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THE FORCING METRIC DIMENSION OF A TOTAL GRAPH OF NON-ZERO ANNIHILATING IDEALS

ФОРСОВАНА МЕТРИЧНА РОЗМІРНІСТЬ ТОТАЛЬНОГО ГРАФА НЕНУЛЬОВИХ АНІГІЛЯЦІЙНИХ ІДЕАЛІВ

Let R be a commutative ring with identity, which is not an integral domain. An ideal I of a ring R is called an annihilating ideal if there exists $r \in R - \{0\}$ such that $Ir = (0)$. The total graph of non-zero annihilating ideals of R, denoted by $\Omega(R)$, is a graph with the vertex set $A(R)^*$, the set of all non-zero annihilating ideals of R, and two distinct vertices I and J are joined if and only if $I + J$ is also an annihilating ideal of R. We study the forcing metric dimension of $\Omega(R)$ and determine the forcing metric dimension of $\Omega(R)$. It is shown that the forcing metric dimension of $\Omega(R)$ is equal either to zero or to the metric dimension.

Нехай R — комутативне кільце з одиницею, яке не є цілісною областю. Ідеал I кільця R називається анігіляційним ідеалом, якщо існує таке $r \in R - \{0\}$, що $Ir = (0)$. Тотальний граф ненульових анігіляційних ідеалів R, позначений як $\Omega(R)$, це граф із множиною вершин $A(R)^*$, множиною всіх ненульових анігіляційних ідеалів R. Крім того, дві різні вершини I, J графа з'єднані тоді й лише тоді, коли $I + J$ також є анігіляційним ідеалом R. Ми вивчаємо форсовану метричну розмірність $\Omega(R)$ і визначаємо форсовану метричну розмірність $\Omega(R)$. Показано, що форсована метрична розмірність $\Omega(R)$ дорівнює або нулю, або його метричній розмірності.

1. Introduction. Assigning a metric dimension to a graph was first introduced by Harary and Melter in [8] and it has been studied for a wide variety of graphs, e.g., trees and unicyclic graphs [4], wheel graphs [15] and Cartesian product graphs [9]. Also, a number of results have been presented regarding the strong metric dimension of Cartesian product graphs and Cayley graphs [11], distance-hereditary graphs [10]. Later, the concept of metric dimension and strong metric dimension were applied to graphs associated to commutative rings (see, for example [6, 7, 12, 14]). The fixing number and metric dimension of the zero-divisor graph have been calculated in [16] and the forcing dimensions of some well-known graphs have been studied in [5].

In [1], the authors have studied the metric dimension of a total graph of non-zero annihilating ideals. In this paper, we study the forcing metric dimension of a total graph of non-zero annihilating ideals.

Throughout this paper, all rings are assumed to be commutative with identity and they are not integral domains. The sets of all zero-divisors, nilpotent elements, minimal prime ideals, maximal ideals and Jacobson radical of R are denoted by $Z(R)$, $\mathrm{Nil}(R)$, $\mathrm{Min}(R)$, $\mathrm{Max}(R)$ and $J(R)$, respectively. For a subset T of a ring R we let $T^* = T - \{0\}$. An ideal with non-zero annihilator is called an *annihilating ideal*. The set of annihilating ideals of R is denoted by $\mathbb{A}(R)$. For every subset I of R, we denote the *annihilator* of I by $\mathrm{ann}(I)$. Some more definitions about commutative rings can be find in [2, 3].

We use the standard terminology of graphs following [18]. By $G = (V, E)$, we mean a graph, where V and E are the set of vertices and edges, respectively. If we can find at least one path between two any vertices of G, then G is called *connected*. Also, the length of the shortest path between two

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distinct vertices x and y is denoted by $d(x, y)$ (note that $d(x, y) = \infty$, if there is no path between x and y) and $\operatorname{diam}(G) = \max\big\{d(x, y) \mid x, y \in V\big\}$ is called the *diameter* of G.

