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MORE ON STABILITY OF TWO FUNCTIONAL EQUATIONS

БІЛЬШЕ ПРО СТІЙКІСТЬ ДВОХ ФУНКЦІОНАЛЬНИХ РІВНЯНЬ

We prove the generalized stability of the functional equations $\|f(x+y)\| = \|f(x)+f(y)\|$ and $\|f(x-y)\| = \|f(x)-f(y)\|$ in p-uniformly convex spaces with $p \geq 1$.

Доведено узагальнену стійкість функціональних рівнянь $\|f(x+y)\| = \|f(x)+f(y)\|$ і $\|f(x-y)\| = \|f(x)-f(y)\|$ у p-рівномірно опуклих просторах з $p \geq 1$.

1. Introduction. The Hyers – Ulam stability problem of functional equations whether for a function satisfies some functional equations approximately there exists a function satisfying it exactly and being uniformly close to the former one was proposed by Ulam [27]. One years later, Hyers [11] first partially resolved the Ulam problem for the Cauchy functional equation on Banach spaces. This stability phenomenon of functional equations is called Hyers – Ulam stability. Since then Ulam's problem has attracted a large number of mathematicians to investigate this subject for a broad class of functional equations. See, for example, Jung, Popa and Rassias [14], Brzdek, Popa and Xu [6] for linear functional equation in a single variable; Abdollahpoura, Aghayaria and Rassias [2] for Laguerre differential equations; Miura, Miyajima and Takahasi [20] for first order linear differential operators; Jin, Park and Rassias [13] for hom-derivations in C^*-ternary algebras; Jung et al. [16, 17] for mean value type functional equations. For more background on this topic, we refer to [1, 3, 5, 12, 15, 18, 22, 23] and references therein.

In [21], Rassias generalized the result of Hyers for linear mappings by considering an unbounded Cauchy difference and proved the following theorem.

Theorem 1.1. *Let f be a map form a Banach space E into a Banach space F, and assume that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for some $\theta > 0,\ 0 \leq p < 1$, and for all $x, y \in E$. Then there exists a unique additive map $T\colon E \to F$ which satisfies

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\,\|x\|^p$$

for all $x \in E$.

The functional equations

$$\|f(x+y)\| = \|f(x) + f(y)\|$$

and

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$$\|f(x-y)\| = \|f(x) - f(y)\|$$

have extensively been studied by many mathematicians (see, e.g., [7–9, 24–26]).

In 2003, Tabor [26] proved the following theorem which implies the stability of the functional equation $\|f(x+y)\| = \|f(x) + f(y)\|$.

Theorem 1.2. *Let $(X,+)$ be a group, E be a real Banach space and $f\colon X \to E$ be a surjective map. If*

$$|\,\|f(x+y)\| - \|f(x)+f(y)\|\,| \le \varepsilon \quad \textit{for all} \quad x, y \in X,$$

then

$$\|f(x+y) - f(x) - f(y)\| \le 13\varepsilon \quad \textit{for all} \quad x, y \in X.$$

In 2005, Sikorska [24] proved the following theorem which implies the stability of the functional equation $\|f(x-y)\| = \|f(x) - f(y)\|$.

Theorem 1.3. *Let $(X,+)$ be an Abelian group, E be a real Banach space and $f\colon X \to E$ be a δ-surjective map. If*

$$|\,\|f(x-y)\| - \|f(x)-f(y)\|\,| \le \varepsilon \quad \textit{for all} \quad x, y \in X,$$

then

$$\|f(x+y) - f(x) - f(y)\| \le 5\varepsilon + 5\delta \quad \textit{for all} \quad x, y \in X.$$

Making use of a result of Lindenstrauss and Szankowski [19], Dong [7] generalized these two theorems by large perturbation and proved the following results.

