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SOME REFINEMENTS OF THE HERMITE – HADAMARD INEQUALITY WITH THE HELP OF WEIGHTED INTEGRALS

ДЕЯКІ УТОЧНЕННЯ НЕРІВНОСТІ ЕРМІТА – АДАМАРА ЗА ДОПОМОГОЮ ВАГОВИХ ІНТЕГРАЛІВ

By using the definition of modified (h, m, s)-convex functions of the second type, we present various refinements of the classical Hermite – Hadamard inequality obtained within the framework of weighted integrals. Throughout the paper, we show that various known results available from the literature can be obtained as particular cases of our results.

За допомогою визначення (h, m, s)-опуклих модифікованих функцій другого типу отримано різні уточнення класичної нерівності Ерміта – Адамара в термінах вагових інтегралів. Показано, що різні результати, відомі з літератури, можна отримати як часткові випадки наших результатів.

1. Introduction. The notion of convex function has been the object of attention of many researchers in recent years, due to its multiple applications and links with various mathematical areas. Readers interested in the aforementioned development, can consult [35], where a panorama, practically complete, of these branches is presented.

A function $\psi\colon I \to \mathbb{R}$, $I := [\nu_1, \nu_2]$ is said to be convex if $\psi\big(\tau\xi + (1-\tau)\varsigma\big) \le \tau\psi(\xi) + (1-\tau)\psi(\varsigma)$ holds for all $\xi, \varsigma \in I, \tau \in [0, 1]$. And they say that the function ψ is concave on $[\nu_1, \nu_2]$ if the inequality is the opposite.

One of the most important inequalities, for convex functions, is the famous Hermite – Hadamard inequality

$$\psi\left(\frac{\nu_1+\nu_2}{2}\right) \le \frac{1}{\nu_2-\nu_1}\int\limits_{\nu_1}^{\nu_2}\psi(\xi)d\xi \le \frac{\psi(\nu_1)+\psi(\nu_2)}{2} \tag{1}$$

true for any convex function f on $[\nu_1, \nu_2]$. Hermite in [21] and, independently of him, Hadamard in [20] published this inequality. It gives an estimate for the mean value of a convex function, and, furthermore, provides a refinement of Jensen's inequality. Several results can be consulted in [4, 8 – 10, 14 – 16, 19, 22, 29, 30, 32, 37, 49] and references therein for more information and other extensions of the Hermite – Hadamard inequality.

Toader in [55] defined m-convexity in the following way.

Definition 1. *Let* $\psi\colon [0, \nu_2] \to \mathbb{R}$ *and* $m \in [0, 1]$. *If inequality*

$$\psi\big(\tau\xi + m(1-\tau)\varsigma\big) \le \tau\psi(\xi) + m(1-\tau)\psi(\varsigma)$$

holds for all $\xi, \varsigma \in [0, \nu_2]$, $\tau \in [0, 1]$, *then is said* ψ *on* $[0, \nu_2]$ *is a* m*-convex function.*

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If the above inequality holds in reverse, then we say that the function ψ is m-concave.

The following definitions are successive extensions of the concept of convex function and, as we will see later, they are particular cases of our definition.

Definition 2 [10, 23]. *Let* $\psi\colon [0,\nu_2]\to[0,+\infty)$ *and* $s\in(0,1]$. *If inequality*

$$\psi\big(\tau\xi+(1-\tau)\varsigma\big)\leq\tau^s\psi(\xi)+(1-\tau^s)\psi(\varsigma)$$

is holds for all $\xi,\varsigma\in[0,\nu_2]$, $0<\tau<1$, *then is said* ψ *on* $[0,\nu_2]$ *is a* s*-convex function* (*in the first sense*).

Definition 3 [10, 23]. *Let* $\psi\colon [0,\nu_2]\to[0,+\infty)$ *and* $s\in(0,1]$. *If inequality*

$$\psi\big(\tau\xi+(1-\tau)\varsigma\big)\leq\tau^s\psi(\xi)+(1-\tau)^s\psi(\varsigma)$$

is holds for all $\xi,\varsigma\in[0,\nu_2]$, $0<\tau<1$, *then is said* ψ *on* $[0,\nu_2]$ *is a* s*-convex function* (*in the second sense*).

Definition 4 [60]. *Let* $\psi\colon [0,\nu_2]\to[0,+\infty)$ *and* $s\in[-1,1]$. *If inequality*

$$\psi\big(\tau\xi+(1-\tau)\varsigma\big)\leq\tau^s\psi(\xi)+(1-\tau)^s\psi(\varsigma),$$

is holds for all $\xi,\varsigma\in[0,\nu_2]$, $0<\tau<1$, *then is said* ψ *on* $[0,\nu_2]$ *is a extended* s*-convex function.*

In [31], present the class of (α,m)-convex functions as follows.

Definition 5. *Let* $\psi\colon [0,\nu_2]\to\mathbb{R}$ *and* $\alpha,m\in[0,1]$. *If inequality*

$$\psi\big(\tau\xi+m(1-\tau)\varsigma\big)\leq\tau^\alpha\psi(\xi)+m(1-\tau^\alpha)\psi(\varsigma)$$

holds for all $\xi,\varsigma\in[0,\nu_2]$, $\tau\in[0,1]$, *then is said* ψ *on* $[0,\nu_2]$ *is a* (α,m)*-convex function.*

In [30] the following definition is introduced.

Definition 6. *Let* $h\colon[0,1]\to\mathbb{R}$ *be a nonnegative function*, $h\neq 0$. *The nonnegative function* $\psi\colon [0,\nu_2]\to[0,+\infty)$ *is said to be* (h,m)*-convex on* $[0,\nu_2]$ *if inequality*

$$\psi\big(\tau\xi+m(1-\tau)\varsigma\big)\leq h(\tau)\psi(\xi)+mh(1-\tau)\psi(\varsigma)$$

is fulfilled for $m\in[0,1]$, *for all* $\xi,\varsigma\in I$ *and* $\tau\in[0,1]$.

