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ON THE MEAN VALUE OF THE GENERALIZED DEDEKIND SUM AND CERTAIN GENERALIZED HARDY SUMS WEIGHTED BY THE KLOOSTERMAN SUM

ПРО СЕРЕДНЄ ЗНАЧЕННЯ УЗАГАЛЬНЕНОЇ СУМИ ДЕДЕКІНДА ТА ДЕЯКИХ УЗАГАЛЬНЕНИХ СУМ ГАРДІ, ЗВАЖЕНИХ ЗА СУМОЮ КЛОСТЕРМАНА

We study a hybrid mean-value problem related to the generalized Dedekind sum, certain generalized Hardy sums, and Kloosterman sum and obtain several meaningful conclusions by means of the analytic method and the properties of the character sum and the Gauss sum.

Вивчено гібридну проблему про середнє значення, пов'язану з узагальненою сумою Дедекінда, деякими узагальненими сумами Гарді та сумою Клостермана. Кілька значущих висновків отримано за допомогою аналітичного методу, властивостей суми характерів та суми Гаусса.

1. Introduction. For a positive integer q and integers $h,\ m,\ n,$ the generalized Dedekind sum $s(h,m,n,q)$ is defined by [13]

$$s(h,m,n,q)=\sum_{a=1}^{q}\overline{B}_m\left(\frac{a}{q}\right)\overline{B}_n\left(\frac{ha}{q}\right),$$

where

$$\overline{B}_m(x)=\begin{cases}B_m(x-[x]), & \text{if} \quad x\in\mathbb{R}\backslash\mathbb{Z},\\ 0, & \text{if} \quad x\in\mathbb{Z},\end{cases}$$

with the Bernoulli polynomial $B_m(x)$, can be generated by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad |t|<2\pi.$$

In particular, if $m=n=1$, we have the classical Dedekind sum $s(h,1,1,q)=s(h,q)$, arising in the theory of Dedekind η-function. The most important result for Dedekind sum is the reciprocity law for $(h,q)=1$:

$$s(h,q)+s(q,h)=-\frac{1}{4}+\frac{1}{12}\left(\frac{h}{q}+\frac{q}{h}+\frac{1}{hq}\right).$$

Walum [18] derived a relation between the mean square value of $s(h,p)$ and the fourth power mean of the Dirichlet L-function. Following this, Conrey et al. [4] considered the mean-value problem of the Dedekind sum and obtained an asymptotic formula for ${\sum}_{h=1}^{\prime q}\left|s(h,q)\right|^{2m}$, where the dash denotes the summation over all $1\leq h\leq q$ such that $(h,q)=1$.

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In parallel with the generalized Dedekind sum $s(h,m,n,q)$, Liu and Zhang [13] have generalized Hardy sums as follows:

$$s_1(h,m,q)=\sum_{a=1}^{q}(-1)^{[ha/q]}\overline{B}_m\left(\frac{a}{q}\right),$$

$$s_2(h,m,n,q)=\sum_{a=1}^{q}(-1)^{a}\overline{B}_m\left(\frac{a}{q}\right)\overline{B}_n\left(\frac{ha}{q}\right),$$

$$s_3(h,n,q)=\sum_{a=1}^{q}(-1)^{a}\overline{B}_n\left(\frac{ha}{q}\right),$$

$$s_5(h,m,q)=\sum_{a=1}^{q}(-1)^{a+[ha/q]}\overline{B}_m\left(\frac{a}{q}\right).$$

In particular, $s_1(h,1,q)=s_1(h,q)$, $s_2(h,1,1,q)=s_2(h,q)$, $s_3(h,1,q)=s_3(h,q)$ and $s_5(h,1,q)=s_5(h,q)$ are classical Hardy sums, encountered in the theory of logarithms of the classical theta functions [2, 8]. These sums and several generalizations have been considered in [3, 5, 6, 10–12, 16].

In recent years, various types of mean-value problem of the Dedekind or Hardy sums and Kloosterman sum, defined by [7]

$$S(u,v,q)=\sum_{a=1}^{q}{}'e\left(\frac{ua+v\overline{a}}{q}\right)$$

have been extensively studied (see [9, 14, 15, 19–21]). Here $\bar{a}$ denotes the solution of the congruence $xa\equiv 1 \pmod q$, the dash denotes the summation over all $1\leq a\leq q$ such that $(a,q)=1$ and $e(x)=e^{2\pi ix}$.

For example, with the help of the properties of the Gauss sums and mean-value formulas of Dirichlet L-function, the hybrid mean-value problem containing generalized Dedekind sum, generalized Hardy sums and Kloosterman sum has been discussed and several exact computational formulas have been obtained in [17]. These results are generalizations of the formulas in [15, 20].

In this paper, helped by the properties of the Gauss sums and mean-value identities of Dirichlet L-function, we achieve the following conclusions for generalized Dedekind sum, certain generalized Hardy sums and Kloosterman sum in order to help to reach additional relations between these sums.