Theorem 1.4. *Let $(X,+)$ be an Abelian group, and E be a real Banach space. Assume that $f\colon X \to E$ is a surjective map. Put*

$$\phi_f(t) = \sup\{|\,\|f(x)-f(y)\| - \|f(x-y)\|\,| : \|f(x)-f(y)\| \le t \textit{ or } \|f(x-y)\| \le t\}$$

for $t \ge 0$. If

$$\int_1^\infty \frac{\phi_f(t)}{t^2}\,dt < \infty, \tag{1.1}$$

then, for any $x \in X$, we have

$$\|f(x+y) - f(x) - f(y)\| = o(\|f(y)\|) \quad \textit{as} \quad \|f(y)\| \to \infty.$$

Theorem 1.5. *Let $(X,+)$ be an Abelian group and E be a real Banach space. Assume that $f\colon X \to E$ is a surjective map. Put*

$$\overline{\phi}_f(t) = \sup\{|\,\|f(x)+f(y)\| - \|f(x+y)\|\,| : \|f(x)+f(y)\| \le t \textit{ or } \|f(x+y)\| \le t\}$$

for $t \ge 0$. If

$$\int_1^\infty \frac{\overline{\phi}_f(t)}{t^2}\,dt < \infty, \tag{1.2}$$

then, for any $x \in X$, *we have*

$$\|f(x+y) - f(x) - f(y)\| = o(\|f(y)\|) \quad as \quad \|f(y)\| \to \infty.$$

Let f be a mapping form a group X to a real Banach space E. Put

$$\alpha_f(t) = \sup\{|\,\|f(x) - f(y)\| - \|f(x-y)\|\,| : \|f(x) - f(y)\| \leq t\} \tag{1.3}$$

and

$$\overline{\alpha}_f(t) = \sup\{|\,\|f(x) + f(y)\| - \|f(x+y)\|\,| : \|f(x) + f(y)\| \leq t\}. \tag{1.4}$$

In this paper, we first show that the integral convergence conditions (1.1) is equivalent to

$$\int\limits_1^\infty \frac{\alpha_f(t)}{t^2} dt < \infty, \tag{1.5}$$

and the integral convergence conditions (1.2) is equivalent to

$$\int\limits_1^\infty \frac{\overline{\alpha}_f(t)}{t^2}\, dt < \infty. \tag{1.6}$$

Moreover, we generalize the Theorems 1.2 and 1.3 by large perturbation in p-uniformly convex spaces with $p \geq 1$.

2. Main results. To begin with, we show the following proposition.

Proposition 2.1. *Let* $(X, +)$ *be an Abelian group,* E *be a real Banach space and* $f : X \to E$ *be a surjective map. Let* α_f *be as in* (1.3). *If*

$$\int\limits_1^\infty \frac{\alpha_f(t)}{t^2}\, dt < \infty,$$

then, for any $x \in X$, *we have*

$$\|f(x+y) - f(x) - f(y)\| = o(\|f(y)\|) \quad as \quad \|f(y)\| \to \infty.$$

Proof. Suppose that

$$\int\limits_1^\infty \frac{\alpha_f(t)}{t^2}\, dt < \infty.$$

We claim that there exists a constant $M > 0$ such that $t < 2(t - \alpha_f(t))$ for every $t > M$. If not, for every positive integer n we can find $t_n > n$ such that $\dfrac{t_n}{2} \leq \alpha_f(t_n)$. Then we obtain

$$\int\limits_{t_n}^{2t_n} \frac{\alpha_f(t)}{t^2}\, dt \geq \int\limits_{t_n}^{2t_n} \frac{\alpha_f(t_n)}{t^2}\, dt = \alpha_f(t_n)\, \frac{1}{2t_n} \geq \frac{1}{4},$$

which leads to a contradiction.

Let $\|f(x-y)\| \leq t$. If $\|f(x)-f(y)\| > M$, then we obtain

$$\|f(x)-f(y)\| < 2(\|f(x)-f(y)\| - \alpha_f(\|f(x)-f(y)\|)) \leq 2\|f(x-y)\| \leq 2t,$$

this yields

$$|\,\|f(x)-f(y)\| - \|f(x-y)\|\,| \leq \alpha_f(2t). \tag{2.1}$$

If $\|f(x)-f(y)\| \leq M$, then

$$|\,\|f(x)-f(y)\| - \|f(x-y)\|\,| \leq \alpha_f(M). \tag{2.2}$$

Now let $\|f(x)-f(y)\| \leq t$, then

$$|\,\|f(x)-f(y)\| - \|f(x-y)\|\,| \leq \alpha_f(t). \tag{2.3}$$

So, if $\phi_f(t)$ is given in Theorem 1.4, (2.1), (2.2) with (2.3) together implies

$$\phi_f(t) \leq \max\{\alpha_f(M), \alpha_f(2t)\} \quad \text{for} \quad t \geq 0.$$