If the above inequality is reversed, then ψ is said to be (h,m)-concave. Note that if $h(\tau)=\tau$ then the f above definition reduces to the definition of m-convex function; if, in addition, $m=1$ then we obtain the definition of convex function.

In [34] the authors presented the class of $s-(\alpha,m)$-convex functions as follows ("redefined" in [61]).

Definition 7. *Let* $\psi\colon [0,+\infty)\to[0,+\infty)$ *and* $\alpha,m\in[0,1]$. *If inequality*

$$\psi\big(t\nu_1+m(1-t)\nu_2\big)\leq t^{\alpha s}\psi(\nu_1)+m\big(1-t^{\alpha s}\big)\psi(\nu_2)$$

holds for all $\xi,\varsigma\in[0,+\infty)$, $\tau\in[0,1]$ *and* $s\in(0,1]$, *then is said* ψ *is a* $s-(\alpha,m)$*-convex function in the first sense.*

Definition 8. *Let* $\psi\colon[0,+\infty)\to[0,+\infty)$ *and* $\alpha,m\in[0,1]$. *If inequality*

$$\psi\big(t\nu_1+m(1-t)\nu_2\big)\le\big(t^\alpha\big)^s\psi(\nu_1)+m\big(1-t^\alpha\big)^s\psi(\nu_2)$$

holds for all $\xi,\varsigma\in[0,+\infty)$, $\tau\in[0,1]$ *and* $s\in(0,1]$, *then is said* ψ *is a* $s-(\alpha,m)$*-convex function in the second sense.*

On the basis of these definitions, we will present the classes of functions that will be the basis of our work (see [36]).

Definition 9. *Let* $h\colon[0,1]\to[0,1]$ *be a nonnegative function*, $h\neq0$ *and* $\psi\colon I=[0,+\infty)\to[0,+\infty)$. *If inequality*

$$\psi\big(\tau\xi+m(1-\tau)\varsigma\big)\le h^s(\tau)\psi(\xi)+m\big(1-h^s(\tau)\big)\psi(\varsigma)$$

is fulfilled for all $\xi,\varsigma\in I$ *and* $\tau\in[0,1]$, *where* $m\in[0,1]$, $s\in[-1,1]$, *then is said function* ψ *is a* (h,m,s)*-convex modified of first type on* I.

Definition 10. *Let* $h\colon[0,1]\to[0,1]$ *nonnegative functions*, $h\neq0$ *and* $\psi\colon I=[0,+\infty)\to[0,+\infty)$. *If inequality*

$$\psi\big(\tau\xi+m(1-\tau)\varsigma\big)\le h^s(\tau)\psi(\xi)+m(1-h(\tau))^s\psi(\varsigma) \tag{2}$$

is fulfilled for all $\xi,\varsigma\in I$ *and* $\tau\in[0,1]$, *where* $m\in[0,1]$, $s\in[-1,1]$. *then is said function* ψ *is a* (h,m,s)*-convex modified of second type on* I.

***Remark* 1.** From Definitions 9 and 10 we can define $N^s_{h,m}[a,b]$, where $a,b\in[0,+\infty)$, as the set of functions (h,m,s)-convex modified, for which $\psi(a)\ge0$, characterized by the triple $(h(\tau),m,s)$. Note that if:

1) $(h(\tau),0,0)$, we have the increasing functions [11],
2) $(\tau,0,s)$, we have the s-starshaped functions [11],
3) $(\tau,0,1)$, we have the starshaped functions [11],
4) $(\tau,m,1)$, then ψ is a m-convex function on $[0,+\infty)$ [55],
5) $(\tau,1,s)$, $s\in(0,1]$, then ψ is a s-convex function on $[0,+\infty)$,
6) $(\tau,1,s)$, $s\in[-1,1]$, then ψ is a extended s-convex function on $[0,+\infty)$,
7) $(\tau^\alpha,1,s)$ with $a\in(0,1]$, then ψ is a $s-(\alpha,m)$-convex function on $[0,+\infty)$,
8) $(h(\tau),m,1)$, then ψ is an (h,m)-convex function on $[0,+\infty)$,
9) $(\tau,1,1)$, then ψ is a convex function on $[0,+\infty)$ [11].

All through the work we utilize the functions Γ (see [44, 46, 62, 63]) and Γ_k (see [12]):

$$\Gamma(z)=\int\limits_0^\infty\tau^{z-1}e^{-\tau}\,d\tau,\quad \Re(z)>0,$$

$$\Gamma_k(z)=\int\limits_0^\infty\tau^{z-1}e^{-\tau^k/k}d\tau,\quad k>0.$$

Unmistakably, if $k\to1$ we have $\Gamma_k(z)\to\Gamma(z)$, $\Gamma_k(z)=(k)^{\frac{z}{k}-1}\Gamma\Big(\dfrac{z}{k}\Big)$ and $\Gamma_k(z+k)=z\Gamma_k(z)$.

The following functions will also be required:

$$B_\xi(a,b) = \int_0^\xi \tau^{a-1}(1-\tau)^{b-1}d\tau,$$

$$B_1(a,b) = B(a,b) = \int_0^1 \tau^{a-1}(1-\tau)^{b-1}d\tau.$$

To encourage comprehension of the subject, we present the definition of the Riemann – Liouville fractional integral (with $0 \leq \nu_1 < \tau < \nu_2 \leq \infty$). The first is the classic Riemann – Liouville fractional integrals.

