\|u\|^2_{H_0^2(\Omega)} - C_{30}\|u\|^r_{L^2(\Omega)}\|u\|^{p-r}_{H_0^2(\Omega)} - C_{29},
\end{aligned} \tag{2.18}$$

where $\dfrac{r}{2} + \dfrac{p-r}{2_*} = 1$.

From Lemma 2.10, we get

$$\max_{u\in\mathbb{W}_k} \overline{\Phi}(\gamma(u)) \geq \overline{\Phi}(\omega) \geq \inf_{u\in\partial B_\rho\bigcap\mathbb{V}_{k-1}^\perp} \overline{\Phi}(u) \quad \forall \omega \in \gamma(\mathbb{W}_k)\bigcap\partial B_\rho\bigcap\mathbb{V}_{k-1}^\perp. \tag{2.19}$$

Moreover, by $u \in \mathbb{V}_{k-1}^\perp$, we have

$$\|u\|_{L^2(\Omega)} \leq \lambda_k^{\frac{-1}{2}}\|u\|_{H_0^2(\Omega)}. \tag{2.20}$$

Combining (2.18) and (2.20), for any $u \in \partial B_\rho \bigcap \mathbb{V}_{k-1}^\perp$, we get

$$\overline{\Phi}(u) \geq \left(\frac{1}{2} - C_{30}\lambda_k^{-r/2}\rho^{p-2}\right)\rho^2 - C_{29}. \tag{2.21}$$

From (2.16), (2.19) and (2.21), for big enough k, we obtain

$$\alpha_k \geq \sup_{\rho>0} \inf_{u\in\partial B_\rho\bigcap\mathbb{V}_{k-1}^\perp} \overline{\Phi}(u)$$

$$\geq \sup_{\rho>0}\left(\left(\frac{1}{2} - C_{30}\lambda_k^{-r/2}\rho^{p-2}\right)\rho^2 - C_{29}\right)$$

$$\geq C_{31}\lambda_k^{\frac{r}{p-2}} = C_{31}\lambda_k^{\frac{p}{p-2}-\frac{N}{4}}. \tag{2.22}$$

On the other hand, it follows from Agmon's generalization [2] of Weyl's formula [13], which in fact is an extension of earlier work of Pleijel [8] for $N = 2$, we have

$$\lambda_k \geq C_{32} k^{\frac{4}{N}}. \tag{2.23}$$

Combining (2.22) and (2.23), we arrive at the conclusion of the lemma.

Next we can construct critical values of $\overline{\Phi}$ as follows.

Lemma 2.12. *Suppose that $\beta_k > \alpha_k \geq M_2$. Let $\delta \in (0, \beta_k - \alpha_k)$ and*

$$\Lambda_k(\delta) = \left\{\Psi \in \Lambda_k : \overline{\Phi}(\Psi) \leq \alpha_k + \delta \ \ on \ \ \mathbb{W}_k\right\}.$$

Let

$$\beta_k(\delta) = \inf_{\Psi\in\Lambda_k(\delta)} \max_{u\in\mathbb{U}_k} \overline{\Phi}(\Psi(u)), \quad k \in \mathbb{N}. \tag{2.24}$$

Then $\beta_k(\delta)$ is a critical value of $\overline{\Phi}$.

Proof. By using Deformation theorem in [3], we can prove this lemma similarly as in the proof of Lemma 1.57 in [9]. We omit the details.

***Proof of Theorem* 1.1.** From (2.16), (2.17), (2.24) and Lemma 2.11, we get that

$$\beta_k(\delta) \geq \beta_k \geq \alpha_k \to \infty \quad \text{as} \quad k \to \infty.$$

Theorem 1.1 will be proved if we can show the existence of a subsequence of β_k's which satisfies $\beta_k > \alpha_k$. Indeed, suppose that $\beta_k = \alpha_k$ for all $k \geq k_1$. For any $\epsilon > 0$, choose $\Psi \in \Lambda_k$ such that

$$\max_{u\in\mathbb{U}_k} \overline{\Phi}(\Psi(u)) \leq \alpha_k + \epsilon. \tag{2.25}$$

Let $\widehat{\Psi}(u) = \Psi(u)$ if $u \in \mathbb{U}_k$ and $\widehat{\Psi}(u) = -\Psi(-u)$ if $-u \in \mathbb{U}_k$. Since $\Psi\Big|_{B_{R_{k+1}}\bigcap\mathbb{V}_k}$ is odd and continuous and $\mathbb{W}_{k+1} = \mathbb{U}_k \bigcup(-\mathbb{U}_k)$, then $\widehat{\Psi}$ is well defined on $\mathbb{W}_{k+1}$ and $\widehat{\Psi} \in \Gamma_{k+1}$. Therefore,

$$\alpha_{k+1} \leq \max_{u\in\mathbb{W}_{k+1}} \overline{\Phi}\big(\widehat{\Psi}(u)\big). \tag{2.26}$$

By Lemma 2.4 and (2.25), we obtain

$$\max_{-\mathbb{U}_k} \overline{\Phi}(\widehat{\Psi}(u)) \leq \alpha_k + \epsilon + C_{11}\left(|\alpha_k + \epsilon|^{\frac{\theta+1}{\mu}} + 1\right). \tag{2.27}$$

From (2.25) – (2.27) it follows that

$$\alpha_{k+1} \leq \alpha_k + \epsilon + C_{11}\left(|\alpha_k + \epsilon|^{\frac{\theta+1}{\mu}} + 1\right).$$

Since ϵ is arbitrary, we have

$$\alpha_{k+1} \le \alpha_k \left[1 + C_{21}\left(\alpha_k^{\frac{\theta+1-\mu}{\mu}} + \alpha_k^{-1}\right)\right] \quad \text{for all} \quad k \ge k_1.$$

Therefore, by iteration, we obtain

$$\alpha_{k_1+\ell} \le \alpha_{k_1} \prod_{k=k_1}^{k_1+\ell-1} \left[1 + C_{10}\left(\alpha_k^{\frac{\theta+1-\mu}{\mu}} + \alpha_k^{-1}\right)\right]$$

$$\le \alpha_{k_1} \exp C_{10} \left[\sum_{k=k_1}^{k_1+\ell-1} \left(\alpha_k^{\frac{\theta+1-\mu}{\mu}} + \alpha_k^{-1}\right)\right].$$

Combining (2.16), (1.3) and $p \in (2, 2_*)$, we get that

$$\alpha_{k_1+\ell} \le \alpha_{k_1} \exp\left[C_{10}\left(\sum_{k=k_1}^{\infty} k^{\frac{\theta+1-\mu}{\mu}\left(\frac{4p}{N(p-2)}-1\right)} + k^{-\left(\frac{4p}{N(p-2)}-1\right)}\right)\right] < \infty \quad \text{for all} \quad \ell \in \mathbb{N},$$

which yields a contradiction, that concludes the proof of Lemma 2.11.

This paper was supported by The International Center for Research and Postgraduate Training in Mathematics — Institute of Mathematics, Vietnam Academy of Science and Technology (Grant ICRTM04-2021.03).

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Received 21.10.21