Then

$$\int\limits_M^\infty \frac{\phi_f(t)}{t^2}\,dt \leq \int\limits_M^\infty \frac{\alpha_f(2t)}{t^2}\,dt = 2\int\limits_{2M}^\infty \frac{\alpha_f(t)}{t^2}\,dt < \infty.$$

Therefore,

$$\int\limits_1^\infty \frac{\phi_f(t)}{t^2}\,dt < \infty,$$

and, hence, the result follows from Theorem 1.4.

Proposition 2.1 is proved.

***Remark* 2.1.** If $0 \leq \alpha_f \leq \phi_f$, then

$$\int\limits_1^\infty \frac{\alpha_f(t)}{t^2}\,dt \leq \int\limits_1^\infty \frac{\phi_f(t)}{t^2}\,dt.$$

On the other hand, according to the proof of Proposition 2.1, if $\displaystyle\int_1^\infty \frac{\alpha_f(t)}{t^2}\,dt < \infty$, then $\displaystyle\int_1^\infty \frac{\phi_f(t)}{t^2}\,dt < \infty$. The integral convergence conditions (1.1) and (1.5) are therefore equivalent. Similarly, the conditions (1.2) and (1.6) are equivalent.

In what follows, we show the generalized stability of the functional equations $\|f(x+y)\| = \|f(x)+f(y)\|$ and $\|f(x-y)\| = \|f(x)-f(y)\|$ in p-uniformly convex spaces with $p \geq 1$. We first recall that the modulus of convexity of a Banach space E is the function $\delta_E\colon [0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x-y\| \geq \varepsilon,\ x,y \in B_E\right\}.$$

A Banach space E is said to be uniformly convex provided $\delta_E(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$.

Definition 2.1. *A uniformly convex Banach space E is called p-uniformly convex if there exists a constant $C > 0$ such that $\delta_E(\varepsilon) \geq C\varepsilon^p$ for all $0 < \varepsilon \leq 2$.*

Recently, Cheng et al. [4] introduced the following perturbation function for a map f from a Banach space E_1 into a Banach space E_2 with $f(0) = 0$:

$$\varepsilon_f(t) = \sup\{|\,\|f(x) - f(y)\| - \|x - y\|\,| : \|x - y\| \leq t\}, \quad t \geq 0,$$

and showed the following celebrated theorem without the surjective assumption condition.

Theorem 2.1 [4, Theorem 2.5]. *Let f be a map from the Banach space E_1 into the p-uniformly convex space E_2. Assume that $f(0) = 0$ and*

$$\int_1^\infty \frac{\varepsilon_f(t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\, dt < \infty.$$

Then there exists a linear isometry $U : E_1 \to E_2$ such that

$$\|f(x) - U(x)\| = o(\|x\|) \quad \textit{as} \quad \|x\| \to \infty.$$

It is easy to check that the map f in the above Theorem 2.1 satisfies that $\varepsilon_f(t) = o(t)$ as $t \to \infty$. Such a map $f : E_1 \to E_2$ is named as coarse isometry in [4].

We are now ready to show the main result of this paper, which is a generalization of Theorem 1.3.

Theorem 2.2. *Let $(X, +)$ be an Abelian group, E be a p-uniformly convex space and $f : X \to E$ be a surjective map. Let α_f be as in* (1.3). *If*

$$\int_1^\infty \frac{\alpha_f(t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\, dt < \infty, \tag{2.4}$$

then, for any $x \in X$, we have

$$\|f(x + y) - f(x) - f(y)\| = o(\|f(y)\|) \quad \textit{as} \quad \|f(y)\| \to \infty.$$

Proof. Fix $x \in X$ and define a set-valued map $\Psi_x : E \to 2^E$ by

$$\Psi_x(u) = \big\{f(a_u + x) - f(x) : a_u \in f^{-1}(u)\big\}, \quad u \in E.$$

