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QUATERNIONIC DAVIS – WIELANDT SHELL IN A RIGHT QUATERNIONIC HILBERT SPACE

КВАТЕРНІОННА ОБОЛОНКА ДЕВІСА – ВІЛАНДТА В ПРАВОМУ КВАТЕРНІОННОМУ ГІЛЬБЕРТОВОМУ ПРОСТОРІ

We derive some results concerning the quaternionic Davis – Wielandt shell for a bounded right linear operator in a right quaternionic Hilbert space. The relations between the geometric properties of the quaternionic Davis – Wielandt shells and the algebraic properties of quaternionic operators are obtained.

Отримано деякі результати щодо кватерніонної оболонки Девіса – Віландта для обмеженого правого лінійного оператора в правому кватерніонному гільбертовому просторі. Виведено співвідношення між геометричними властивостями кватерніонних оболонок Девіса – Віландта та алгебраїчними властивостями кватерніонних операторів.

1. Introduction. Let us first establish the relevant notations and terminologies to be used throughout the article. As usual, let $\mathbb{C}$ and $\mathbb{R}$ denote the fields of the complex and real numbers, respectively. Let $\mathbb{H}$ be a four-dimensional vector space over $\mathbb{R}$ with an ordered basis, denoted by $\{1, i, j, k\}$ and $\mathbb{H}^*$ the group (under quaternionic multiplication) of all invertible quaternions. A general quaternion can be written as

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1 i + \mathbf{q}_2 j + \mathbf{q}_3 k \quad \text{for all} \quad \mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \mathbb{R},$$

where i, j, k are the three quaternionic imaginary units, satisfying

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = k = -ji, \qquad jk = i = -kj, \qquad ki = j = -ik.$$

For a given $\mathbf{q} \in \mathbb{H}$, we define the real part, $\mathrm{Re}(\mathbf{q}) := \mathbf{q}_0$ and the imaginary part, $\mathrm{Im}(\mathbf{q}) := \mathbf{q}_1 i + \mathbf{q}_2 j + \mathbf{q}_3 k$. The quaternionic conjugate of $\mathbf{q}$ is

$$\overline{\mathbf{q}} = \mathbf{q}_0 - i\mathbf{q}_1 - j\mathbf{q}_2 - k\mathbf{q}_3,$$

while $|\mathbf{q}| = \sqrt{\mathbf{q}\overline{\mathbf{q}}}$ denotes the usual norm of the quaternion $\mathbf{q}$. If $\mathbf{q}$ is a non-zero element, it has inverse

$$\mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}.$$

For every $\mathbf{q} \in \mathbb{H}$, define

$$\mathbf{p} \sim \mathbf{q} \quad \text{if and only if} \quad p = \beta^{-1} q \beta \quad \text{for some} \quad \beta \in \mathbb{H}\backslash\{0\}.$$

It is an equivalence relation on $\mathbb{H}$. The equivalence class of $\mathbf{q}$, denoted by $[\mathbf{q}]$, is given by

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$$[\mathbf{q}] = \Big\{\mathbf{p} \in \mathbb{H}\colon \ \mathrm{Re}(\mathbf{q}) = \mathrm{Re}(\mathbf{p}), \big|\,\mathrm{Im}(\mathbf{q})\big| = \big|\,\mathrm{Im}(\mathbf{p})\big|\Big\}.$$

The set of all imaginary unit quaternions, denoted by $\mathbf{S}$, is defined as

$$\mathbf{S} := \big\{\mathbf{q} \in \mathbb{H}\colon \overline{\mathbf{q}} = -\mathbf{q} \quad \text{and } |\mathbf{q}| = 1\big\} = \big\{\mathbf{q} \in \mathbb{H}\colon \mathbf{q}^2 = -1\big\}.$$

Let $V_{\mathbb{H}}^R$ be a vector space under right multiplication by quaternions. For $\phi, \psi, \omega \in V_{\mathbb{H}}^R$ and $q \in \mathbb{H}$, the inner product $\langle\cdot,\cdot\rangle\colon V_{\mathbb{H}}^R \times V_{\mathbb{H}}^R \longrightarrow \mathbb{H}$ satisfies the following properties:

(i) $\overline{\langle\phi,\psi\rangle} = \langle\psi,\phi\rangle$,

(ii) $\|\phi\|^2 = \langle\phi,\phi\rangle > 0$ unless $\phi = 0$, a real norm,

(iii) $\langle\phi,\omega+\psi\rangle = \langle\phi,\omega\rangle + \langle\phi,\psi\rangle$,

(iv) $\langle\phi,\psi\mathbf{q}\rangle = \langle\phi,\psi\rangle\mathbf{q}$,

(v) $\langle\phi\mathbf{q},\psi\rangle = \overline{\mathbf{q}}\langle\phi,\psi\rangle$, where $\overline{\mathbf{q}}$ stands for the quaternionic conjugate.

