DOI: 10.37863/umzh.v75i6.7120

UDC 517.9

Bilender P. Allahverdiev (Department of Mathematics, Khazar University, Baku, Azerbaijan and Research Center of Econophysics, UNEC-Azerbaijan State University of Economics, Baku, Azerbaijan),
Hüseyin Tuna[1] (Department of Mathematics, Mehmet Akif Ersoy University, Burdur, Turkey)

IMPULSIVE DIRAC SYSTEM ON TIME SCALES
ІМПУЛЬСНА СИСТЕМА ДІРАКА НА ЧАСОВИХ ШКАЛАХ

We consider an impulsive Dirac system on Sturmian time scales. An existence theorem is given for this system. A maximal, minimal and self-adjoint operators generated by the impulsive dynamic Dirac system are constructed. We also construct the Green function for this problem. Finally, an eigenfunction expansion is obtained.

Розглядається імпульсна система Дірака на часових шкалах Штурма. Наведено теорему існування для цієї системи. Побудовано максимальний, мінімальний та самоспряжений оператори, що породжені імпульсною динамічною системою Дірака, а також функцію Гріна для цієї задачі. Насамкінець отримано розклад за власними функціями.

1. Introduction. Dirac systems received extensive studies in the last century [21, 34, 35]. These systems provide a natural description of the electron spin, predict the existence of antimatter, and can reproduce accurately the spectrum of the hydrogen atom.

Impulsive differential equations arise as the mathematical modeling of various areas of science. For example, population dynamics, economics, optimal control, and chemotherapy. It is well-known that these equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Recently, these equations have been studied by several authors (see [22 – 27, 29, 33]).

On the other hand, the theory of dynamic equations aims to unify continuous-time and discrete-time equations. Dynamic equations have become a very active area of research (see [4 – 7, 11, 12, 14 – 16, 30, 31]). Therefore the idea to apply the time scale analysis to the investigation of the Dirac system looks natural and the results could be of interest. To our best knowledge, although there are some results for the impulsive Dirac problems [10, 13, 19], there are not any results on the impulsive dynamic Dirac system. In this paper, we investigate some properties of this system, so we will unify the discrete impulsive Dirac system [9, 20] and continuous impulsive Dirac system [2, 3, 8, 17, 32].

This paper is organized as follows. In the second section, an existence and uniqueness theorem for the impulsive dynamic Dirac system is proved. In the third section, a maximal, minimal and self-adjoint operators is constructed. Finally, we construct the Green function for this problem and an eigenfunction expansion is given in the last section.

[1] Corresponding author, e-mail: hustuna@gmail.com.

2. Impulsive Dirac system on Sturmian time scales. Let $\mathbb{T}$ be a Sturmian time scale. We define the jump operators $\sigma,\ \rho\colon \mathbb{T}\to\mathbb{T}$ by

$$\sigma(t)=\inf\{s\in\mathbb{T}\colon s>t\}$$

and

$$\rho(t)=\sup\{s\in\mathbb{T}\colon s<t\},$$

where $t\in\mathbb{T}$. If $\rho(t)\neq t$, then $t\in\mathbb{T}$ is called left scattered point. If $\rho(t)=t$, then $t\in\mathbb{T}$ is called left dense point. A point $t\in\mathbb{T}$ is right scattered if $\sigma(t)\neq t$ and right dense if $\sigma(t)=t$. Moreover, $\sigma(\rho(t))=\rho(\sigma(t))=t,\ t\in\mathbb{T}$ (see [1]).

Consider the impulsive dynamic Dirac system defined by the formula

$$\Gamma z=\lambda z,\quad t\in J=[a,c)\cup(c,b], \tag{1}$$

with impulsive condition

$$z(c+)=Cz(c-), \tag{2}$$

where $J\subset\mathbb{T},\ \lambda\in\mathbb{C}$,

$$\Gamma z:=\begin{cases}-z_2^{\nabla}+p(t)z_1,\\ z_1^{\Delta}+r(t)z_2,\end{cases}$$
$$z=\begin{pmatrix}z_1\\ z_2\end{pmatrix},\quad C=\begin{pmatrix}d_1 & 0\\ 0 & d_2\end{pmatrix}, \tag{3}$$

d_1,d_2 are real numbers such that $\det C=d_1d_2>0,\ p(.)$ and $r(.)$ are real-valued functions defined on J, and $p(.),\ r(.)\in L^1_{\Delta}(\mathbb{T})$. Note that $z(c+)$ and $z(c-)$ represent right and left limits with respect to the time scale, and in addition, the points $a,b,c\in J$ are left dense.

***Remark* 1.** The reason for choosing Γ operator and (2) conditions in this way is to obtain a self-adjoint operator.

Denote by $\mathcal{H}:=L^2_{\Delta}\big(J;\mathbb{C}^2\big)$ the Hilbert space with the inner product

$$\langle f,g\rangle_{\mathcal{H}}:=\int_a^c(f(t),g(t))_{\mathbb{C}^2}\Delta t+\delta\int_c^b(f(t),g(t))_{\mathbb{C}^2}\Delta t,$$
$$=\int_a^c g^*(t)f(t)\Delta t+\delta\int_c^b g^*(t)f(t)\Delta t,$$

where $f,g\in\mathcal{H}$ and $\delta=\dfrac{1}{d_1d_2}$.

Theorem 1. *Eq.* (1) *with the conditions*

$$z(c+,\lambda)=Cz(c-,\lambda),\quad z_1(t_0,\lambda)=c_1,\quad z_2^{\rho}(t_0,\lambda)=c_2 \tag{4}$$

has a unique solution

$$z(t,\lambda)=\begin{pmatrix} z_1(t,\lambda)\\ z_2(t,\lambda)\end{pmatrix}$$

in $\mathcal{H}$.