Definition 11. *Let $\psi \in L_1[\nu_1,\nu_2]$. Then the Riemann – Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ are defined by* (*right and left*, *respectively*)

$$I^\alpha_{\nu_1+}\psi(\xi) = \frac{1}{\Gamma(\alpha)}\int_{\nu_1}^\xi (\xi-\tau)^{\alpha-1}\psi(\tau)\,dt, \quad \xi > \nu_1,$$

$$I^\alpha_{\nu_2-}\psi(\xi) = \frac{1}{\Gamma(\alpha)}\int_\xi^{\nu_2} (\tau-\xi)^{\alpha-1}\psi(\tau)\,dt, \quad \xi < \nu_2.$$

Next we present the weighted integral operators, which will be the basis of our paper.

Definition 12. *Let $\psi \in L_1(\nu_1,\nu_2)$ and $w\colon [0,\infty) \to [0,\infty)$ be a continuous function with first and second order derivatives piecewise continuous on $[0,\infty)$. Then the weighted fractional integrals are defined by* (*right and left*, *respectively*)

$${}^{\varrho+1}J^w_{\nu_1+}\psi(\xi) = \int_{\nu_1}^\xi w'\left(\frac{\xi-\tau}{\dfrac{\nu_2-\nu_1}{\varrho+1}}\right)\psi(\tau)\,dt, \quad \xi > \nu_1,$$

$${}^{\varrho+1}J^w_{\nu_2-}\psi(\xi) = \int_\xi^{\nu_2} w'\left(\frac{\tau-\xi}{\dfrac{\nu_2-\nu_1}{\varrho+1}}\right)\psi(\tau)\,dt, \quad \xi < \nu_2.$$

Definition 13. *Let $\psi \in L_1(\nu_1,\nu_2)$ and $w\colon [0,\infty) \to [0,\infty)$ be a continuous function with first and second order derivatives piecewise continuous on $[0,\infty)$. Then the weighted fractional integrals are defined by* (*right and left*, *respectively*)

$${}^{2(\varrho+1)}J^w_{\nu_1+}\psi(\xi) = \int_{\nu_1}^\xi w'\left(\frac{\xi-\tau}{\dfrac{\nu_2-\nu_1}{2(\varrho+1)}}\right)\psi(\tau)\,dt, \quad \xi > \nu_1,$$

$${}^{2(\varrho+1)}J^w_{\nu_2-}\psi(\xi) = \int_\xi^{\nu_2} w'\left(\frac{\tau-\xi}{\dfrac{\nu_2-\nu_1}{2(\varrho+1)}}\right)\psi(\tau)\,dt, \quad \xi < \nu_2.$$

***Remark* 2.** To have a clearer idea of the amplitude of the Definition 12 (and obviously of Definition 13), let us consider some particular cases:

1. Putting $w'(\tau) \equiv 1,\ \varrho = 0$, we obtain the classical Riemann integral.

2. If $w'(\tau) = \dfrac{\tau^{(\alpha-1)}(\nu_2 - \nu_1)^{(\alpha-1)}}{\Gamma(\alpha)}$ for $\varrho = 0$, then we obtain the Riemann – Liouville fractional integral, right and left.

3. If $w'(\tau) = \dfrac{\tau^{(\frac{\alpha}{k}-1)}(\nu_2 - \nu_1)^{(\alpha-1)}}{k\Gamma_k(\alpha)}$ for $\varrho = 0$, then we obtain the k-Riemann – Liouville fractional integral, right and left [33].

4. If

$$w'(\xi - \tau) = \frac{(h(\xi) - h(\tau))^{(\frac{\alpha}{k}-1)} h'(\tau)(\nu_2 - \nu_1)^{(\alpha-1)}}{k\Gamma_k(\alpha)}$$

and

$$w'(\tau - \xi) = \frac{(h(\tau) - h(\xi))^{(\frac{\alpha}{k}-1)} h'(\tau)(\nu_2 - \nu_1)^{(\alpha-1)}}{k\Gamma_k(\alpha)}, \quad \varrho = 0,$$

then we obtain the right- and left-hand sided fractional integrals of a function ψ with respect to another function h on $[\nu_1, \nu_2]$ (see [3]).

In this paper, we obtain different variants of the Hermite – Hadamard inequality, in the framework of the (h, m, s)-convex modified functions, via generalized operators of the Definition 13.

2. Hermite – Hadamard-type inequalities for (h, m, s)-convex modified functions of second type. A version of the Hermite – Hadamard inequality can be represented, using the weighted integral operators of the Definition 13, as follows:

Theorem 1. *Let $\psi\colon [0, +\infty) \to \mathbb{R}$ be a (h, m, s)-convex modified function of second type with $m \in (0, 1]$. If $0 \le \nu_1 < m\nu_2 < +\infty$, $\psi \in L^1[\nu_1, m\nu_2]$ and $h \in L^1[0, 1]$, then we have the inequality*

$$\begin{aligned}
&\psi\left(\frac{\nu_1 + \nu_2}{2}\right)(w(1) - w(0)) \\
&\quad \le \frac{\varrho + 1}{\nu_2 - \nu_1}\left[h^s\left(\frac{1}{2}\right)\, {}^{\varrho+1}J^w_{\left(\frac{\nu_1 + \varrho\nu_2}{\varrho+1}\right)+}\psi(v_2) + \left(1 - h\left(\frac{1}{2}\right)\right)^s\, {}^{\varrho+1}J^w_{\left(\frac{\nu_2 + \varrho\nu_1}{\varrho+1}\right)-}\psi(v_1)\right] \\
&\quad \le \psi(\nu_1)\left[h^s\left(\frac{1}{2}\right)\int\limits_0^1 w'(\tau) h^s\left(\frac{\tau}{\varrho + 1}\right) dt\right. \\
&\qquad \left. + \left(1 - h\left(\frac{1}{2}\right)\right)^s \int\limits_0^1 w'(\tau)\left(1 - h\left(\frac{\varrho + 1 - \tau}{\varrho + 1}\right)\right)^s dt\right] \\
&\qquad + m\psi\left(\frac{\nu_2}{m}\right)\left[h^s\left(\frac{1}{2}\right)\int\limits_0^1 w'(\tau)\left(1 - h\left(\frac{\varrho + 1 - \tau}{\varrho + 1}\right)\right)^s dt\right.
\end{aligned}$$

$$+\left(1-h\left(\frac{1}{2}\right)\right)^s \int_0^1 w' h^s\left(\frac{\tau}{\varrho+1}\right)dt\Bigg]. \tag{3}$$