Let $V_{\mathbb{H}}^R$ be a right quaternionic Hilbert space. A right $\mathbb{H}$-linear operator, for simplicity, right linear operator, is a map $T\colon \mathcal{D}(T) \subseteq V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ such that

$$T(\phi a + \psi b) = (T\phi)a + (T\psi)b, \quad \text{if} \quad \phi,\psi \in \mathcal{D}(T) \quad \text{and} \quad a,b \in \mathbb{H},$$

where the domain $\mathcal{D}(T)$ of T is a right $\mathbb{H}$-linear subspace of $V_{\mathbb{H}}^R$. A right linear operator $T\colon V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ is said to be bounded if there exists $C \geq 0$ such that

$$\|T\phi\| \leq C\|\phi\| \quad \text{for all} \quad \phi \in \mathcal{D}(T).$$

As in the complex case, if $T\colon \mathcal{D}(T) \subseteq V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ is any right linear operator, one defines $\|T\|$ by setting

$$\|T\|_{\mathcal{D}(T)} = \sup_{\phi\in\mathcal{D}(T)\setminus\{0\}} \frac{\|T\phi\|}{\|\phi\|} = \inf\Big\{C \in \mathbb{R}\colon \|T\phi\| \leq C\|\phi\| \text{ for all } \phi \in \mathcal{D}(T)\Big\}.$$

Denote by $\mathcal{B}(V_{\mathbb{H}}^R)$ the set of all bounded right linear operators of $V_{\mathbb{H}}^R$:

$$\mathcal{B}(V_{\mathbb{H}}^R) = \Big\{T\colon V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R \text{ right linear operator: } \|T\| < +\infty\Big\}.$$

It is immediate to verify that, if T and S are right linear operators in $\mathcal{B}\big(V_{\mathbb{H}}^R\big)$, then the same is true for $T+S$ and TS, and it holds:

$$\|T+S\| \leq \|T\| + \|S\| \ \text{ and } \quad \|TS\| \leq \|T\|\|S\|.$$

We define the natural domains of the sum $T+S$ and of the composition TS by setting

$$\mathcal{D}(T+S) = \mathcal{D}(T)\cap\mathcal{D}(S) \quad \text{and} \quad \mathcal{D}(TS) = \{\phi \in \mathcal{D}(S)\colon S\phi \in \mathcal{D}(T)\}.$$

The adjoint $T^*\colon D(T^*) \longrightarrow V_{\mathbb{H}}^R$ of T is the unique right operator with the following properties:

$$\mathcal{D}(T^*) = \Big\{\psi \in V_{\mathbb{H}}^R\colon \exists\ \varphi \quad \text{such that} \quad \langle\psi, T\phi\rangle = \langle\varphi,\phi\rangle\Big\}$$

and

$$\langle\psi, T\phi\rangle = \langle T^*\psi,\phi\rangle \quad \text{for all} \quad \phi \in \mathcal{D}(T),\ \psi \in \mathcal{D}(T^*).$$

Here we list out some of the properties of quaternions, which we need later. For every $\phi,\psi \in V_{\mathbb{H}}^R$ and $\mathbf{q},\mathbf{p} \in \mathbb{H}$, we have

(i) $\mathbf{q}(\phi+\psi)=\mathbf{q}\phi+\mathbf{q}\psi$ and $\mathbf{q}(\phi\mathbf{p})=(\mathbf{q}\phi)\mathbf{p}$,
(ii) $\|\mathbf{q}\phi\|=|\mathbf{q}|\|\phi\|$,
(iii) $\mathbf{q}(\mathbf{p}\phi)=(\mathbf{q}\mathbf{p})\phi$,
(iv) $\langle\overline{\mathbf{q}}\phi,\psi\rangle=\langle\phi,\mathbf{q}\psi\rangle$,
(v) $r\phi=\phi r$ for all $r\in\mathbb{R}$,
(vi) $\mathbf{q}\varphi_k=\varphi_k\mathbf{q}$ for all $k\in\mathbb{N}$,

One of the main obstacle to develop a spectral theory of quaternionic linear operators was the lack of a precise notion of quaternionic spectrum. This fact had consequences on the precise formulation and on the proof of the spectral theorem for quaternionic linear operators. As the authors mention in 1936, Birkhoff and von Neumann showed that quantum mechanics can be formulated only on real, complex and quaternionic numbers. So since that time started an increasing interest for the quaternionic spectral theory and the theory of quaternionic groups.