The *girth* of a graph G, denoted by $\operatorname{girth}(G)$, is the length of the shortest cycle in G. The graph $H = (V_0, E_0)$ is a subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced* subgraph by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \big\{\{u, v\} \in E \mid u, v \in V_0\big\}$. Let $x \in V$, then $N(x) = \big\{y \in V \mid \{x, y\} \in E\big\}$ and $N[x] = N(x) \cup \{x\}$.

Let $G = (V, E)$ be a connected graph, $S = \{v_1, v_2, \ldots, v_k\}$ be an ordered subset of V and $v \in V(G) \setminus S$. The metric representation of v with respect to S is the k-vector $D(v|S) = \big(d(v, v_1), d(v, v_2), \ldots, d(v, v_k)\big)$. For $S \subseteq V$, if, for every $v, u \in V(G) - S$, $D(u|S) = D(v|S)$ implies that $u = v$, then S is called the resolving set for G. The metric basis for G is a resolving set S of minimum cardinality and the number of elements in S is called the metric dimension of G $(\dim_M(G))$. For a resolving set W of G, a subset S of W is called the forcing subset of W if W is the unique resolving set containing S. The forcing number $f\big(W, \dim(G)\big)$ of W in G is the minimum cardinality of a forcing subset for W, while the forcing metric dimension, $f_{\dim}(G)$, of G is the smallest forcing number among all resolving sets of G.

For a graph G with $|V(G)| \geq 2$, if, for all $x \in V(G) - \{u, v\}$, $d(u, x) = d(v, x)$ $(u, v$ are two distinct vertices), then u, v are distance similar. Clearly, if either $u - v \notin E(G)$ and $N(u) = N(v)$ or $u - v \in E(G)$ and $N[u] = N[v]$, then two distinct vertices u and v are distance similar.

An k*-partite* graph is one whose vertex set can be partitioned into k subsets so that an edge has both ends in no subset. A *complete* k*-partite* graph is an k-partite graph in which each vertex is adjacent to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with part sizes m and n is denoted by $K^{m,n}$. If $m = 1$, then the bipartite graph is called star graph. A *complete* graph is a graph such that there exists an edge between each pair of vertices and is denoted by K^n.

Let R be a commutative ring with identity which is not an integral domain. An ideal I of a ring R is called an annihilating ideal if there exists $r \in R - \{0\}$ such that $Ir = (0)$. S. Visweswaran and H. D. Patel [17] associated a graph with the set of all non-zero annihilating ideals of R, denoted by $\Omega(R)$ as the graph with the vertex-set $A(R)^*$, the set of all non-zero annihilating ideals of R and two distinct vertices I, J are joined if and only if $I + J$ is also an annihilating ideal of R. In this paper, we study the forcing metric dimension of $\Omega(R)$ and the forcing metric dimension of $\Omega(R)$ is determined. It is shown that the forcing metric dimension of $\Omega(R)$ is zero or equal to its metric dimension.

2. Forcing metric dimension of a total graph of a reduced ring. Let R be a commutative ring. In this section, we discuss the forcing metric dimension for a total graph of non-zero annihilating ideals when R is reduced. It is shown that the forcing metric dimension of $\Omega(R)$ is equal to zero, when $|\operatorname{Max}(R)| \geq 4$.

First, some results on the metric dimension and forcing metric dimension which will be used later.

Lemma 2.1. *Let G be a connected graph and Σ be the set of all resolving sets of G. Then*:

(1) $f_{\dim}(G) = 0$ *if and only if* $|\Sigma| = 1$ *(G has unique resolving set)*;

(2) $f_{\dim}(G) = 1$ *if and only if* $|\Sigma| \geq 2$ *and there exists* $W \in \Sigma$ *such that* $W \nsubseteq \cup_{W_i \in \Sigma, W_i \neq W} W_i$ (*in other words*, G *has at least two distinct resolving sets and some vertices belong to exactly one of them*);

(3) $f_{\dim}(G) = \dim(G)$ *if and only if*, *for every* $W \in \Sigma$ *and* $\varnothing \neq S \subsetneq W$ $(S \neq W)$, *there exists* $W' \in \Sigma$ *such that* $S \subsetneq W'$ *with* $W \neq W'$ (*no resolving set of* G *is the unique resolving set of* G *containing any of its proper subsets*).