Proof. For $\xi, \varsigma \in [0,+\infty)$, $\tau = \dfrac{1}{2}$ and $m=1$, we obtain

$$\psi\left(\frac{\xi+\varsigma}{2}\right) \le \psi\left(\frac{\xi+\varsigma}{2}\right) \le h^s\left(\frac{1}{2}\right)\psi(\xi) + \left(1-h\left(\frac{1}{2}\right)\right)^s \psi(\varsigma).$$

If we choose $\xi = \dfrac{\tau}{\varrho+1}\nu_1 + \left(\dfrac{\varrho+1-\tau}{\varrho+1}\right)\nu_2$ and $\varsigma = \dfrac{\tau}{\varrho+1}\nu_2 + \left(\dfrac{\varrho+1-\tau}{\varrho+1}\right)\nu_1$ with $\tau \in [0,1]$, we get

$$\psi\left(\frac{\nu_1+\nu_2}{2}\right) \le h^s\left(\frac{1}{2}\right)\psi\left(\frac{\tau}{\varrho+1}\nu_1 + \left(\frac{\varrho+1-\tau}{\varrho+1}\right)\nu_2\right)$$
$$+\left(1-h\left(\frac{1}{2}\right)\right)^s \psi\left(\frac{\tau}{\varrho+1}\nu_2 + \left(\frac{\varrho+1-\tau}{\varrho+1}\right)\nu_1\right). \tag{4}$$

Multiplying both members of the previous inequality by $w'(\tau)$, integrating with respect to τ from 0 to 1, we have

$$\psi\left(\frac{\nu_1+\nu_2}{2}\right)(w(1)-w(0))) \le h^s\left(\frac{1}{2}\right)\int_0^1 w'(\tau)\psi\left(\frac{\tau}{\varrho+1}\nu_1 + \left(\frac{\varrho+1-\tau}{\varrho+1}\right)\nu_2\right)dt$$
$$+\left(1-h\left(\frac{1}{2}\right)\right)^s \int_0^1 w'(\tau)\psi\left(\frac{\tau}{\varrho+1}\nu_2 + \left(\frac{\varrho+1-\tau}{\varrho+1}\right)\nu_1\right)dt,$$

and changing variables, we obtain

$$\psi\left(\frac{\nu_1+\nu_2}{2}\right)(w(1)-w(0)) \le h^s\left(\frac{1}{2}\right)\frac{\varrho+1}{\nu_2-\nu_1}\int_{\frac{\nu_1+\varrho\nu_2}{\varrho+1}}^{\nu_2} w'\left(\frac{\nu_2-\xi}{\frac{\nu_2-\nu_1}{\varrho+1}}\right)\psi(\xi)d\xi$$
$$+\left(1-h\left(\frac{1}{2}\right)\right)^s \frac{\varrho+1}{\nu_2-\nu_1}\int_{\nu_1}^{\frac{\nu_2+\varrho\nu_1}{\varrho+1}} w'\left(\frac{\xi-\nu_1}{\frac{\nu_2-\nu_1}{\varrho+1}}\right)\psi(\xi)d\xi$$

or

$$\psi\left(\frac{\nu_1+\nu_2}{2}\right)(w(1)-w(0))$$
$$\le \frac{\varrho+1}{\nu_2-\nu_1}\left[h^s\left(\frac{1}{2}\right)\, {}^{\varrho+1}J^w_{\left(\frac{\nu_1+\varrho\nu_2}{\varrho+1}\right)+}\psi(v_2) + \left(1-h\left(\frac{1}{2}\right)\right)^s\, {}^{\varrho+1}J^w_{\left(\frac{\nu_2+\varrho\nu_1}{\varrho+1}\right)-}\psi(v_1)\right].$$

Thus, we obtained the first inequality of (3).

From right member of (4), we get

$$h^s\left(\frac{1}{2}\right)\psi\left(\frac{\tau}{\varrho+1}\nu_1+\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\nu_2\right)+\left(1-h\left(\frac{1}{2}\right)\right)^s\psi\left(\frac{\tau}{\varrho+1}\nu_2+\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\nu_1\right)$$
$$=h^s\left(\frac{1}{2}\right)\psi\left(\frac{\tau}{\varrho+1}\nu_1+m\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\frac{\nu_2}{m}\right)$$
$$+\left(1-h\left(\frac{1}{2}\right)\right)^s\psi\left(\frac{\tau}{\varrho+1}\nu_2+m\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\frac{\nu_1}{m}\right)$$
$$\le h^s\left(\frac{1}{2}\right)\left(h\left(\frac{\tau}{\varrho+1}\right)\psi(\nu_1)+m\left(1-h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s\psi\left(\frac{\nu_2}{m}\right)\right)$$
$$+\left(1-h\left(\frac{1}{2}\right)\right)^s\left(h\left(\frac{\tau}{\varrho+1}\right)\psi(\nu_2)+m\left(1-h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s\psi\left(\frac{\nu_1}{m}\right)\right).$$

Multiplying this by $w'(\tau)$ and integrating with respect to τ, between 0 and 1, allows us to have the right member of (3).