Only in 2006 F. Colombo and I. Sabadini introduced the notion of S-spectrum, S-resolvent operator that allowed to fully develop quaternionic operator theory. The spectral theorem based on the S-spectrum took several other years and in 2016, D. Alpay, F. Colombo, D. P. Kimsey (see [1]) gave a full proof of this fundamental theorem for both bounded and unbounded operators.

The S-functional calculus, and in general the spectral theory on the S-spectrum, started its development only in 2006. The discovery of the S-spectrum and of the S-functional calculus is well explained in the introduction of the book [6] with a complete list of the references and it is also described how hypercomplex analysis methods were used to identify the appropriate notion of quaternionic spectrum whose existence was suggested by quaternionic quantum mechanics.

Let $T\in\mathcal{B}(V_{\mathbb{H}}^R)$. Define $R_{\mathbf{q}}(T)=T^2-2\operatorname{Re}(\mathbf{q})T+|\mathbf{q}|^2I_{V_{\mathbb{H}}^R}$ for $\mathbf{q}\in\mathbb{H}$. Then the S-spectrum of T, denoted by $\sigma^S(T)$, defined as

$$\sigma^S(T):=\Big\{\mathbf{q}\in\mathbb{H}\colon R_{\mathbf{q}}(T)\text{ has no inverse in }\ \mathcal{B}(V_{\mathbb{H}}^R)\Big\}.$$

The S-point spectrum is defined as

$$\sigma_{\mathbf{q}}^S(T):=\Big\{\mathbf{q}\in\mathbb{H}\colon R_{\mathbf{q}}(T)\text{ is not one-to-one}\Big\}.$$

The approximate S-point spectrum of T, denoted by $\sigma_{ap}^S(T)$, is defined as

$$\sigma_{ap}^S(T)=\Big\{\mathbf{q}\in\mathbb{H}\colon\exists(\phi_n)\text{ such that }\ \|\phi_n\|=1\text{ and }\lim_{n\to+\infty}\|R_{\mathbf{q}}(T)\phi_n\|=0\Big\}.$$

All the above mentioned material can be found in [4 – 6].

Motivated by theoretical study and applications, researchers considered different generalizations of the quaternionic numerical range. One of these generalizations is the quaternionic Davis – Wielandt shell of a right linear operator T. For a given right linear operator $T\colon V_{\mathbb{H}}^R\longrightarrow V_{\mathbb{H}}^R$. The quaternionic Davis – Wielandt shell of T, denoted by $\mathcal{DW}_{\mathbb{H}}(T)$, is defined as

$$\mathcal{DW}_{\mathbb{H}}(T)=\Big\{(\mathbf{q},\mathbf{s})=(\langle T\phi,\phi\rangle_{\mathbb{H}},\langle T\phi,T\phi\rangle_{\mathbb{H}})\colon\phi\in V_{\mathbb{H}}^R,\|\phi\|=1\Big\}\subseteq\mathbb{H}\times\mathbb{R}.$$

Also, the quaternionic Davis – Wielandt radius of $T\in\mathcal{B}\big(V_{\mathbb{H}}^R\big)$ is defined as

$$d\omega_{\mathbb{H}}(T)=\sup\Big\{\sqrt{|\langle T\phi,\phi\rangle_{\mathbb{H}}|^2+\|T\phi\|^4}\Big\}.$$

Evidently, the projection of the set $\mathcal{DW}(T)$ on the first coordinate is the quaternionic numerical range of $T \in \mathcal{B}(V_{\mathbb{H}}^R)$, denoted by $\mathcal{W}_{\mathbb{H}}(T)$, is defined as

$$\mathcal{W}_{\mathbb{H}}(T) = \Big\{\mathbf{q} = \langle T\phi, \phi\rangle_{\mathbb{H}} : \phi \in V_{\mathbb{H}}^R, \|\phi\| = 1\Big\} \subseteq \mathbb{H}.$$

Since $\mathcal{W}_{\mathbb{H}}(T)$ is the image of $\mathcal{DW}_{\mathbb{H}}(T)$ under the projection $(\mathbf{q}, \mathbf{s}) \to \mathbf{q}$, one expects that $\mathcal{DW}_{\mathbb{H}}(T)$ can tell and gives us more information about $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ than $\mathcal{W}_{\mathbb{H}}(T)$.