Fix $u, v \in E$ and take $z_u \in \Psi_x(u)$ and $z_v \in \Psi_x(v)$. This implies that there exist $a_u \in f^{-1}(u)$ and $a_v \in f^{-1}(v)$ such that $z_u = f(a_u + x) - f(x)$ and $z_v = f(a_v + x) - f(x)$. Then we obtain

$$\begin{aligned}
|\|z_u - z_v\| - \|u - v\|| &= |\|f(a_u + x) - f(a_v + x)\| - \|u - v\|| \\
&\leq |\|f(a_u + x) - f(a_v + x)\| - \|f(a_u - a_v)\|| + |\|f(a_u - a_v)\| - \|u - v\|| \\
&\leq \alpha_f(\|z_u - z_v\|) + \alpha_f(\|u - v\|).
\end{aligned} \tag{2.5}$$

We assert that: there exists a positive constant $M(\alpha)$ such that $t < 2(t - \alpha_f(t))$ for every $t > M(\alpha)$. Indeed, if for every positive integer n we can find $t_n > n$ such that $\dfrac{t_n}{2} \leq \alpha_f(t_n)$, then we get

$$\int_{t_n}^{2t_n} \frac{\alpha_f(t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\,dt \ge \int_{t_n}^{2t_n} \frac{\alpha_f(t_n)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\,dt = \alpha_f(t_n)^{\frac{1}{p}} \frac{p\left(2^{\frac{1}{p}}-1\right)}{2^{\frac{1}{p}} t_n^{\frac{1}{p}}}$$

$$\ge \frac{t_n^{\frac{1}{p}}}{2^{\frac{1}{p}}} \frac{p\left(2^{\frac{1}{p}}-1\right)}{2^{\frac{1}{p}} t_n^{\frac{1}{p}}} = \frac{p\left(2^{\frac{1}{p}}-1\right)}{2^{\frac{2}{p}}} > 0,$$

which contradicts to (2.4).

Let $\|u-v\| \le t$. Then we have

$$\|f(a_u - a_v)\| \le \|u-v\| + \alpha_f(\|u-v\|) \le t + \alpha_f(t).$$

If $\|z_u - z_v\| > M(\alpha)$, then

$$\|z_u - z_v\| < 2(\|z_u - z_v\| - \alpha_f(\|z_u - z_v\|)) \le 2f(a_u - a_v) \le 2(t + \alpha_f(t)). \tag{2.6}$$

If $\|z_u - z_v\| \le M(\alpha)$, then (2.5) implies

$$|\|z_u - z_v\| - \|u-v\|| \le \alpha_f(M(\alpha)) + \alpha_f(t). \tag{2.7}$$

Let $s_x\colon E \to E$ be an arbitrary selection of Ψ_x. (2.6) and (2.7) together implies that

$$\varepsilon_{s_x}(t) \le \max\{\alpha_f(M(\alpha)) + \alpha_f(t),\ \alpha_f(2(t+\alpha_f(t))) + \alpha_f(t)\} \quad \text{for} \quad t \ge 0.$$

Note that for $t > M(\alpha)$, we obtain $t < 2(t - \alpha_f(t))$, i.e., $t \ge 2\alpha_f(t) \ge \alpha_f(t)$. Thus, we get

$$\int_{M(\alpha)}^{\infty} \frac{\varepsilon_{s_x}(t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\,dt \le \int_{M(\alpha)}^{\infty} \frac{[\alpha_f(2(t+\alpha_f(t))) + \alpha_f(t)]^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\,dt$$

$$\le \int_{M(\alpha)}^{\infty} \frac{(2\alpha_f(4t))^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\,dt \le 2^{\frac{1}{p}} \int_{M(\alpha)}^{\infty} \frac{\alpha_f(4t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\,dt$$

$$= 2^{\frac{3}{p}} \int_{4M(\alpha)}^{\infty} \frac{\alpha_f(t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\,dt < \infty.$$

Therefore,

$$\int_{1}^{\infty} \frac{\varepsilon_{s_x}(t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\,dt < \infty.$$