Theorem 1 is proved.

***Remark* 3.** If in the previous theorem we consider convex functions, i.e., $s=m=1$ and $h(\tau)=\tau$ and we put $w(\tau)=\tau$, with $\varrho=0$, from (3) we obtain the classic Hermite–Hadamard inequality (1).

***Remark* 4.** Analogously, working with convex functions ($s=m=1$ and $h(\tau)=\tau$) and taking $w'(\tau)=\tau^{\alpha-1}$ with $\varrho=1$, we have Theorem 4 of [51].

***Remark* 5.** If we consider $w(\tau)=\tau^{-\alpha}$, the previous result gives us new results for non conformable integral operators (see [35]). It is clear that if we consider other kernels $w'(\tau)$ the results derived from the previous theorem are new.

The following result will be basic from now on.

Lemma 1. *Let ψ be a real function defined on some interval $[\nu_1,\nu_2]\subset\mathbb{R}$, differentiable on (ν_1,ν_2). If $\psi'\in L_1(a,b)$, then we have the equality*

$$-\frac{1}{2}\left\{w(1)\left(\psi\left(\frac{\nu_1+\nu_2+2\varrho\nu_2}{2(\varrho+1)}\right)+\psi\left(\frac{\nu_1+\nu_2+2\varrho\nu_1}{2(\varrho+1)}\right)\right)-w(0)\big(\psi(\nu_1)+\psi(\nu_2)\big)\right\}$$
$$+\left(\frac{\varrho+1}{\nu_2-\nu_1}\right)\left({}^{2(\varrho+1)}J^w_{\left(\frac{\nu_1+\nu_2+2\varrho\nu_2}{2(\varrho+1)}\right)+}\psi(\nu_2)+{}^{2(\varrho+1)}J^w_{\left(\frac{\nu_1+\nu_2+2\varrho\nu_2}{2(\varrho+1)}\right)-}\psi(\nu_1)\right)$$
$$=\frac{\nu_2-\nu_1}{4(\varrho+1)}\int_0^1 w(\tau)\left[\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)\right.$$
$$\left.-\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right]dt. \tag{5}$$

Proof. First, note that

$$\int_0^1 w(\tau)\left[\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)-\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right]dt$$

$$= \int_0^1 w(\tau)\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)dt$$

$$-\int_0^1 w(\tau)\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)dt = I_1 - I_2.$$

Integrating by parts, we obtain

$$I_1 = -\frac{2(\varrho+1)}{\nu_2-\nu_1}\left[w(1)\psi\left(\frac{\nu_1+\nu_2+2\varrho\nu_2}{2(\varrho+1)}\right)-w(0)\psi(\nu_2)\right]$$

$$+\frac{2(\varrho+1)}{\nu_2-\nu_1}\int_0^1 w'(\tau)\psi\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)dt$$

$$= -\frac{2(\varrho+1)}{\nu_2-\nu_1}\left[w(1)\psi\left(\frac{\nu_1+\nu_2+2\varrho\nu_2}{2(\varrho+1)}\right)-w(0)\psi(\nu_2)\right]$$

$$+\frac{4(\varrho+1)^2}{(\nu_2-\nu_1)^2}\,{}^{2(\varrho+1)}J^w_{\left(\frac{\nu_1+\nu_2+2\varrho\nu_2}{2(\varrho+1)}\right)+}\psi(\nu_2),$$

since

$$\int_0^1 w'(\tau)\psi\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)dt$$

$$= \frac{4(\varrho+1)^2}{(\nu_2-\nu_1)^2}\int_{\left(\frac{\nu_1+\nu_2+2\varrho\nu_2}{2(\varrho+1)}\right)+}^{\nu_2} w'\left(\frac{\nu_2-u}{\frac{\nu_2-\nu_1}{2(\varrho+1)}}\right)\psi(\tau)dt.$$

Analogously,

$$I_2 = \frac{2(\varrho+1)}{\nu_2-\nu_1}\left[w(1)\psi(\frac{\nu_1+\nu_2+2\varrho\nu_1}{2(\varrho+1)})-w(0)\psi(\nu_1)\right]$$

$$-\frac{4(\varrho+1)^2}{(\nu_2-\nu_1)^2}\,{}^{2(\varrho+1)}J^w_{\left(\frac{\nu_1+\nu_2+2\varrho\nu_1}{\varrho+1}\right)-}\psi(\nu_1).$$

From $I_1 - I_2$, and grouping appropriately, we have the required inequality.

Lemma 1 is proved.

Before analyzing some particular cases of this result, we want to clarify some details.

***Remark* 6.** In [26], starting from Lemma 2.1 of [28] (see Remark 9 below), through a change of variables obtains, for convex functions and with $w(\tau) = \tau$, the integral of the right member of (5), we must point out that the + sign in said work is incorrect. In this way, it is clear that equation (2) of [26] is obtained from the previous lemma for $\varrho = 0,\ w(\tau) = \tau$.

We want to point out various results that can be obtained from the previous lemma, which shows the scope and generality of said result.

***Remark* 7.** Putting $w(\tau) = \tau^{\alpha}$ and considering convex functions and $\varrho = 0$, we obtain the Lemma 3 of [51].

***Remark* 8.** If $w(\tau) = \tau,\ \varrho = 0$ we obtain a new result for classic integral.

***Remark* 9.** Let us consider $\varrho = 0$. For various choices of the weight $w'(\tau)$ and taking, not only the right member of (5) but only one of the integrals, they can be obtained without difficulty a variant of Lemma 1 of [5], Lemma 2.1 of [13] (also see Lemma 2.1 of [25]), Lemma 2.3 of [18], Lemma 1 of [24], Lemma 1 of [27], Lemma 2.1 of [28], Lemma 1 of [38], Lemma 1 of [41], Lemma 3.1 of [53] and Lemma 2 of [50] (see also [40]) are obtained.