Proof. Using the formula

$$z_2^\rho+(\sigma(t)-t)(z_2^\rho)^\Delta=(z_2^\rho)^\sigma=z_2, \tag{5}$$

we obtain

$$z_1^\Delta=(\lambda+r(t))(z_2^\rho)^\sigma=(\lambda+r(t))\Big(z_2^\rho+(\sigma(t)-t)(z_2^\rho)^\Delta\Big),$$

$$z_2^\nabla=(z_2^\rho)^\Delta=-(\lambda+p(t))z_1.$$

Hence

$$\begin{aligned} &z_1^\Delta=(\sigma(t)-t)(\lambda+p(t))(\lambda+r(t))z_1+(\lambda+r(t))z_2^\rho,\\ &(z_2^\rho)^\Delta=-(\lambda+p(t))z_1.\end{aligned} \tag{6}$$

It follows from (6) that

$$(\widehat{z}(t))^\Delta=H(t,\lambda)\widehat{z}(t),$$

where

$$\widehat{z}(t)=\begin{pmatrix} z_1\\ z_2^\rho\end{pmatrix},$$

and

$$H(t,\lambda):=\begin{pmatrix} -(\sigma(t)-t)(\lambda+p(t))(\lambda+r(t)) & (\lambda+r(t))\\ -(\lambda+p(t)) & 0\end{pmatrix}.$$

Since

$$\det[I_2+(\sigma(t)-t)H(t,\lambda)]$$

$$=\begin{vmatrix} 1-(\sigma(t)-t)^2(\lambda+p(t))(\lambda+r(t)) & (\sigma(t)-t)(\lambda+r(t))\\ -(\sigma(t)-t)(\lambda+p(t)) & 1\end{vmatrix}\neq 0,$$

the matrix $H(t,\lambda)$ is regressive. By [11, Theorem 5.8], we infer that Eq. (1) has a unique solution $z(t,\lambda)$ with initial conditions (4).

3. Construction of self-adjoint operator. Consider the sets

$$D_{\max}=\left\{ z\in\mathcal{H}: \begin{array}{c} z_1 \text{ and } z_2 \text{ are } \Delta \text{-absolutely}\\ \text{continuous on all subintervals of } J,\\ \text{one-sided limits } z_1(c\pm) \text{ and } z_2(c\pm) \text{ exist}\\ \text{and finite, } z(c+)=Cz(c-) \text{ and } \Gamma z\in\mathcal{H}\end{array}\right\},$$

$$D_{\min}=\{z\in D_{\max}: z_1(a)=z_2^\rho(a)=z_1(b)=z_2^\rho(b)=0\}. \tag{7}$$

We define the *maximal* operator $T_{\max}$ on $D_{\max}$ with the rule $T_{\max}z=\Gamma z$. The operator $T_{\min}$, that is, the restriction of the operator $T_{\max}$ to $D_{\min}$ is called the *minimal operator*.

For the functions $u, v \in D_{\max}$,we obtain the Green formula

$$\int_a^b [((\Gamma u)(t), v(t))_{\mathbb{C}^2} - (u(t), (\Gamma v)(t))_{\mathbb{C}^2}]\Delta t$$

$$= [u,v](b) - [u,v](c+) + [u,v](c-) - [u,v](a), \tag{8}$$

where

$$[u,v](t) = W_\Delta(u,\overline{v})(t) := u_1(t)\overline{v_2^\rho(t)} - u_2^\rho(t)\overline{v_1(t)}.$$

From (7) and (8), we obtain the following lemmas.

Lemma 1. *The operator $T_{\min}$ is Hermitian.*

Lemma 2. *Let $v \in D_{\max}$ and $u \in D_{\min}$. Then we have the following relation:*

$$\langle T_{\min} u, v\rangle_{\mathcal{H}} = \langle u, T_{\max} v\rangle_{\mathcal{H}}.$$

Lemma 3. *Let us denote by* $\mathrm{Nul}(T)$ *and* $\mathrm{Ran}(T)$ *the null space and the range of an operator* T, *respectively. Then we have*

$$\mathrm{Ran}\,(T_{\min}) = \mathrm{Nul}\,(T_{\max})^{\perp}.$$

Proof. Let $\xi \in \mathrm{Ran}(T_{\min})$. There exists $u \in D_{\min}$ such that $T_{\min}u = \xi$. It follows from Lemma 3 that, for each $v \in \mathrm{Nul}(T_{\max})$,

$$\langle \xi, v\rangle_{\mathcal{H}} = \langle T_{\min} u, v\rangle_{\mathcal{H}} = \langle u, T_{\max} v\rangle_{\mathcal{H}} = 0,$$

i.e., $\mathrm{Ran}(T_{\min}) \subset \mathrm{Nul}(T_{\max})^{\perp}$.