Proof. 1. If $|\Sigma| = 1$, then it is clear that $f_{\dim}(G) = 0$. Assume that $f_{\dim}(G) = 0$. Then, for some $W \in \Sigma$, $f\big(W, \dim(G)\big) = 0$. If $W \subseteq \cup_{W_i \in \Sigma, W_i \neq W} W_i$, then we have $f\big(W, \dim(G)\big) \geq 1$, a contradiction and if $W \nsubseteq \cup_{W_i \in \Sigma, W_i \neq W} W_i$, then $f\big(W, \dim(G)\big) = 1$ again a contradiction. So, $|\Sigma| = 1$.

2. Assume that $f_{\dim}(G) = 1$. By Part 1, $|\Sigma| \geq 2$. Now, we show that there exists $W \in \Sigma$ such that $W \nsubseteq \cup_{W_i \in \Sigma, W_i \neq W} W_i$. But if for every $W \in \Sigma$, $W \subseteq \cup_{W_i \in \Sigma, W_i \neq W} W_i$. Then, for every $x \in W$, there exists $W' \in \Sigma$ with $W \neq W'$ such that $x \in W'$. This implies that $f\big(W, \dim(G)\big) \geq 2$ and hence $f_{\dim}(G) \geq 2$, a contradiction. Conversely, since $|\Sigma| \geq 2$, by Part 1, $f_{\dim}(G) \geq 1$. On the other hand, since there exists $W \in \Sigma$ such that $W \nsubseteq \cup_{W_i \in \Sigma, W_i \neq W} W_i$, $f\big(W, \dim(G)\big) \leq 1$, and hence $f_{\dim}(G) \leq f\big(W, \dim(G)\big) \leq 1$.

3. First, suppose that $f_{\dim}(G) = \dim(G)$. If there exist $W \in \Sigma$ and $S \subsetneq W$ $(S \neq W)$ such that there is no $W' \in \Sigma$ with $S \subsetneq W'$, then $f_{\dim}(G) \leq f\big(W, \dim(G)\big) \leq |S| < |W| = \dim(G)$, a contradiction. Conversely, since, for every $W \in \Sigma$ and $\varnothing \neq S \subsetneq W$ $(S \neq W)$, there exists $W' \in \Sigma$ such that $S \subsetneq W'$ with $W \neq W'$, we have $f\big(W, \dim(G)\big) = |W| = \dim(G)$ and hence $f_{\dim}(G) = \dim(G)$.

Lemma 2.1 is proved.

In [5], it has been shown that for all integers a, b with $0 \leq a \leq b$ and $b \geq 1$, there exists a nontrivial connected graph G with $f_{\dim}(G) = a$ and $\dim(G) = b$ if and only if $\{a, b\} \neq \{0, 1\}$. In connection with this result, it is shown that for all integers b with $b \geq 1$, there exists a reduced ring R such that $\dim(\Omega(R)) = b$ but $f_{\dim}(\Omega(R)) \in \{0, 1, 2\}$.

If R is a reduced ring with finitely many ideals, then R is Artinian ring and so by [2, Theorem 8.7], R is direct product of finitely many fields. Using this, we calculate the forcing metric dimension of $\Omega(R)$.

Theorem 2.1. *Suppose that* R *is a reduced ring with identity. If* $\dim_M(\Omega(R))$ *is finite*, *then*:

(1) *if* $|\mathrm{Max}(R)| = 2$, *then* $f_{\dim}(\Omega(R)) = \dim(\Omega(R)) = 1$;

(2) *if* $|\mathrm{Max}(R)| = 3$, *then* $f_{\dim}(\Omega(R)) = \dim_M(\Omega(R)) = 2$;

(3) *if* $|\mathrm{Max}(R)| = n \geq 4$, *then* $f_{\dim}(\Omega(R)) = 0$ *and* $\dim_M(\Omega(R)) = n$.