Theorem 1.1. *Assume that q is a square-full number. Then, for $m\equiv n\equiv 1 \pmod 2$ and for any integer u with $(u,q)=1$, we have*

$$\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'S(u,a,q)\overline{S(u,b,q)}s(a\bar{b},m,n,q)$$

$$=\frac{q^3}{\phi(q)}\sum_{l=0}^{m+n}r_{m,n,l}\cdot q^{l-m-n+1}\prod_{p|q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m-n+1}}\right),$$

where $\sum_{a=1}^{\prime q}$ *denotes the summation over all* $1 \le a \le q$ *such that* $(a,q)=1$, $\prod_{p|q}$ *denotes the product over all different prime divisors of* q, $\phi(q)$ *is the Euler function,* $\overline{f(n)}$ *denotes the complex conjugation of* $f(n)$ *and*

$$r_{m,n,l} = B_{m+n-l}\sum_{a=0}^{m}\sum_{b=0}^{n} B_{m-a}B_{n-b}\frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{m+n-l}}{a+b+1}$$

with the Bernoulli number B_m.

***Remark* 1.1.** If we take $m=n=1$ in Theorem 1.1, then, using the values of Bernoulli numbers $B_1=-1/2,\ B_0=1$ and the fact

$$\phi(q) = q\prod_{p|q}\left(1-\frac{1}{p}\right) \quad \text{for} \quad q \ge 1,$$

Theorem 1.1 coincides with [14, Theorem 3.1]. In conclusion, our formula given above is a generalization of Theorem 3.1 of [14].

Theorem 1.2. *Let* q *be a square-full number. Then, for* $m \equiv 1 \pmod 2$ *and for any integer* u *with* $(u,q)=1$, *we have*

$$\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}' S(u,a,q)\overline{S(u,b,q)}s_5(2a\bar{b},m,q)$$
$$= \frac{q^3}{\phi(q)}\sum_{l=0}^{m+1} r_{m,1,l}\cdot q^{l-m}\left(\frac{-4.2^m-6+2^l}{2^{m-1}+1}\right)\prod_{p|q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m}}\right).$$

Theorem 1.3. *Let* q *be a square-full number. Then, for* $n \equiv 1 \pmod 2$ *and for any integer* u *with* $(u,q)=1$, *we obtain*

$$\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}' S(u,a,q)\overline{S(u,b,q)}s_3(a\bar{b},n,q)$$
$$= \frac{q^3}{\phi(q)}\sum_{l=0}^{n+1} r_{1,n,l}\cdot q^{l-n}\left(\frac{5.2^n+6-2^{l+1}}{2^{n-1}+1}\right)\prod_{p|q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-n}}\right).$$

2. Some lemmas. In order to prove our theorems, we require some lemmas. In the sequel, we will use many properties of Gauss sums $\tau(\chi)$, which can be found in [1].

Lemma 2.1. *Let* h,q *be positive integers with* $q \ge 3$. *Then, for* $m \equiv n \equiv 1 \pmod 2$ *and* $(h,q)=1$, *we have*

$$s(h,m,n,q) = \frac{-4m!n!}{(2\pi i)^{m+n}q^{m+n-1}}\sum_{d|q}\frac{d^{m+n}}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\overline{\chi}(h)L(m,\chi)L(n,\overline{\chi}),$$

where $L(m,\chi)$ *is the Dirichlet* L*-function corresponding to the character* $\chi \bmod d$, $\sum_{d|q}$ *denotes the summation over all divisors of* q *and* $\phi(q)$ *is the Euler function.*

Proof. See Theorem 2.3 of [13].

Lemma 2.2. *For a square-full number* q, *we have*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}{}^{*} L(m,\chi)L(n,\overline{\chi})$$

$$= -\frac{(2\pi i)^{m+n}}{4m!n!}\sum_{l=0}^{m+n} r_{m,n,l}\cdot q^{l-m-n+1}\prod_{p|q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m-n+1}}\right), \tag{2.1}$$

and for a square-full odd number q, *we obtain*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}{}^{*} \overline{\chi}(2)L(m,\chi)L(n,\overline{\chi})$$

$$= -\frac{(2\pi i)^{m+n}}{4m!n!}\sum_{l=0}^{m+n} r_{m,n,l}\cdot q^{l-m-n+1}\left(\frac{2^l-2^{m+n}-2}{2^m+2^n}\right)$$

$$\times \prod_{p|q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m-n+1}}\right), \tag{2.2}$$

where $\sum^{*}_{\substack{\chi \bmod q \\ \chi(-1)=-1}}$ *denotes the summation over all odd primitive characters* $\chi \bmod q$, $\prod_{p|q}$ *denotes the product over all different prime divisors of* q *and*

$$r_{m,n,l} = B_{m+n-l}\sum_{a=0}^{m}\sum_{b=0}^{n} B_{m-a}B_{n-b}\frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{m+n-l}}{a+b+1}$$

with the Bernoulli number B_m.

Proof. The statements (2.1) and (2.2) are Lemma 2.6 and Lemma 2.8 of [17], respectively.

Lemma 2.3 [17, Lemma 2.2]. *Let* $q \geq 3$ *be an odd number. Then, for odd numbers* h, q *with* $(h,q)=1$,

$$s_5(h,m,q) = 2s(h,m,1,q) - 4s(\bar{2}h,m,1,q),$$

where $2\bar{2} \equiv 1 \pmod q$.