***Remark* 10.** Also, the reader will be able to verify, without much difficulty, that under different variants of the weight $w(\tau)$ we can obtain Lemma 2 of [42], Lemma 1.1 of [54] (see also Lemma 2 of [39]), Lemma 2.1 from [48], Lemma 2.1 from [65], Lemma 2.1 from [59], Lemma 1.6 from [47], Lemma 2.1 from [1], Lemma 1 of [6], Lemma 2.1 of [52], and Lemma 2.1 of [45].

***Remark* 11.** With $w(\tau) = (1-\tau)^{\alpha}$ and $\varrho = 0$, we obtain a new result for the Riemann – Liouville integrals.

***Remark* 12.** If $\varrho = 1$, Lemma 1 of [2] and Lemma 1 of [58] can be obtained from our result, under the appropriate definition of $w(\tau) = w_1 + w_2$ (see also [57]).

Our first main result is the following.

Theorem 2. *Let $\psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° such that $\psi' \in L_1\Big[\nu_1, \dfrac{\nu_2}{m}\Big]$. Under the assumptions of Lemma 1 if $|\psi'|$ is modified (h, m, s)-convex of second type on $\Big[\nu_1, \dfrac{\nu_2}{m}\Big]$, we have the inequality*

$$\left|\mathbb{A} + \left({}^{2(\varrho+1)}J^{w}_{\nu_1+}\psi\left(\frac{\nu_1+\nu_2+2\varrho\nu_2}{2(\varrho+1)}\right) + {}^{2(\varrho+1)}J^{w}_{\nu_2-}\psi\left(\frac{\nu_1+\nu_2+2\varrho\nu_1}{2(\varrho+1)}\right)\right)\right|$$

$$\leq \frac{(\nu_2-\nu_1)^2}{(\varrho+1)^2}\left\{\left(\left|\psi'\left(\frac{\nu_1+\nu_2}{2}\right)\right|\mathbb{B} + m\mathbb{C}\left[\left|\psi'\left(\frac{\nu_1}{m}\right)\right| + \left|\psi'\left(\frac{\nu_2}{m}\right)\right|\right]\right)\right\} \tag{6}$$

with

$$\mathbb{A} = -\frac{1}{2}\left\{w(1)\left(\psi\left(\frac{\nu_1+\nu_2+2\varrho\nu_2}{2(\varrho+1)}\right) + \psi\left(\frac{\nu_1+\nu_2+\varrho\nu_1}{2(\varrho+1)}\right)\right) - w(0)\big(\psi(\nu_1)+\psi(\nu_2)\big)\right\},$$

$$\mathbb{B} = \int\limits_0^1 w(\tau)\left[h^s\left(\frac{\tau}{\varrho+1}\right) + h^s\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right]dt,$$

and

$$\mathbb{C} = \int\limits_0^1 w(\tau)\left(1 - h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s dt.$$

Proof. From Lemma 1 we obtain

$$\left|\int\limits_0^1 w(\tau)\left[\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right) + \frac{\varrho+1-\tau}{\varrho+1}\nu_2\right) - \psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right) + \frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right]dt\right|$$

$$\leq \int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)\right|dt$$

$$+\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right|dt.$$

Using the modified (h,m,s)-convexity of $|\psi'|$, we get

$$\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)\right|dt$$

$$\leq \int_0^1 w(\tau)\left[h^s\left(\frac{\tau}{\varrho+1}\right)\left|\psi'\left(\frac{\nu_1+\nu_2}{2}\right)\right|+m\left(1-h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s\left|\psi'\left(\frac{\nu_2}{m}\right)\right|\right]dt$$

$$=\left|\psi'\left(\frac{\nu_1+\nu_2}{2}\right)\right|\int_0^1 w(\tau)h^s\left(\frac{\tau}{\varrho+1}\right)dt+m\left|\psi'\left(\frac{\nu_2}{m}\right)\right|\int_0^1 w(\tau)\left(1-h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s dt. \tag{7}$$

In the same way

$$\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right|dt$$

$$\leq\left|\psi'\left(\frac{\nu_1+\nu_2}{2}\right)\right|\int_0^1 w(\tau)h^s\left(\frac{\tau}{\varrho+1}\right)dt+m\left|\psi'\left(\frac{\nu_1}{m}\right)\right|\int_0^1 w(\tau)\left(1-h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s dt. \tag{8}$$

From (7) and (8) we easily obtain (6).

Theorem 2 is proved.

***Remark* 13.** Considering $\varrho=0$, the following results can be derived (and/or complemented, since they only use the extremes of the interval and not the midpoint) from the theorem above:

Theorem 1 of [5], only for I_2 and

$$w(\tau)=\begin{cases}\tau, & \tau\in\left[0,\dfrac{\nu_2-\xi}{\nu_2-\nu_1}\right],\\[2ex] \tau-1, & \tau\in\left[\dfrac{\nu_2-\xi}{\nu_2-\nu_1},1\right],\end{cases}$$

for convex functions.

Theorem 2.1 from [7] (case $q=1$), for m-convex functions, $h(\tau)=\tau$ and $s=1$.

Theorem 2.2 from [13], obtained for convex functions, using $w(\tau)=1-2\tau$ and using only I_2.

Theorem 2.4 of [18] for convex functions, $h(\tau) = \tau$ and $s = m = 1$, a known result for k-fractional integrals.

Theorem 2.2 of [28], obtained for convex functions and

$$w(\tau) = \begin{cases} \tau, & \left[0, \dfrac{1}{2}\right), \\ \tau - 1, & \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

Theorem 2.3 of [34] with I_2, $w(\tau) = 1 - 2\tau$ and $s-(\alpha, m)$-convex functions are considered.