Proof. 1. By [1, Lemma 2.1], since $\dim_M(\Omega(R))$ is finite, R has finitely many ideals and so R is direct product of finitely many fields. If $|\mathrm{Max}(R)| = n = 2$, then $R \cong F_1 \times F_2$, where F_i is a field for every $1 \leq i \leq 2$. So $\Omega(R) = \overline{K}_2$ and hence $f_{\dim}(\Omega(R)) = \dim(\Omega(R)) = 1$.

If $n = 3$, then $R \cong F_1 \times F_2 \times F_3$, where F_i is a field for every $1 \leq i \leq 3$. Now, we put

$$W_1 = \big\{(0) \times F_2 \times F_3, F_1 \times (0) \times F_3\big\},$$

$$W_2 = \big\{(0) \times F_2 \times F_3, F_1 \times F_2 \times (0)\big\},$$

$$W_3 = \big\{F_1 \times (0) \times F_3, F_1 \times F_2 \times (0)\big\}.$$

By Figure 1, we can easily get

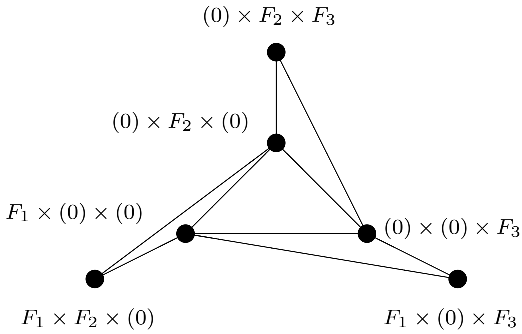

Fig. 1. $\Omega(F_1 \times F_2 \times F_3)$.

$$f\big(W_1, \dim(\Omega(R))\big) = f(W_1, 2) = 2,$$

$$f\big(W_2, \dim(\Omega(R))\big) = f(W_2, 2) = 2,$$

$$f\big(W_3, \dim(\Omega(R))\big) = f(W_3, 2) = 2.$$

Therefore, $f_{\dim}(\Omega(R)) = \dim(\Omega(R)) = 2$.

2. First assume that $n = 4$. So $R \cong F_1 \times F_2 \times F_3 \times F_4$, where F_i is a field for every $1 \le i \le 4$. By proof of [1, Theorem 2.1], $W = \mathrm{Max}(R)$ is a metric basis of $\Omega(R)$. We show that W is the unique resolving set of $\Omega(R)$. For this, let W' be a metric basis for $\Omega(R)$ with $W \neq W'$ and let

$$A = \big\{(F_1, 0, 0, 0), (0, F_2, 0, 0), (0, 0, F_3, 0), (0, 0, 0, F_4)\big\},$$

$$B = \big\{(F_1, F_2, 0, 0), (F_1, 0, F_3, 0), (F_1, 0, 0, F_4), (0, F_2, F_3, 0), (0, F_2, 0, F_4), (0, 0, F_3, F_4)\big\}.$$