By Theorem 2.1, there exists a linear isometry $I_{s_x}\colon E \to E$ such that

$$\|s_x(u) - s_x(0) - I_{s_x}(u)\| = o(\|u\|) \quad \text{as} \quad \|u\| \to \infty.$$

Therefore,

$$\|s_x(u) - I_{s_x}(u)\| = o(\|u\|) \quad \text{as} \quad \|u\| \to \infty. \tag{2.8}$$

Let $h_x \colon E \to E$ be another selection of Ψ_x. Combining (2.6) with (2.7), we have

$$\|h_x(u) - s_x(u)\| \le \alpha_f(M(\alpha)) + \alpha_f(0) \quad \text{for all} \quad u \in E.$$

Thus,

$$\begin{aligned}\|I_{s_x}(u) - I_{h_x}(u)\| &\le \|I_{s_x}(u) - s_x(u)\| + \|s_x(u) - h_x(u)\| + \|h_x(u) - I_{h_x}(u)\| \\ &\le o(\|u\|) + \alpha_f(M(\alpha)) + \alpha_f(0) \quad \text{as} \quad \|u\| \to \infty.\end{aligned}$$

This implies that $I_{s_x} = I_{h_x}$. Then we can denote I_{s_x} by I_x. By taking $u = f(y)$ in (2.8), we get

$$\|f(y + x) - f(x) - I_x(f(y))\| = o(\|f(y)\|) \quad \text{as} \quad \|f(y)\| \to \infty. \tag{2.9}$$

Fix $x_1, x_2 \in X$. By (2.5), (2.6) and (2.7), we have

$$\begin{aligned}\|I_{x_1} f(y) - I_{x_2} f(y)\| &\le \|I_{x_1} f(y) - (f(y + x_1) - f(x_1))\| \\ &\quad + \|(f(y + x_1) - f(x_1)) - (f(y + x_2) - f(x_2))\| \\ &\quad + \|(f(y + x_2) - f(x_2)) - I_{x_2} f(y)\| \\ &\le o(\|f(y)\|) + \big|\, \|f(y + x_1) - f(y + x_2)\| - \|f(x_1) - f(x_2)\| \,\big| \\ &\quad + 2\|f(x_1) - f(x_2)\| \le o(\|f(y)\|) + 2\|f(x_1) - f(x_2)\| \\ &\quad + \max\Big\{\alpha_f(M(\alpha)) + \alpha_f(\|f(x_1) - f(x_2)\|), \\ &\alpha_f(2(\|f(x_1) - f(x_2)\| + \alpha_f(\|f(x_1) - f(x_2)\|))) + \alpha_f(\|f(x_1) - f(x_2)\|)\Big\}\end{aligned}$$

as $\|f(y)\| \to \infty$. Thus, $I_{x_1} = I_{x_2}$. Taking $x = 0$ in (2.9), we obtain

$$\|f(y) - I_0(f(y))\| = o(\|f(y)\|) \quad \text{as} \quad \|f(y)\| \to \infty.$$

This implies $I_x = I$ for any $x \in X$. Therefore, the result follows from (2.9) by substituting $I_x = I$.

Theorem 2.2 is proved.

Theorem 2.3. *Let $(X, +)$ be an Abelian group, E be a p-uniformly convex space and $f\colon X \to E$ be a surjective map. Let $\overline{\alpha}_f$ be as in* (1.4). *If*

$$\int_1^\infty \frac{\overline{\alpha}_f(t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\, dt < \infty,$$

then, for any $x \in X$, we have

$$\|f(x + y) - f(x) - f(y)\| = o(\|f(y)\|) \quad \textit{as} \quad \|f(y)\| \to \infty.$$

Proof. Suppose that

$$\int\limits_1^\infty \frac{\overline{\alpha}_f(t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\,dt < \infty,$$

then there exists a positive constant $M(\overline{\alpha})$ such that $t < 2\big(t - \overline{\alpha}_f(t)\big)$ for every $t > M(\overline{\alpha})$.