Theorem 7 from [39], for s-convex functions.

Theorem 3.1 from [45], where I_2 is used and the interval $[0,1]$ is divided, using $w_1 = \lambda - \tau$ for $\left[0, \dfrac{1}{2}\right]$ and $w_2 = \mu - \tau$ for $\left[\dfrac{1}{2}, 1\right]$, where λ and μ real numbers such that $0 \leq \lambda \leq \dfrac{1}{2} \leq \mu \leq 1$.

Theorem 3 of [50], for convex functions and taking $w(\tau) = (1-\tau)^\alpha - \tau^\alpha$ and I_2, a result for the Riemann – Liouville fractional integrals.

The first part of Theorem 5 of [56], statement for fractional integrals of the Riemann – Liouville type, using only I_2 and $w(\tau) = (1-\tau)^\alpha - \tau^\alpha$, in function class h-convex.

Theorem 5 of [64] working with $w(\tau) = (1-\tau)^\alpha - \tau^\alpha$ and using I_2, a valid inequality for fractional integrals.

***Remark* 14.** Considering $\varrho = 1$ and $m = 1$, $h(\tau) = \tau$ (that is, we consider s-convex functions in the second sense), from the Theorem 2 we obtain Theorem 10 of [58], putting $w(\tau) = w_1(\tau) + w_2(\tau)$, where

$$w_1(\tau) = \frac{1}{4}\big(t\nu_2 + (1-\tau)\nu_1\big) \quad \text{and} \quad w_2(\tau) = \frac{1}{4}\big(t\nu_1 + (1-\tau)\nu_2\big).$$

Refinements of the previous results, can be obtained by imposing new additional conditions on $|\psi'|^q$.

Theorem 3. *Let $\psi\colon I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° such that $\psi' \in L_1\left[\nu_1, \dfrac{\nu_2}{m}\right]$. Under the assumptions of Lemma 1 if $\left|\psi'\right|^q$, $q \geq 1$, is modified (h, m, s)-convex of second type on $\left[\nu_1, \dfrac{\nu_2}{m}\right]$, we have the inequality*

$$\left|\mathbb{A} + \left({}^{2(\varrho+1)}J^w_{\nu_1+}\psi\left(\frac{\nu_1+\nu_2+2\varrho\nu_2}{2(\varrho+1)}\right) + {}^{2(\varrho+1)}J^w_{\nu_2-}\psi\left(\frac{\nu_1+\nu_2+2\varrho\nu_1}{2(\varrho+1)}\right)\right)\right|$$
$$\leq \frac{(\nu_2-\nu_1)^2}{(\varrho+1)^2} B_q\left\{(p_{12}C_{11} + mp_2C_{12})^{\frac{1}{q}} + (p_{12}C_{11} + mp_1C_{12})^{\frac{1}{q}}\right\}$$

with $\mathbb{A}$ as before, $B_q = \left(\displaystyle\int_0^1 w^p(\tau)dt\right)^{\frac{1}{p}}$, $p_1 = \left|\psi'\left(\dfrac{\nu_1}{m}\right)\right|^q$, $p_{12} = \left|\psi'\left(\dfrac{\nu_1+\nu_2}{2}\right)\right|^q$, $p_2 = \left|\psi'\left(\dfrac{\nu_2}{m}\right)\right|^q$, $C_{11} = \displaystyle\int_0^1 h^s\left(\frac{\tau}{\varrho+1}\right)dt$, and $C_{12} = \displaystyle\int_0^1 \left(1 - h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s dt$.

Proof. As previous result, from Lemma 1 we obtain

$$\left|\int\limits_0^1 w(\tau)\left[\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right) + \frac{\varrho+1-\tau}{\varrho+1}\nu_2\right) - \psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right) + \frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right]dt\right|$$

$$\leq \int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)\right|dt$$

$$+\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right|dt.$$

From Hölder's inequality, we obtain

$$\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)\right|dt$$

$$\leq \left(\int_0^1 w^p(\tau)dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)\right|^q dt\right)^{\frac{1}{q}} \tag{9}$$

and

$$\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right|dt$$

$$\leq \left(\int_0^1 w^p(\tau)dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right|^q dt\right)^{\frac{1}{q}} \tag{10}$$

for $\dfrac{1}{p}+\dfrac{1}{q}=1$. Using the (h,m,s)-convexity of the second type of $|\psi'|^q$, we obtain, from (9) and (10),

$$\int_0^1\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)\right|^q dt$$

$$\leq \left|\psi'\left(\frac{\nu_1+\nu_2}{2}\right)\right|^q\int_0^1 h^s\left(\frac{\tau}{\varrho+1}\right)dt+m\left|\psi'\left(\frac{\nu_2}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s dt, \tag{11}$$

$$\int_0^1\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right|^q dt$$

$$\leq \left|\psi'\left(\frac{\nu_1+\nu_2}{2}\right)\right|^q\int_0^1 h^s\left(\frac{\tau}{\varrho+1}\right)dt+m\left|\psi'\left(\frac{\nu_1}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s dt. \tag{12}$$

Denoting, for brevity $B_q=\left(\int_0^1 w^p(\tau)\,dt\right)^{\frac{1}{p}}$, substituting (11), (12) in (9) and (10), we obtain the required inequality.

Theorem 3 is proved.

***Remark* 15.** The Theorem 11 of [58] and Theorem 5 of [51] as particular cases of the above result, if we taking $\varrho = 1$. Other known results from the literature that can be derived of the above theorem are the following: Theorem 3.2 of [17], Theorem 6 of [64], Theorem 2.1 (second part) and Theorem 2.2 of [7], Theorem 11 of [58], Theorem 2.3 of [13], Theorem 6 of [57], Theorem 8 of [39], the second part of Theorem 1 of [41] , Theorem 1 of [43], Theorem 2.11 of [34], Theorem 5 of [56] and Theorem 2 of [5].