If $W' \cap A \neq \varnothing$, then without loss of generality, we assume that $(F_1, 0, 0, 0) \in W' \cap A$. Since the only vertex I that $d\big((F_1, 0, 0, 0), I\big) = 2$ is $I = (0, F_2, F_3, F_4)$, $d\big((F_1, 0, 0, 0), J\big) = 1$ for all $J \in V(\Omega(R)) \setminus \big\{(F_1, 0, 0, 0), (0, F_2, F_3, F_4)\big\}$. Let $W' = \big\{(F_1, 0, 0, 0), w_2, w_3, w_4\big\}$. Hence, for every $J \in V(\Omega(R)) \setminus \big\{(0, F_2, F_3, F_4)\big\}$, the first component of the 4-vector $D(J|W')$ must be 1 and hence there are only 8 possibilities for $D(J|W')$. This implies that $\big|V(\Omega(R)) \setminus \big\{(0, F_2, F_3, F_4)\big\}\big| - 4 \le 8$, a contradiction. Therefore, $W' \cap A = \varnothing$. Now, we show that $W' \cap B = \varnothing$ too. If $W' \cap B \neq \varnothing$, then we let $W' = \big\{(0, 0, F_3, F_4), w_2, w_3, w_4\big\}$. Since $D((F_1, 0, 0, 0)|W') \neq D((0, F_2, 0, 0) \mid W')$, $\{w_2, w_3, w_4\} \cap \mathrm{Max}(R) \neq \varnothing$. So we let $w_2 = (F_1, 0, F_3, F_4)$. Since $D((F_1, 0, 0, 0)|W') \neq D((0, 0, F_3, 0) \mid W')$, we must have $w_3 = (F_1, F_2, 0, F_4)$. Similarly, $w_4 = (F_1, F_2, F_3, 0)$ and so $W' = \big\{(0, 0, F_3, F_4), (F_1, 0, F_3, F_4), (F_1, F_2, 0, F_4), (F_1, F_2, F_3, 0)\big\}$. Now, we can easily get $D\big((0, 0, 0, F_4)|W'\big) = (1, 1, 1, 2) = D\big((F_1, 0, 0, F_4) \mid W'\big)$, a contradiction. Therefore, $W' \cap B = \varnothing$. In fact, $W' = \varnothing$ and $W = \mathrm{Max}(R)$ is the unique resolving set of $\Omega(R)$ and hence, by Lemma 2.1, $f_{\dim}(\Omega(R)) = 0$.

Now, assume that $n \ge 5$ and $R \cong F_1 \times \ldots \times F_n$, where F_i is a field for every $1 \le i \le n$. We show that $f_{\dim}(\Omega(R)) = 0$.

By proof of Theorem 2.1 in [1], $W = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$ is a metric basis for $\Omega(R)$, where $\mathfrak{m}_i \in \mathrm{Max}(R)$ for every $1 \le i \le n$. We show that W is the only possible set for a metric basis of $\Omega(R)$ and hence by Lemma 2.1, $f_{\dim}(\Omega(R)) = 0$. For this, let W' be a metric basis for $\Omega(R)$ with $W \neq W'$ and let $(w'_1, w'_2, \ldots, w'_n) = w' \in W' \setminus W$. By $ZC(w')$ we mean the number of zero components of w'. Since $W = \mathrm{Max}(R)$, $w' \notin \mathrm{Max}(R)$ and so at least two of the components of w' must be zero and hence $ZC(w') \geq 2$. We show that $ZC(w') \geq 3$. If not, $ZC(w') = 2$ and so without loss of generality, we can assume that $w'_1 = 0$ and $w'_2 = 0$. Now, let $A = \{I = (I_1, I_2, \ldots, I_n) \mid I \in V(\Omega(R)) \text{ and } I_1 = F_1, I_2 = F_2\}$. Then we can easily get for every $J \in V(\Omega(R))$, $d(w', J) = 2$ if and only if $J \in A$. This, together with $\operatorname{diam}(\Omega(R)) \in \{1, 2\}$ (see [17]), imply that for every $K \in V(\Omega(R))$, $d(w', K) = 1$ if and only if $K \in B$, where $B = V(\Omega(R)) \setminus \{A \cup \{w'\}\}$. Since $|A| = 2^{n-2} - 1$ and $|V(\Omega(R))| = 2^n - 2$, $|B| = 2^n - 2^{n-2} - 2$. Now, without loss of generality, we can assume that $W' = \{w', v_2, \ldots, v_n\}$. Hence, for every $I \in B$, the first component of the n-vector $D(I|W')$ must be 1 and there are only 2^{n-1} possibilities for $D(I|W')$. This implies that $|B| - n \leq 2^{n-1}$. Since $n \geq 5$, a contradiction. Therefore, $ZC(w') \geq 3$. Now, by induction on $ZC(w')$, we get that $W' = \varnothing$ and so W is the only possible set for a metric basis of $\Omega(R)$.