For any given $\xi \in \mathrm{Nul}(T_{\max})^{\perp}$ and for all $v \in \mathrm{Nul}(T_{\max})$, we have $(\xi, v) = 0$. Let us consider the following problem:

$$\begin{aligned} &(\Gamma x)(t) = \xi(t), \quad t \in J, \\ &\widehat{x}(a) = 0, \quad x(c+,\lambda) = Cx(c-,\lambda). \end{aligned} \tag{9}$$

It follows from Theorem 1 that the problem (9) has a unique solution on J. Let $\Phi(t) = (\phi_1(t), \phi_2(t))$ be the fundamental solution of the system

$$\Gamma x = 0, \quad t \in J,$$

$$\widehat{\Phi}(b) = I_2, \quad \Phi(c+,\lambda) = C\Phi(c-,\lambda).$$

It is clear that $\phi_i \in \mathrm{Nul}(T_{\max})$ for $1 = 1,2$. From (8), for $i = 1,2$, we conclude that

$$0 = \langle \xi, \phi_i\rangle_{\mathcal{H}} = \int_a^c (\xi(t), \phi_i(t))_{\mathbb{C}^2}\Delta t + \delta \int_c^b (\xi(t), \phi_i(t))_{\mathbb{C}^2}\Delta t$$

$$= \int_a^c (\xi(t), \phi_i(t))_{\mathbb{C}^2}\Delta t + \delta \int_c^b (\xi(t), \phi_i(t))_{\mathbb{C}^2}\Delta t$$

$$= \int_a^c \phi_i^*(t)(\Gamma x)(t)\Delta t + \delta \int_c^b \phi_i^*(t)(\Gamma x)(t)\Delta t$$

$$- \int_a^c ((\Gamma \phi_i)(t))^* x(t)\, dt - \delta \int_c^b ((\Gamma \phi_i)(t))^* x(t)\, dt$$

$$= [x, \phi_i](b) + [x, \phi_i](c-) - \delta[x, \phi_i](c+) - \delta[x, \phi_i](a) = [x, \phi_i](b).$$

This implies that

$$[x, \phi_1](b) = -x_2^\rho(b) = 0, \quad [x, \phi_2](b) = x_1(b) = 0,$$

i.e., $\xi \in \mathrm{Ran}(T_{\min})$.

Theorem 2. *The operator* $T_{\min}$ *is a densely defined operator and the operator* $T_{\min}$ *is symmetric. Furthermore,* $T_{\min}^* = T_{\max}$.

Proof. Let $\xi \in D_{\min}^\perp$. Then we have $\langle \xi, v\rangle_{\mathcal{H}} = 0$ for all $v \in D_{\min}$. Define $T_{\min}v(t) = \phi(t)$. Let $u(t)$ be any solution of the system

$$\Gamma u = \xi(t), \quad t \in J, \quad u(c+, \lambda) = Cu(c-, \lambda).$$

It follows from (8) that

$$\langle u, \phi\rangle_{\mathcal{H}} - \langle \xi, v\rangle_{\mathcal{H}} = \int_a^c \phi^*(t)u(t)\Delta t + \delta \int_c^b \phi^*(t)u(t)\Delta t$$

$$- \int_a^c v^*(t)\xi(t)\Delta t - \delta \int_c^b v^*(t)\xi(t)\Delta t$$

$$= \int_a^c (\Gamma v)^*(t)u(t)\Delta t + \delta \int_c^b (\Gamma v)^*(t)(t)u(t)\Delta t$$

$$- \int_a^c v^*(t)(\Gamma u)(t)\Delta t - \delta \int_c^b v^*(t)(\Gamma u)(t)\Delta t$$

$$= -[v, u](a) - [v, u](c-) + [v, u](c+) + [v, u](b) = 0.$$

From Lemma 4, we see that

$$u \in \mathrm{Ran}(T_{\min})^\perp = \mathrm{Nul}(T_{\max}).$$

Thus, $\xi = 0$, i.e., $D_{\min}^\perp = \{0\}$.

Let us denote by $D_{\min}^*$ the domain of the operator $T_{\min}^*$. Now, we will prove that $D_{\min}^* = D_{\max}$ and $T_{\min}^* u = T_{\max} u$ for all $u \in D_{\min}^*$. It follows from Lemma 3 that

$$\langle u, T_{\min} v\rangle_{\mathcal{H}} = \langle T_{\max} u, v\rangle_{\mathcal{H}},$$

where $u \in D_{\min}$ and $v \in D_{\max}$. Hence, the functional $\langle u, T_{\min}(.)\rangle_{\mathcal{H}}$ is continuous on $D_{\min}$ and $v \in D^*_{\min}$, i.e., $D_{\max} \subset D^*_{\min}$.

Now, we will show that $D^*_{\min} \subset D_{\max}$. If $u \in D^*_{\min}$, then $u, \phi \in \mathcal{H}$, where $\phi := T^*_{\min} u$. Assume that y is a solution of the equation

$$\Gamma y = \phi. \tag{10}$$

From Lemma 3, we deduce that

$$\langle \phi, v\rangle_{\mathcal{H}} = \langle T_{\max} y, v\rangle_{\mathcal{H}} = \langle y, T_{\min} v\rangle_{\mathcal{H}}.$$

Thus

$$\langle u - y, T_{\min} v\rangle_{\mathcal{H}} = \langle u, T_{\min} v\rangle_{\mathcal{H}} - \langle y, T_{\min} v\rangle_{\mathcal{H}} = \langle T^*_{\min} u, v\rangle_{\mathcal{H}} - \langle \phi, v\rangle_{\mathcal{H}} = 0,$$

i.e., $u - y \in \mathrm{Ran}(T_{\min})^{\perp}$. By Lemma 4, we see that $u - y \in \mathrm{Nul}(T_{\max})$.

From (10), we conclude that $\Gamma u = \Gamma y = \phi$. Then we obtain $T_{\max} u = \phi = T^*_{\min} u$ and $u \in D_{\max}$ due to $u, \phi \in \mathcal{H}$.