It is known that there exist a close relationship between the quaternionic Davis – Wielandt shell of T and the family of $\mathbb{H}$-joint numerical ranges $\mathcal{JW}_{\mathbb{H}}(T, T^*T)$ of $T \in \mathcal{B}(V_{\mathbb{H}}^R)$. The latter is defined by

$$\mathcal{JW}_{\mathbb{H}}(T, T^*T) = \Big\{(\mathbf{q}, \mathbf{s}) = (\langle T\phi, \phi\rangle_{\mathbb{H}}, \langle T^*T\phi, \phi\rangle_{\mathbb{H}}) : \phi \in V_{\mathbb{H}}^R, \|\phi\| = 1\Big\} \subseteq \mathbb{H} \times \mathbb{R}.$$

Now, let $T = A + iB$ such that $A = A^*$ and $B = B^*$. Then $\mathcal{DW}_{\mathbb{H}}(T)$ can be identified with the $\mathbb{H}$-joint numerical range. Identifying $\mathbb{H} \times \mathbb{R}$ with $\mathbb{R}^5$, we have

$$\mathcal{JW}_{\mathbb{H}}(T) = \Big\{(\langle A\phi, \phi\rangle_{\mathbb{H}}, \langle B\phi, \phi\rangle_{\mathbb{H}}, \langle T^*T\phi, \phi\rangle_{\mathbb{H}}) : \phi \in V_{\mathbb{H}}^R, \|\phi\| = 1\Big\} \subseteq \mathbb{H} \times \mathbb{R},$$

which is a $\mathbb{H}$-joint numerical range of self-adjoint operators A, B and T^*T. For more results on numerical ranges, joint numerical ranges and Davis – Wielandt shells in the complex case the reader is refereed to, e.g., [2, 3, 7 – 12].

The purpose of this paper is to develop and study corresponding results for the quaternionic Davis – Wielandt shell $\mathcal{DW}_{\mathbb{H}}(T)$ for a bounded right linear operators $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ and some related geometric and analytic properties. Moreover, we study the relation between the geometrical properties of $\mathcal{DW}_{\mathbb{H}}(T)$ and the algebraic properties of bounded right linear operators $T \in \mathcal{B}(V_{\mathbb{H}}^R)$. Complete descriptions are obtained for the Davis – Wielandt shells of several classes of quaternion operators. The principal objective of the present expository article, therefore, is to unite and to reflect upon some related issues that are crucial to the study of linear algebra over the quaternions, for example, quaternionic numerical range, quaternionic Davis – Wielandt shell, and $\mathbb{H}$-joint numerical ranges.

This paper is organized as follows. In the first section, we give necessary details of quaternionic Hilbert spaces and right linear operators on such spaces. In the second section, we establish more results showing that the quaternionic Davis – Wielandt shell is useful in studying quaternion operators. Also, we present some basic results for the quaternionic Davis – Wielandt shell of a bounded right linear operator in a right quaternionic Hilbert space.

2. The Davis – Wielandt shell and their properties. In this section, we define the quaternionic Davis – Wielandt shell for quaternionic operators in a right quaternionic Hilbert space. We begin with the following definition.

Definition 2.1. *Let $T \in \mathcal{B}(V_{\mathbb{H}}^R)$. The quaternionic Davis – Wielandt shell of $T \in \mathcal{B}(V_{\mathbb{H}}^R)$, denoted by $\mathcal{DW}_{\mathbb{H}}(T)$, is defined as*

$$\mathcal{DW}_{\mathbb{H}}(T) = \Big\{(\mathbf{q}, \mathbf{s}) = (\langle T\phi, \phi\rangle_{\mathbb{H}}, \langle T\phi, T\phi\rangle_{\mathbb{H}}) : \phi \in V_{\mathbb{H}}^R, \|\phi\| = 1\Big\},$$

and the quaternionic Davis – Wielandt radius of $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ is computable with the formula

$$d\omega_{\mathbb{H}}(T) = \sup\Big\{\sqrt{|\langle T\phi, \phi\rangle_{\mathbb{H}}|^2 + \|T\phi\|^4}\Big\}.$$