Theorem 2.1 is proved.

The following is an immediate consequence of the above results.

***Corollary* 2.1.** *Suppose that R is a reduced ring with identity. If $\dim_M(\Omega(R))$ is finite, then $f_{\dim}(\Omega(R)) \in \{0, 1, 2\}$.*

3. Forcing metric dimension of a total graph of a non-reduced ring. In this section, we study the forcing metric dimension of $\Omega(R)$ when R is non-reduced. We show that for a non-reduced ring R the metric dimension and the forcing metric dimension of $\Omega(R)$ are equal.

Theorem 3.1. *Suppose that $R \cong R_1 \times \ldots \times R_n$, where R_i is an Artinian local ring such that, for every $1 \leq i \leq n$, $|A(R_i)^*| \geq 1$. Then $f_{\dim}(\Omega(R)) = \dim_M(\Omega(R)) = |A(R)^*| - 2^n + 1$.*

Proof. Suppose that $I = (I_1, \ldots, I_n)$ and $J = (J_1, \ldots, J_n)$ are vertices of $\Omega(R)$. Define the relation $\sim$ on $V(\Omega(R))$ as follows: $I \sim J$, whenever for each $1 \leq i \leq n$, "$I_i \subseteq \operatorname{Nil}(R_i)$ if and only if $J_i \subseteq \operatorname{Nil}(R_i)$".

Clearly, $\sim$ is an equivalence relation on $V(\Omega(R))$. The equivalence class of I is denoted by $[I]$. Suppose that X and Y are two elements of the equivalence class of I. Let $K \in N(X)$. Then, since $K + X$ is an annihilating ideal and $X \sim Y$, we have that $K + Y$ is also an annihilating ideal and hence $K \in N(Y)$. This means that $N(X) \subseteq N(Y)$. Similarly, $N(Y) \subseteq N(X)$. Therefore, $N(X) = N(Y)$. Also, the number of equivalence classes is $2^n - 1$. Now, let $[I]$ be an arbitrary equivalence class and $X, Y \in [I]$. Since $N(X) = N(Y)$, we obtain that $X \in W$ or $Y \in W$, where W is the metric basis for the graph $\Omega(R)$. This implies that $[I] \setminus \{I\} \subseteq W$ (also see the proof of Theorem 3.2 in [1]). We show that there is no proper forcing subset S of W such that W is the unique resolving set containing S. Assume to the contrary, there is a proper forcing subset S of W such that W is the unique resolving set containing S. Assume that $K \in W \setminus S$. We have $[K] \setminus \{K\} \subseteq W$. Since $|[K]| \geq 2$, we put $K' \in [K] \setminus \{K\}$ and $W' = \{W \cup \{K'\}\} \setminus \{K\}$. In fact, W' is obtained from W by replacing K with K'. Since $K \sim K'$, $N(K) = N(K')$. So we get that W' is the metric basis for the graph $\Omega(R)$ such that $S \subseteq W'$, a contradiction. Therefore, there is no resolving set of $\Omega(R)$ that is the unique resolving set of $\Omega(R)$ containing any of its proper subsets. Hence $f_{\dim}(\Omega(R)) = \dim_M(\Omega(R))$. On the other hand, since $\dim_M(\Omega(R)) = |A(R)^*| - 2^n + 1$ by [1, Theorem 3.2], $f_{\dim}(\Omega(R)) = \dim_M(\Omega(R)) = |A(R)^*| - 2^n + 1$.

Theorem 3.1 is proved.