Now we define the operator T by $T\colon D \to \mathcal{H}$, where

$$D = \left\{ z \in D_{\max} : \begin{array}{c} k_{11} z_1(a) + k_{12} z_2^{\rho}(a) = 0, \\ k_{21} z_1(a) + k_{22} z_2^{\rho}(a) = 0, \\ \text{where } k_{ij},\ i,j = 1,2, \text{ are real} \\ \text{numbers such that } k_{11}^2 + k_{12}^2 \neq 0,\ k_{21}^2 + k_{22}^2 \neq 0 \end{array} \right\}.$$

From (8), we obtain the following theorem.

Theorem 3. *The operator T is self-adjoint on $\mathcal{H}$. Hence it has the following properties:*

(i) *all eigenvalues are real and simple,*

(ii) *the eigenfunctions corresponding to distinct eigenvalues are orthogonal.*

4. Eigenfunction expansion. Consider the boundary-value problem

$$-z_2^{\nabla} + \{-\lambda + p(t)\} z_1 = h_1(t), \tag{11}$$

$$z_1^{\Delta} + \{-\lambda + r(t)\} z_2 = h_2(t), \tag{12}$$

together with the conditions

$$k_{11} z_1(a) + k_{12} z_2^{\rho}(a) = 0, \tag{13}$$

$$z_1(c+) - d_1 z_1(c-) = 0, \tag{14}$$

$$z_2(c+) - d_2 z_2(c-) = 0, \tag{15}$$

$$k_{21} z_1(b) + k_{22} z_2^{\rho}(b) = 0, \tag{16}$$

where $t \in J$ and

$$h(.) = \begin{pmatrix} h_1(.) \\ h_2(.) \end{pmatrix} \in \mathcal{H}.$$

Denote by

$$\varphi(t,\lambda)=\begin{pmatrix}\varphi_1(t,\lambda)\\ \varphi_2(t,\lambda)\end{pmatrix},$$

$$\varphi_1(t,\lambda)=\begin{cases}\varphi_{11}(t,\lambda), & t\in[a,c),\\ \varphi_{12}(t,\lambda), & t\in(c,b],\end{cases}$$

$$\varphi_2(t,\lambda)=\begin{cases}\varphi_{21}(t,\lambda), & t\in[a,c),\\ \varphi_{22}(t,\lambda), & t\in(c,b],\end{cases}$$

and

$$\psi(t,\lambda)=\begin{pmatrix}\psi_1(t,\lambda)\\ \psi_2(t,\lambda)\end{pmatrix},$$

$$\psi_1(t,\lambda)=\begin{cases}\psi_{11}(t,\lambda), & t\in[a,c),\\ \psi_{12}(t,\lambda), & t\in(c,b],\end{cases}$$

$$\psi_2(t,\lambda)=\begin{cases}\psi_{21}(t,\lambda), & t\in[a,c),\\ \psi_{22}(t,\lambda), & t\in(c,b],\end{cases}$$

two basic solutions of the problem (11), (12) which satisfy the initial conditions

$$\varphi(a,\lambda)=\begin{pmatrix}-k_{12}\\ -k_{11}\end{pmatrix},\quad \psi(b,\lambda)=\begin{pmatrix}-k_{22}\\ -k_{21}\end{pmatrix}$$

and both transmission conditions (14), (15). It is clear that

$$D(\lambda)=-W_\Delta(\varphi,\psi)=-W_{\Delta,1}(\varphi_{11},\psi_{11})=-\delta W_{\Delta,2}(\varphi_{22},\psi_{22})\neq 0.$$

Then we have the following theorem.

Theorem 4. *If* $D(\lambda)\neq 0$, *then the nonhomogeneous boundary-value problem* (11)–(15) *has a unique solution* $z(t,\lambda)$ *defined by the formula*

$$z(t,\lambda)=\langle G(t,.,\lambda),\overline{h(.)}\rangle_{\mathcal{H}},\quad t\in J, \tag{17}$$

where

$$G(t,x,\lambda)=\frac{1}{D(\lambda)}\begin{cases}\psi(t,\lambda)\varphi^T(x,\lambda), & a\le x\le t\le b,\ t\neq c,\ x\neq c,\\ \varphi(t,\lambda)\psi^T(x,\lambda), & a\le t\le x\le b,\ t\neq c,\ x\neq c.\end{cases} \tag{18}$$

Proof. From (17) and (18), we obtain

$$z_1(t,\lambda)=\frac{1}{D(\lambda)}\begin{cases}\psi_{11}(t,\lambda)\displaystyle\int_a^t(\varphi_{11}(x,\lambda)h_1(x)+\varphi_{21}(x,\lambda)h_2(x))\Delta x\\ +\varphi_{11}(t,\lambda)\displaystyle\int_t^c(\psi_{11}(x,\lambda)h_1(x)+\psi_{21}(x,\lambda)h_2(x))\Delta x\\ +\varphi_{11}(t,\lambda)\delta\displaystyle\int_c^b(\psi_{12}(x,\lambda)h_1(x)+\psi_{22}(x,\lambda)h_2(x))\Delta x, \quad t\in[a,c),\\ \psi_{12}(t,\lambda)\displaystyle\int_a^c(\varphi_{11}(x,\lambda)h_1(x)+\varphi_{21}(x,\lambda)h_2x))\Delta x\\ +\psi_{12}(t,\lambda)\delta\displaystyle\int_c^t(\varphi_{12}(x,\lambda)h_1(x)+\varphi_{22}(x,\lambda)h_2(x))\Delta x\\ +\varphi_{12}(t,\lambda)\delta\displaystyle\int_t^b(\psi_{12}(x,\lambda)h_1(x)+\psi_{22}(x,\lambda)h_2(x))\Delta x, \quad t\in(c,b],\end{cases} \tag{19}$$