Theorem 4. *Let $\psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° such that $\psi' \in L_1\left[\nu_1, \frac{\nu_2}{m}\right]$. Under the assumptions of Lemma 1 if $|\psi'|^q$, $q > 1$, is modified (h, m, s)-convex of second type on $\left[\nu_1, \frac{\nu_2}{m}\right]$, we have the inequality*

$$\left|\mathbb{A} + \left({}^{2(\varrho+1)}J^w_{\nu_1+}\psi\left(\frac{\nu_1+\nu_2+2\varrho\nu_2}{2(\varrho+1)}\right) + {}^{2(\varrho+1)}J^w_{\nu_2-}\psi\left(\frac{\nu_1+\nu_2+2\varrho\nu_1}{2(\varrho+1)}\right)\right)\right|$$

$$\leq \frac{(\nu_2-\nu_1)^2}{(\varrho+1)^2}\Delta\left\{(p_{12}D_{11} + mp_2D_{12})^{\frac{1}{q}} + (p_{12}D_{21} + mp_2D_{22})^{\frac{1}{q}}\right\} \tag{13}$$

with $\mathbb{A}$, p_1, p_{12} and p_2 as before,

$$\Delta = \left(\int_0^1 w(\tau)\,dt\right)^{1-\frac{1}{q}}, \qquad D_{11} = \int_0^1 w(\tau)h^s\left(\frac{\tau}{\varrho+1}\right)dt,$$

$$D_{12} = \int_0^1 w(\tau)\left(1 - h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s dt, \qquad D_{21} = \int_0^1 w(\tau)h^s\left(\frac{\varrho+1-\tau}{\varrho+1}\right)dt$$

and

$$D_{22} = \int_0^1 w(\tau)\left(1 - h\left(\frac{\tau}{\varrho+1}\right)\right)^s dt.$$

Proof. As before, from the Lemma 1 we have

$$\left|\int_0^1 w(\tau)\left[\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right) + \frac{\varrho+1-\tau}{\varrho+1}\nu_2\right) - \psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right) + \frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right]dt\right|$$

$$\leq \int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right) + \frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)\right|dt$$

$$+ \int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right) + \frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right|dt,$$

and using well-known power mean inequality, we obtain

$$\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right) + \frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)\right|dt$$

$$\leq \left(\int_0^1 w(\tau)dt\right)^{1-\frac{1}{q}} \left(\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)\right|^q dt\right)^{\frac{1}{q}} \tag{14}$$

and

$$\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right|dt$$

$$\leq \left(\int_0^1 w(\tau)dt\right)^{1-\frac{1}{q}} \left(\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right|^q dt\right)^{\frac{1}{q}}. \tag{15}$$

Using the modified (h,m,s)-convexity of $|\psi'|^q$, we get

$$\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_2\right)\right|^q dt$$

$$\leq \int_0^1 w(\tau)\left[h^s\left(\frac{\tau}{\varrho+1}\right)\left|\psi'\left(\frac{\nu_1+\nu_2}{2}\right)\right|^q + m\left(1-h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s\left|\psi'\left(\frac{\nu_2}{m}\right)\right|^q\right]dt$$

$$= \left|\psi'\left(\frac{\nu_1+\nu_2}{2}\right)\right|^q \int_0^1 w(\tau)h^s\left(\frac{\tau}{\varrho+1}\right)dt + m\left|\psi'\left(\frac{\nu_2}{m}\right)\right|^q \int_0^1 w(\tau)\left(1-h\left(\frac{\varrho+1-\tau}{\varrho+1}\right)\right)^s dt. \tag{16}$$

Similarly,

$$\int_0^1 w(\tau)\left|\psi'\left(\frac{\tau}{\varrho+1}\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\varrho+1-\tau}{\varrho+1}\nu_1\right)\right|^q dt$$

$$\leq \left|\psi'\left(\frac{\nu_1+\nu_2}{2}\right)\right|^q \int_0^1 w(\tau)h^s\left(\frac{\varrho+1-\tau}{\varrho+1}\right)dt + m\left|\psi'\left(\frac{\nu_2}{m}\right)\right|^q \int_0^1 w(\tau)\left(1-h\left(\frac{\tau}{\varrho+1}\right)\right)^s dt. \tag{17}$$

If we put (16) and (17) in (14) and (15), then we obtain the inequality (13).

Theorem 4 is proved.

***Remark* 16.** Theorem 12 of [58] can be obtained from Theorem 3 putting $\varrho = 1$ and considering s-convex functions. Additionally, the following results: Theorem 3.2 of [45], Theorem 2.13 of [34], Theorem 5 of [5], Theorem 9 of [39], Theorem 7 of [57], Theorem 2 of [41], Theorem 2.3 of [7], Theorem 7 of [64] and Theorem 3.6 of [17], can be obtained as particular cases of the previous.

***Remark* 17.** For $\varrho = 1$ this result complements Theorem 6 of [51].

3. Conclusions. In this paper, a generalization of the Hermite – Hadamard inequality is obtained in the framework of the heavy operators of the Definition 12, later, we obtained a new identity for the heavy fractional integrals of the Definition 13, based on this result, various refinements of the classic Hadamard inequality are obtained.

Throughout the work, it is shown that many known results from the literature are obtained as particular cases of ours.

The generality of the results obtained, taking into account the Remark 1 and the operators of the Definition 13, allow us to derive new results for various classes of functions, either convex, h-convex, m-convex and s-convex in the second sense, defined in a closed interval of nonnegative values of real numbers. It is clear that the problem of extending these results to the case of (h, m, s)-convex functions of the first type remains open.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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