Let $\|f(x+y)\| \leq t$. If $\|f(x)+f(y)\| > M(\overline{\alpha})$, then

$$\|f(x)+f(y)\| < 2(\|f(x)+f(y)\| - \overline{\alpha}_f(\|f(x)+f(y)\|)) \leq 2\|f(x+y)\| \leq 2t.$$

Therefore,

$$\|f(x)+f(y)\| \leq \max\{2\|f(x+y)\|,\ M(\overline{\alpha})\}. \tag{2.10}$$

By substituting $y=-x$ in (2.10), we obtain

$$\|f(x)+f(-x)\| \leq \max\{2\|f(0)\|, M(\overline{\alpha})\} \equiv \Lambda$$

for all $x \in X$.

For any $x, y \in X$,

$$\begin{aligned}\|f(x)-f(y)\| - \|f(x-y)\| &= \|f(x)+f(-y)-f(-y)-f(y)\| - \|f(x-y\| \\ &\leq \|f(x)+f(-y)\| - \|f(x-y\| + \|f(-y)+f(y)\| \\ &\leq |\,\|f(x)+f(-y)\| - \|f(x-y\|\,| + \Lambda. \end{aligned} \tag{2.11}$$

On the other hand,

$$\begin{aligned}\|f(x)-f(y)\| - \|f(x-y)\| &= \|f(x)+f(-y)-f(-y)-f(y)\| - \|f(x-y\| \\ &\geq \|f(x)+f(-y)\| - \|f(x-y\| - \|f(-y)+f(y)\| \\ &\geq -|\,\|f(x)+f(-y)\| - \|f(x-y\|\,| - \Lambda. \end{aligned} \tag{2.12}$$

Combining (2.11) with (2.12), we have

$$|\,\|f(x)-f(y)\| - \|f(x-y)\|\,| \leq |\,\|f(x)+f(-y)\| - \|f(x-y\|\,| + \Lambda. \tag{2.13}$$

Note that

$$\|f(x)+f(-y)\| \leq \|f(x)-f(y)\| + \|f(y)-f(-y)\| \leq \|f(x)-f(y)\| + \Lambda. \tag{2.14}$$

(2.13) and (2.14) together implies

$$\alpha_f(t) \leq \overline{\alpha}_f(t+\Lambda) + \Lambda \quad \text{for all} \quad t \geq 0.$$

Therefore,

$$\int\limits_1^\infty \frac{\alpha_f(t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\,dt < \infty.$$

Theorem 2.3 is proved.

Making use of the Hanner estimates [10]: L_p is 2-uniformly convex, if $1 < p \leq 2$; p-uniformly convex, if $2 < p < \infty$, we can get the following results.

***Corollary* 2.1.** *Let* $(X, +)$ *be an Abelian group and* $E = L_p$, $1 < p < \infty$. *Assume that* $f\colon X \to E$ *is a surjective map. Let* α_f *be as in* (1.3). *If*

$$\int\limits_1^\infty \frac{\alpha_f(t)^{\frac{1}{2}}}{t^{1+\frac{1}{2}}}\, dt < \infty \quad (\textit{for} \quad 1 < p \leq 2)$$

or

$$\int\limits_1^\infty \frac{\alpha_f(t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\, dt < \infty \quad (\textit{for} \quad 2 < p < \infty),$$

then, for any $x \in X$, *we have*

$$\|f(x+y) - f(x) - f(y)\| = o(\|f(y)\|) \quad \textit{as} \quad \|f(y)\| \to \infty.$$

***Corollary* 2.2.** *Let* $(X, +)$ *be an Abelian group, and* $E = L_p$, $1 < p < \infty$. *Assume that* $f\colon X \to E$ *is a surjective map. Let* $\overline{\alpha}_f$ *be as in* (1.4). *If*

$$\int\limits_1^\infty \frac{\overline{\alpha}_f(t)^{\frac{1}{2}}}{t^{1+\frac{1}{2}}}\, dt < \infty \quad (\textit{for} \quad 1 < p \leq 2)$$

or

$$\int\limits_1^\infty \frac{\overline{\alpha}_f(t)^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}\, dt < \infty \quad (\textit{for} \quad 2 < p < \infty),$$

then, for any $x \in X$, *we have*

$$\|f(x+y) - f(x) - f(y)\| = o(\|f(y)\|) \quad \textit{as} \quad \|f(y)\| \to \infty.$$

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