and

$$z_2(t,\lambda)=\frac{1}{D(\lambda)}\begin{cases}\psi_{21}(t,\lambda)\displaystyle\int_a^t(\varphi_{11}(x,\lambda)h_1(x)+\varphi_{21}(x,\lambda)h_2(x))\Delta x\\ +\varphi_{21}(t,\lambda)\displaystyle\int_t^c(\psi_{11}(x,\lambda)h_1(x)+\psi_{21}(x,\lambda)h_2(x))\Delta x\\ +\varphi_{21}(t,\lambda)\delta\displaystyle\int_c^b(\psi_{12}(x,\lambda)h_1(x)+\psi_{22}(x,\lambda)h_2(x))\Delta x, t\in[a,c),\\ \psi_{22}(t,\lambda)\displaystyle\int_a^c(\varphi_{11}(x,\lambda)h_1(x)+\varphi_{21}(x,\lambda)h_2(x))\Delta x\\ +\psi_{22}(t,\lambda)\delta\displaystyle\int_c^t(\varphi_{12}(x,\lambda)h_1(x)+\varphi_{22}(x,\lambda)h_2(x))\Delta x\\ +\varphi_{22}(t,\lambda)\delta\displaystyle\int_t^b(\psi_{12}(x,\lambda)h_1(x)+\psi_{22}(x,\lambda)h_2(x))\Delta x, \quad t\in(c,b].\end{cases} \tag{20}$$

It follows from (19) that

$$z_1^{\Delta}(t,\lambda)=\frac{1}{D(\lambda)}\begin{cases}\psi_{11}^{\Delta}(t,\lambda)\displaystyle\int_a^t(\varphi_{11}(x,\lambda)h_1(x)+\varphi_{21}(x,\lambda)h_2(x))\Delta x\\ +\varphi_{11}^{\Delta}(t,\lambda)\displaystyle\int_t^c(\psi_{11}(x,\lambda)h_1(x)+\psi_{21}(x,\lambda)h_2(x))\Delta x\\ +\varphi_{11}^{\Delta}(t,\lambda)\delta\displaystyle\int_c^b(\psi_{12}(x,\lambda)h_1(x)+\psi_{22}(x,\lambda)h_2(x))\Delta x\\ +W_{\Delta,1}(\psi,\varphi)(\sigma(t))h_2(t), \quad t\in[a,c),\\ \psi_{12}^{\Delta}(t,\lambda)\displaystyle\int_a^c(\varphi_{11}(x,\lambda)h_1(x)+\varphi_{21}(x,\lambda)h_2(x))\Delta x\\ +\psi_{12}^{\Delta}(t,\lambda)\delta\displaystyle\int_c^t(\varphi_{12}(x,\lambda)h_1(x)+\varphi_{22}(x,\lambda)h_2(x))\Delta x\\ +\varphi_{12}^{\Delta}(t,\lambda)\delta\displaystyle\int_t^b(\psi_{12}(x,\lambda)h_1(x)+\psi_{22}(x,\lambda)h_2(x))\Delta x\\ +\delta W_{\Delta,2}(\psi,\varphi)(\sigma(t))h_2(t), \quad t\in(c,b],\end{cases}$$

$$= \frac{1}{D(\lambda)} \begin{cases} \{\lambda - r(t)\}\psi_{21}(t,\lambda)\displaystyle\int_a^t (\varphi_{11}(x,\lambda)h_1(x) + \varphi_{21}(x,\lambda)h_2(x))\Delta x \\ +\{\lambda - r(t)\}\varphi_{21}(t,\lambda)\displaystyle\int_t^c (\psi_{11}(x,\lambda)h_1(x) + \psi_{21}(x,\lambda)h_2(x))\Delta x \\ +\{\lambda - r(t)\}\varphi_{21}(t,\lambda)\delta\displaystyle\int_c^b (\psi_{12}(x)h_1(x) + \psi_{22}(x,\lambda)h_2(x))\Delta x \\ +h_2(t), \quad t \in [a,c), \\ \{\lambda - r(t)\}\psi_{22}(t,\lambda)\displaystyle\int_a^t (\varphi_{11}(x,\lambda)h_1(x) + \varphi_{21}(x,\lambda)h_2(x))\Delta x \\ +\{\lambda - r(t)\}\psi_{22}(t,\lambda)\delta\displaystyle\int_c^t (\varphi_{12}(x,\lambda)h_1(x) + \varphi_{22}(x,\lambda)h_2(x))\Delta x \\ +\{\lambda - r(t)\}\varphi_{22}(t,\lambda)\delta\displaystyle\int_t^b (\psi_{12}(x,\lambda)h_1(x) + \psi_{22}(x,\lambda)h_2(x))\Delta x \\ +h_2(t), \quad t \in (c,b], \end{cases}$$