3. Proofs. In this section, we prove the main results.

***3.1. Proof of Theorem* 1.1.** Let us begin with the identity

$$\sum_{a=1}^{q}{}' \chi(a)S(u,a,q)$$

$$= \sum_{a=1}^{q}{}'\chi(a)\sum_{b=1}^{q}{}' e\left(\frac{ub+a\bar{b}}{q}\right) = \sum_{b=1}^{q}{}' e\left(\frac{ub}{q}\right)\sum_{a=1}^{q}{}'\chi(a)e\left(\frac{a\bar{b}}{q}\right)$$

$$= \tau(\chi) \sum_{b=1}^{q}{}' \chi(b) e\left(\frac{ub}{q}\right) = \overline{\chi}(u)\tau^2(\chi). \tag{3.1}$$

Employing Lemma 2.1 and (3.1), one can write

$$\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' S(u,a,q)\overline{S(u,b,q)} s(a\overline{b}, m, n, q)$$

$$= \frac{-4m!n!}{(2\pi i)^{m+n} q^{m+n-1}} \sum_{d|q} \frac{d^{m+n}}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' S(u,a,q)\overline{S(u,b,q)}\,\overline{\chi}(a\overline{b}) L(m,\chi) L(n,\overline{\chi})$$

$$= \frac{-4m!n!}{(2\pi i)^{m+n} q^{m+n-1}} \sum_{d|q} \frac{d^{m+n}}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \left|\tau\left(\chi\chi_q^0\right)\right|^4 L(m,\chi) L\left(n,\overline{\chi}\right),$$

where χ_q^0 is the principal character modulo q. Note that for an odd square-full number q and any non-primitive character χ modulo q, we have

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right) = 0.$$

From this and the properties of Gauss sums, provided that $d|q$ and $d \neq q$, then $\tau\left(\chi\chi_q^0\right) = 0$; provided that $d = q$ and χ is a primitive character modulo q, then $\left|\tau\left(\chi\chi_q^0\right)\right| = \sqrt{q}$. Thus, gathering these facts and applying equation (2.1) give that

$$\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' S(u,a,q)\overline{S(u,b,q)} s(a\overline{b}, m, n, q)$$

$$= \frac{-4m!n!}{(2\pi i)^{m+n}} \frac{q^3}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} L(m,\chi) L(n,\overline{\chi})$$

$$= \frac{q^3}{\phi(q)} \sum_{l=0}^{m+n} r_{m,n,l} \cdot q^{l-m-n+1} \prod_{p|q} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^l}\right)\left(1 - \frac{1}{p^{l-m-n+1}}\right),$$

which completes the proof.

***3.2. Proof of Theorem* 1.2.** From Lemma 2.3, one has

$$\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' S(u,a,q)\overline{S(u,b,q)} s_5(2a\overline{b}, m, q)$$

$$= \sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' S(u,a,q)\overline{S(u,b,q)} \left\{2s(2a\overline{b}, m, 1, q) - 4s(a\overline{b}, m, 1, q)\right\}$$

$$= 2\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' S(u,a,q)\overline{S(u,b,q)} s(2a\overline{b}, m, 1, q)$$

$$-4\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'S(u,a,q)\overline{S(u,b,q)}s(a\bar{b},m,1,q)=2W_1-4W_2, \tag{3.2}$$

where

$$W_1=\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'S(u,a,q)\overline{S(u,b,q)}s(2a\bar{b},m,1,q)$$

and

$$W_2=\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'S(u,a,q)\overline{S(u,b,q)}s(a\bar{b},m,1,q).$$

Observe that W_2 equals to Theorem 1.1 for $n=1$. So, let us evaluate W_1. By aid of the equations (2.2) and (3.1), and the similar facts given in Theorem 1.1, we have

$$W_1=\frac{-4m!}{(2\pi i)^{m+1}q^m}\sum_{d|q}\frac{d^{m+1}}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'S(u,a,q)\overline{S(u,b,q)}\overline{\chi}(2a\bar{b})L(m,\chi)L(1,\overline{\chi})$$

$$=\frac{-4m!}{(2\pi i)^{m+1}q^m}\sum_{d|q}\frac{d^{m+1}}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|\tau\left(\chi\chi_q^0\right)\right|^4\chi(2)L(m,\chi)L(1,\overline{\chi})$$

$$=\frac{q^3}{\phi(q)}\sum_{l=0}^{m+1}r_{m,1,l}\cdot q^{l-m}\left(\frac{2^l-2^{m+1}-2}{2^m+2}\right)\prod_{p|q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m}}\right).$$

Combine the expressions W_1 and W_2 in (3.2) to complete the proof.

***3.3. Proof of Theorem* 1.3.** In the light of the relation between generalized Hardy sums and generalized Dedekind sum, given in [17, Proposition 1.1] as

$$s_3(h,n,q)=2s(h,1,n,q)-4s(2h,1,n,q),\quad \text{if } q \text{ and } n \text{ is odd number,}$$

the proof follows from the similar steps as the proof of Theorem 1.2, so we omit it.

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