***Example* 2.1.** Let $\mathbf{q} \in \mathbb{H}$ and $T = \begin{pmatrix} \mathbf{q} & 0 \\ 0 & -\mathbf{q} \end{pmatrix}$. Then

$$\mathcal{DW}_{\mathbb{H}}(T) = \big\{(0, |\mathbf{q}|^2) : \mathbf{q} \in \mathbb{H}\big\}.$$

Indeed, for $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, where $\phi_1, \phi_2 \in V_{\mathbb{H}}^R$, we have the first coordinate of the shell as

$$\langle T\phi, \phi\rangle_{\mathbb{H}} = \left\langle \begin{pmatrix} \mathbf{q} & 0 \\ 0 & -\mathbf{q} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\rangle_{\mathbb{H}} = 0.$$

As for the second coordinate, we get

$$\langle T\phi, T\phi\rangle_{\mathbb{H}} = \left\langle \begin{pmatrix} \mathbf{q} & 0 \\ 0 & -\mathbf{q} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \mathbf{q} & 0 \\ 0 & -\mathbf{q} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\rangle_{\mathbb{H}} = |\mathbf{q}|^2.$$

Then

$$\mathcal{DW}_{\mathbb{H}}(T) = \big\{(0, |\mathbf{q}|^2) : \mathbf{q} \in \mathbb{H}\big\}.$$

***Example* 2.2.** Let $\mathbf{q} \in \mathbb{H}$ and $T = \begin{pmatrix} \mathbf{q} & 0 \\ 0 & T_1 - \mathbf{q} \end{pmatrix}$. Then

$$\mathcal{DW}_{\mathbb{H}}(T) = \mathcal{DW}_{\mathbb{H}}(T_1).$$

Indeed, for $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, where $\phi_1, \phi_2 \in V_{\mathbb{H}}^R$, we have the first coordinate of the shell as

$$\langle T\phi, \phi\rangle_{\mathbb{H}} = \left\langle \begin{pmatrix} \mathbf{q} & 0 \\ 0 & T_1 - \mathbf{q} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\rangle_{\mathbb{H}} = \langle T_1\phi, \phi\rangle_{\mathbb{H}}.$$

As for the second coordinate, we obtain

$$\langle T\phi, T\phi\rangle_{\mathbb{H}} = \left\langle \begin{pmatrix} \mathbf{q} & 0 \\ 0 & T_1 - \mathbf{q} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \mathbf{q} & 0 \\ 0 & T_1 - \mathbf{q} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\rangle_{\mathbb{H}} = \langle T_1\phi, T_1\phi\rangle_{\mathbb{H}}.$$

Theorem 2.1. *Let* $T \in \mathcal{B}(V_{\mathbb{H}}^R)$, $(\mathbf{q}_1, \mathbf{s}_1) \in \mathcal{DW}_{\mathbb{H}}(T)$ *and* $(\mathbf{q}_2, \mathbf{s}_2) \in \mathbb{H}^2$ *such that* $\big(\mathrm{Re}(\mathbf{q}_1), \mathrm{Re}(\mathbf{s}_1)\big) = \big(\mathrm{Re}(\mathbf{q}_2), \mathrm{Re}(\mathbf{s}_2)\big)$ *and* $\big(|\mathbf{q}_1|, |\mathbf{s}_1|\big) = \big(|\mathbf{q}_2|, |\mathbf{s}_2|\big)$. *Then*

$$(\mathbf{q}_2, \mathbf{s}_2) \in \mathcal{DW}_{\mathbb{H}}(T).$$

Proof. Let $(\mathbf{q}_1, \mathbf{s}_1) \in \mathcal{DW}_{\mathbb{H}}(T)$. Then there exists $\phi \in V_{\mathbb{H}}^R$, $\|\phi\| = 1$, such that

$$\mathbf{q}_1 = \langle T\phi, \phi\rangle_{\mathbb{H}} \quad \text{and} \quad \mathbf{s}_1 = \langle T\phi, T\phi\rangle_{\mathbb{H}}.$$