$$= \{\lambda - r(t)\}z_2(t,\lambda) + h_2(t).$$

By (20), we have

$$z_2^{\nabla}(t,\lambda) = \frac{1}{D(\lambda)} \begin{cases} \psi_{21}^{\nabla}(t,\lambda)\displaystyle\int_a^t (\varphi_{11}(x,\lambda)h_1(x) + \varphi_{21}(x,\lambda)h_2(x))\Delta x \\ +\varphi_{21}^{\nabla}(t,\lambda)\displaystyle\int_t^c (\psi_{11}(x,\lambda)h_1(x) + \psi_{21}(x,\lambda)h_2(x))\Delta x \\ +\varphi_{21}^{\nabla}(t,\lambda)\delta\displaystyle\int_c^b (\psi_{12}(x,\lambda)h_1(x) + \psi_{22}(x,\lambda)h_2(x))\Delta x \\ +W_{\Delta,1}(\varphi,\psi)h_1(t), \quad t \in [a,c), \\ \psi_{22}^{\nabla}(t,\lambda)\displaystyle\int_a^c (\varphi_{11}(x,\lambda)h_1(x) + \varphi_{21}(x,\lambda)h_2(x))\Delta x \\ +\psi_{22}^{\nabla}(t,\lambda)\delta\displaystyle\int_c^t (\varphi_{12}(x,\lambda)h_1(x) + \varphi_{22}(x,\lambda)h_2(x))\Delta x \\ +\varphi_{22}^{\nabla}(t,\lambda)\delta\displaystyle\int_t^b (\psi_{12}(x,\lambda)h_1(x) + \psi_{22}(x,\lambda)h_2(x))\Delta x \\ +\delta W_{\Delta,2}(\varphi,\psi)h_1(t), \quad t \in (c,b], \end{cases}$$

$$= \frac{1}{D(\lambda)} \begin{cases} \{-\lambda + p(t)\}\psi_{11}(t,\lambda)\displaystyle\int_a^t (\varphi_{11}(x,\lambda)h_1(x) + \varphi_{21}(x,\lambda)h_2(x))\Delta x \\ +\{-\lambda + p(t)\}\varphi_{11}(t,\lambda)\displaystyle\int_t^c (\psi_{11}(x,\lambda)h_1(x) + \psi_{21}(x,\lambda)h_2(x))\Delta x \\ +\{-\lambda + p(t)\}\varphi_{11}(t,\lambda)\delta\displaystyle\int_c^b (\psi_{12}(x,\lambda)h_1(x) + \psi_{22}(x,\lambda)h_2(x))\Delta x \\ +W_{\Delta,1}(\varphi,\psi)h_1(t), \quad t \in [a,c), \\ \{-\lambda + p(t)\}\psi_{12}(t,\lambda)\displaystyle\int_a^c (\varphi_{11}(x,\lambda)h_1(x) + \varphi_{21}(x,\lambda)h_2(x))\Delta x \\ +\{-\lambda + p(t)\}\psi_{12}(t,\lambda)\delta\displaystyle\int_c^t (\varphi_{12}(x,\lambda)h_1(x) + \varphi_{22}(x,\lambda)h_2(x)\Delta x \\ +\{-\lambda + p(t)\}\varphi_{12}(t,\lambda)\delta\displaystyle\int_t^b (\psi_{12}(x,\lambda)h_1(x) + \psi_{22}(x,\lambda)h_2(x))\Delta x \\ +\delta W_{\Delta,2}(\varphi,\psi)h_1(t), \quad t \in (c,b], \end{cases}$$

$$= \{-\lambda + p(t)\}z_1(t,\lambda) - h_1(t).$$

One proves that (17) satisfies the conditions (13) – (16).

Without loss of generality, we can assume that $\lambda = 0$ is not an eigenvalue. Then we obtain

$$G(t,x) = \frac{1}{D(0)}\begin{cases}\psi(t)\varphi^T(x), & a \le x \le t \le b,\ t \ne c,\ x \ne c,\\ \varphi(t)\psi^T(x), & a \le t \le x \le b,\ t \ne c,\ x \ne c.\end{cases} \tag{21}$$

Definition 1. *A matrix-valued function $M(t,x)$ of two variables with $a \le t,\ x \le b$ is called the Δ-Hilbert – Schmidt kernel if*

$$\int_a^b \int_a^b \|M(t,x)\|^2 \Delta t \Delta x < +\infty.$$

Theorem 5. *$G(t,x)$ defined by* (21) *is a Δ-Hilbert – Schmidt kernel.*

Proof. By virtue of (21), we see that

$$\int_a^b \Delta t \int_a^t \|G(t,x)\|^2 \Delta x < +\infty$$

and

$$\int_a^b \Delta t \int_t^b \|G(t,x)\|^2 \Delta x < +\infty,$$

since the inner integral exists and is a linear combination of the products $\varphi_{1i}(x)\psi_{2k}(t),\ i,k = 1,2,$ and these products belong to $\mathcal{H} \times \mathcal{H}$ because each of the factors belongs to $\mathcal{H}$. Hence

$$\int_a^b \int_a^b \|G(t,x)\|^2 \Delta t \Delta x < +\infty. \tag{22}$$

Now, we need the following theorem.

Theorem 6 [28]. *If*

$$\sum_{i,k=1}^{\infty} \|a_{ik}\|^2 < +\infty, \tag{23}$$

then the operator A defined by the formula

$$A\{x_i\} = \{y_i\}, \tag{24}$$

where

$$y_i = \sum_{k=1}^{\infty} a_{ik} x_k, \quad i = 1,2,3,\ldots,$$

is compact in the sequence space l^2.

Theorem 7. *The operator R defined as*

$$(Rh)(t) = \langle G(t,.), \overline{h(.)} \rangle_{\mathcal{H}}, \quad t \in J,$$

is compact and self-adjoint.