***Corollary* 3.1.** *Let R be a non-reduced ring such that for every $m \in \operatorname{Max}(R)$, $\operatorname{ann}(m) \subseteq m$. If $\dim_M(\Omega(R))$ is finite, then $f_{\dim}(\Omega(R)) = \dim_M(\Omega(R)) = |A(R)^*| - 2^n + 1$, where $n = |\operatorname{Max}(R)|$.*

Proof. Since $\dim_M(\Omega(R))$ is finite, by [1, Lemma 2.1], R is an Artinian ring and so $R \cong R_1 \times \ldots \times R_n$, where R_i is an Artinian local ring for every $1 \leq i \leq n = |\operatorname{Max}(R)|$. Also, since for every $m \in \operatorname{Max}(R)$, $\operatorname{ann}(m) \subseteq m$, $|A(R_i)^*| \geq 1$. Now, the proof follows from Theorem 3.1.

Theorem 3.2. *Let* $R \cong R_1 \times \ldots \times R_n \times F_1 \times \ldots \times F_m$, *be a ring,* $n \geq 1,\ m \geq 1$, *where each* R_i *is an Artinian local ring with* $|A(R_i)^*| \geq 1$ *and each* F_i *is a field. Then* $f_{\dim}(\Omega(R)) = \dim(\Omega(R)) = |A(R)^*| - 2^{n+m} + m + 1$.

Proof. Let

$$A = \big\{(R_1, \ldots, R_n, J_{n+1}, \ldots, J_{n+m}) \in V(\Omega(R)) \mid J_i \in \{0, F_i\}\ for\ n+1 \leq i \leq n+m\big\},$$

$$B = V(\Omega(R)) \setminus A,$$

$$C = \big\{K_{n+1}, K_{n+2}, \ldots, K_{n+m}\big\}.$$

Assume that W is a metric basis for the graph $\Omega(R)$. We show that for every resolving set W, there is no proper forcing subset S of W such that W is the unique resolving set containing S. We continue the proof in two cases.

Case 1. For every $n+1 \leq i \leq n+m$, let $K_i = \big(R_1, R_2, \ldots, R_n, J_{n+1}, J_{n+2}, \ldots, J_{n+m}\big)$ such that $J_j = 0$ if $i = j$, and $J_j = F_j$ if $i \neq j$, and let $C = \big\{K_{n+1}, K_{n+2}, \ldots, K_{n+m}\big\}$.

We know that $C \subseteq \mathrm{Max}(R)$. By proof of Theorem 2.1, we can easily get C is the only resolving set for the vertices of A and hence $C \subseteq W$. This implies that $C \subseteq S$ where S is the forcing subset of W.

Case 2. Let $W' = W \setminus C$ and $(I_1, \ldots, I_n, J_{n+1}, \ldots, J_{n+m}) = I \in W'$. Since $I \notin C$, we have, for some $1 \leq i \leq n,\ I_i \subseteq \mathrm{Nil}(R_i)$. This implies that $\big|[I]\big| \geq 2$. Now, by proof of Theorem 3.1, there is no proper forcing subset S' of W' such that W' is the unique resolving set containing S' for the vertices of B. Therefore, by Cases 1 and 2 there is no resolving set of $\Omega(R)$ that is the unique resolving set of $\Omega(R)$ containing any of its proper subsets and hence $f_{\dim}(\Omega(R)) = \dim_M(\Omega(R))$. On the other hand, by [1, Theorem 3.2], $\dim_M(\Omega(R)) = |A(R)^*| - 2^{n+m} + m + 1$ so we have $f_{\dim}(\Omega(R)) = \dim_M(\Omega(R)) = |A(R)^*| - 2^{n+m} + m + 1$.

Theorem 3.2 is proved.

From Theorem 3.2, the Corollary 3.2 can be obtained.

***Corollary* 3.2.** *Suppose that* R *is a ring with identity. If* $\dim_M(\Omega(R))$ *is finite, then* $f_{\dim}(\Omega(R)) \in \big\{0, \dim_M(\Omega(R))\big\}$.

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