Proof. Let $u, v \in \mathcal{H}$ and let $\Psi_i = \Psi_i(s),\ i = 1, 2, 3, \ldots,$ be a complete, orthonormal basis of $\mathcal{H}$. Then we can define

$$x_i = \langle u, \Psi_i \rangle_{\mathcal{H}} = \int_a^c (u(t), \Psi_i(t))_{\mathbb{C}^2} \Delta t + \delta \int_c^b (u(t), \Psi_i(t))_{\mathbb{C}^2} \Delta t,$$

$$y_i = \langle v, \Psi_i \rangle_{\mathcal{H}} = \int_a^c (v(t), \Psi_i(t))_{\mathbb{C}^2} \Delta t + \delta \int_c^b (v(t), \Psi_i(t))_{\mathbb{C}^2} \Delta t,$$

$$a_{ik} = \int_a^c \int_a^c (G(t,x)\Psi_i(t), \Psi_k(t))_{\mathbb{C}^2} \Delta x \Delta t$$

$$+ \delta^2 \int_c^b \int_c^b (G(t,x)\Psi_i(t), \Psi_k(t))_{\mathbb{C}^2} \Delta x \Delta t, \quad i, k = 1, 2, 3, \ldots,$$

due to $G(t,x)$ is a Hilbert – Schmidt kernel. The space $\mathcal{H}$ is mapped isometrically into l^2. Consequently, our integral operator R transforms into the operator A defined by the formula (24) in the space l^2 by this mapping, and the condition (22) is translated into the condition (23). By Theorem 5, A is compact. Therefore, the original operator R is compact.

Now, we shall prove that R is self-adjoint. Since $G(x,t) = G^T(t,x)$ and $G(t,x)$ is a real matrix-valued function defined on $J \times J$, we infer that

$$\langle Ru, v \rangle_{\mathcal{H}} = \int_a^c ((Ru)(t), v(t))_{\mathbb{C}^2} \Delta t + \delta \int_c^b ((Ru)(t), v(t))_{\mathbb{C}^2} \Delta t$$

$$= \int_a^c \left(\int_a^c G(t,x)u(x)\Delta x, v(t) \right)_{\mathbb{C}^2} \Delta t + \delta^2 \int_c^b \left(\int_c^b G(t,x)u(x)\Delta x, v(t) \right)_{\mathbb{C}^2} \Delta t$$

$$= \int_a^c \left(u(x), \int_a^c G^T(t,x)v(t)\Delta t \right)_{\mathbb{C}^2} \Delta x + \delta^2 \int_c^b \left(u(x), \int_c^b G^T(t,x)v(t)\Delta t \right)_{\mathbb{C}^2} \Delta x$$

$$= \int_a^c \left(u(x), \int_a^c G(x,t)v(t)\Delta t \right)_{\mathbb{C}^2} \Delta x$$

$$+ \delta^2 \int_c^b \left(u(x), \int_c^b G(x,t)v(t)\Delta t \right)_{\mathbb{C}^2} \Delta x = \langle u, Rv \rangle_{\mathcal{H}}.$$

Theorem 8. *The eigenvalues of the operator T form an infinite sequence $\{\lambda_n\}_{n=1}^{\infty}$ of real numbers which can be ordered such that*

$$|\lambda_1| < |\lambda_2| < \ldots < |\lambda_n| < \ldots \to \infty \quad \textit{as} \quad n \to \infty.$$

The set of all normalized eigenfunctions of T forms an orthonormal basis for the space $\mathcal{H}$ and for $z \in \mathcal{H},\ Rz = h,\ Th = z,\ T\chi_n = \lambda_n\chi_n,\ n = 1, 2, 3, \ldots,$ the eigenfunction expansion formula

$$Th = \sum_{n=1}^{\infty} \lambda_n \langle h, \chi_n\rangle_{\mathcal{H}}\chi_n$$

is valid.

Proof. It follows from Theorem 6 and the Hilbert–Schmidt theorem (see [18]) that R has an infinite sequence of non-zero real eigenvalues $\{\xi_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} \xi_n = 0.$$

Then

$$\|\lambda_n\| = \frac{1}{\|\xi_n\|} \to \infty \quad \text{as} \quad n \to \infty.$$

Furthermore, let $\{\chi_n\}_{n=1}^{\infty}$ denote an orthonormal set of eigenfunctions corresponding to $\{\xi_n\}_{n=1}^{\infty}$. Thus we have $z \in \mathcal{H},\ Rz = h,\ Th = z,\ T\chi_n = \lambda_n\chi_n,\ n = 1, 2, 3, \ldots,$ and

$$z = Th = \sum_{n=1}^{\infty}\langle z, \chi_n\rangle_{\mathcal{H}}\chi_n = \sum_{n=1}^{\infty}\langle Th, \chi_n\rangle_{\mathcal{H}}\chi_n$$

$$= \sum_{n=1}^{\infty}\langle h, T\chi_n\rangle_{\mathcal{H}}\chi_n = \sum_{n=1}^{\infty} \lambda_n\langle h, \chi_n\rangle_{\mathcal{H}}\chi_n.$$

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. C. Ahlbrandt, M. Bohner, T. Voepel, *Variable change for Sturm–Liouville differential expressions on time scales*, J. Different. Equat. and Appl., **9**, № 1, 93–107 (2003).
2. B. P. Allahverdiev, H. Tuna, *Titchmarsh–Weyl theory for Dirac systems with transmission conditions*, Mediterr. J. Math., **15**, № 4 (2018).
3. B. P. Allahverdiev, H. Tuna, *Spectral expansion for the singular Dirac system with impulsive conditions*, Turkish J. Math., **42**, 2527–2545 (2018).
4. B. P. Allahverdiev, H. Tuna, *One dimensional Dirac operators on time scales*, Casp. J. Math. Sci., **10**, № 2, 195–209 (2021).
5. D. R. Anderson, G. Sh. Guseinov, J. Hoffacker, *Higher-order self-adjoint boundary-value problems on time scales*, J. Comput. and Appl. Math., **194**, № 2, 309–342 (2006).
6. D. R. Anderson, *Titchmarsh–Sims–Weyl theory for complex Hamiltonian systems on Sturmian time scales*, J. Math. Anal. and Appl., **373**, № 2, 709–725 (2011).
7. F. Atici Merdivenci, G. Sh. Guseinov, *On Green's functions and positive solutions for boundary-value problems on time scales*, J. Comput. and Appl. Math., **141**, № 1–2, 75–99 (2002).
8. E. Bairamov, Ş. Solmaz, *Scattering theory of Dirac operator with the impulsive condition on whole axis*, Math. Methods. Appl. Sci., **44**, № 9, 7732–7746 (2021).

9. E. Bairamov, Ş. Solmaz, *Spectrum and scattering function of the impulsive discrete Dirac systems*, Turkish J. Math., **42**, 3182 – 3194 (2018).
10. M. Benchohra M., S. K. Ntouyas, A. Ouahab, *Existence results for second order boundary-value problem of impulsive dynamic equations on time scales*, J. Math. Anal. and Appl., **296**, 69 – 73 (2004).
11. M. Bohner, A. Peterson, *Dynamic equations on time scales*, Birkhäuser, Boston (2001).
12. M. Bohner, A. Peterson (Eds.), *Advances in dynamic equations on time scales*, Birkhäuser, Boston (2003).
13. T. Gulsen, S. M. Sian, E. Yilmaz, H. Koyunbakan, *Impulsive diffusion equation on time scales*, Int. J. Anal. Appl., **16**, № 1, 137 – 148 (2018).
14. T. Gulsen, E. Yılmaz, *Spectral theory of Dirac system on time scales*, Appl. Anal., **96**, № 16, 2684 – 2694 (2017).
15. S. Hilger, *Ein Maβtkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. Thesis, Univ. Würzburg, Germany (1988).
16. G. Hovhannisyan, *On Dirac equation on a time scale*, J. Math. Phys., **52**, № 10, 1 – 17 (2011).
17. B. Keskin, *Inverse problems for impulsive Dirac operators with spectral parameters contained in the boundary and multitransfer conditions*, Math. Methods. Appl. Sci., **38**, № 15, 3339 – 3345 (2015).
18. A. N. Kolmogorov, S. V. Fomin, *Introductory real analysis*, Translated by R. A. Silverman, Dover Publ., New York (1970).
19. Q. Kong, Q. R. Wang, *Using time scales to study multi-interval Sturm – Liouville problems with interface conditions*, Results Math., **63**, 451 – 465 (2013).
20. T. Köprübaşı, *A study of impulsive discrete Dirac system with hyperbolic eigenparameter*, Turkish J. Math., **45**, № 1, 540 – 548 (2021).
21. B. M. Levitan, I. S. Sargsjan, *Sturm – Liouville and Dirac operators*, Mathematics and its Applications (Soviet Series), Kluwer Acad. Publ. Group, Dordrecht (1991) (translated from the Russian).
22. V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of impulsive differential equations*, vol. 6, Ser. Modern Appl. Math., World Sci., Teaneck, NJ, USA (1989).
23. K. R. Mamedov, *On an inverse scattering problem for a discontinuous Sturm – Liouville equation with a spectral parameter in the boundary condition*, Bound.-Value Probl., **2010**, Article ID 171967 (2010).
24. K. R. Mamedov, N. Palamut, *On a direct problem of scattering theory for a class of Sturm – Liouville operator with discontinuous coefficient*, Proc. Jangjeon Math. Soc., **12**, № 2, 243 – 251 (2009).
25. O. Sh. Mukhtarov, *Discontinuous boundary-value problem with spectral parameter in boundary conditions*, Turkish J. Math., **18**, 183 – 192 (1994).
26. O. Sh. Mukhtarov, K. Aydemir, *The eigenvalue problem with interaction conditions at one interior singular point*, Filomat, **31**, № 17, 5411 – 5420 (2017).
27. O. Sh. Mukhtarov, H. Olğar, K. Aydemir, *Resolvent operator and spectrum of new type boundary-value problems*, Filomat, **29**, № 7, 1671 – 1680 (2015).
28. M. A. Naimark, *Linear differential operators*, 2nd ed., Nauka, Moscow (1969).
29. H. Olğar, O. Sh. Mukhtarov, *Weak eigenfunctions of two-interval Sturm – Liouville problems together with interaction conditions*, J. Math. Phys., **58**, Article 042201 (2017); DOI: 10.1063/1.4979615.
30. A. S. Ozkan, I. Adalar, *Half-inverse Sturm – Liouville problem on a time scale*, Inverse Probl., **36**, № 2, Article ID 025015 (2020).
31. A. S. Ozkan, *Ambarzumyan-type theorems on a time scale*, J. Inverse Ill-Posed Probl., **26**, № 5, 633 – 637 (2018).
32. S. Özkan, R. Kh. Amirov, *An interior inverse problem for the impulsive Dirac operator*, Tamkang J. Math., **42**, № 3, 259 – 263 (2011).
33. A. M. Samoilenko, N. A. Perestyuk, *Impulsive differential equations*, vol. 14, World Sci. Ser. Nonlinear Sci., Ser. A Monogr. and Treatises, World Sci., River Edge, NJ, USA (1995).
34. B. Thaller, *The dirac equation*, Springer (1992).
35. J. Weidmann, *Spectral theory of ordinary differential operators*, Lect. Notes Math., **1258**, Springer, Berlin (1987).

Received 20.01.22