Now, let $(\mathbf{q}_2, \mathbf{s}_2) \in \mathbb{H}^2$ such that

$$\big(\mathrm{Re}(\mathbf{q}_1), \mathrm{Re}(\mathbf{s}_1)\big) = \big(\mathrm{Re}(\mathbf{q}_2), \mathrm{Re}(\mathbf{s}_2)\big) \quad \text{and} \quad \big(|\mathbf{q}_1|, |\mathbf{s}_1|\big) = \big(|\mathbf{q}_2|, |\mathbf{s}_2|\big).$$

Then we have $\mathbf{q}_2 = \alpha^* \mathbf{q}_1 \alpha$ and $\mathbf{s}_2 = \alpha^* \mathbf{s}_1 \alpha$ for some $\alpha \in \mathbb{H}, |\alpha| = 1$, so

$$\mathbf{q}_2 = \alpha^* \langle T\phi, \phi\rangle_{\mathbb{H}} \alpha \quad \text{and} \quad \mathbf{s}_2 = \alpha^* \langle T\phi, T\phi\rangle_{\mathbb{H}} \alpha,$$

which yields that

$$(\mathbf{q}_2, \mathbf{s}_2) \in \mathcal{DW}_{\mathbb{H}}(T).$$

Theorem 2.2. *Let* $T \in \mathcal{B}(V_{\mathbb{H}}^R)$. *Then* $(\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T)$ *if and only if* $([\mathbf{q}], [\mathbf{s}]) \in \mathcal{DW}_{\mathbb{H}}(T)$.

Proof. Let $(\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T)$. Then there exists $\phi \in V_{\mathbb{H}}^R$ such that

$$\mathbf{q} = \langle T\phi, \phi\rangle_{\mathbb{H}} \quad \text{and} \quad \mathbf{s} \in \langle T\phi, T\phi\rangle_{\mathbb{H}}.$$

Hence, for every $\mathbf{p} \in \mathbb{H}\backslash\{0\}$, we have

$$\mathbf{pqp^{-1}} = \mathbf{p}\langle T\phi, \phi\rangle_{\mathbb{H}}\mathbf{p^{-1}} = \frac{\mathbf{p}}{|\mathbf{p}|}\langle T\phi, \phi\rangle_{\mathbb{H}}\frac{\overline{\mathbf{p}}}{|\mathbf{p}|} = \left\langle T\phi\frac{\overline{\mathbf{p}}}{|\mathbf{p}|}, \phi\frac{\overline{\mathbf{p}}}{|\mathbf{p}|}\right\rangle_{\mathbb{H}}.$$

Also, we obtain

$$\mathbf{psp^{-1}} = \mathbf{p}\langle T\psi, T\phi\rangle_{\mathbb{H}}\mathbf{p^{-1}} = \frac{\mathbf{p}}{|\mathbf{p}|}\langle T\phi, T\phi\rangle_{\mathbb{H}}\frac{\overline{\mathbf{p}}}{|\mathbf{p}|} = \left\langle T\phi\frac{\overline{\mathbf{p}}}{|\mathbf{p}|}, T\phi\frac{\overline{\mathbf{p}}}{|\mathbf{p}|}\right\rangle_{\mathbb{H}}.$$

Now, if we take $\psi = \phi\dfrac{\overline{\mathbf{p}}}{|\mathbf{p}|}$, then $\|\psi\| = 1$, $\mathbf{pqp^{-1}} = \langle T\psi, \psi\rangle_{\mathbb{H}}$ and $\mathbf{psp^{-1}} = \langle T\psi, T\psi\rangle_{\mathbb{H}}$. This implies that $(\mathbf{pqp^{-1}}, \mathbf{psp^{-1}}) \in \mathcal{DW}_{\mathbb{H}}(T)$. Thus, $\big([\mathbf{q}], [\mathbf{s}]\big) \in \mathcal{DW}_{\mathbb{H}}(T)$.

We next have a relation for the quaternionic Davis – Wielandt shell of sum of two right linear operators.

Theorem 2.3. *Let* $T, S \in \mathcal{B}\big(V_{\mathbb{H}}^R\big)$. *Then*

$$\mathcal{DW}_{\mathbb{H}}(S + T) \subseteq \mathcal{DW}_{\mathbb{H}}(S) + \mathcal{DW}_{\mathbb{H}}(T) + \mathcal{O},$$

where $\mathcal{O} = \big\{(0, \langle(S^*T + T^*S)\phi, \phi\rangle_{\mathbb{H}}) : \phi \in V_{\mathbb{H}}^R, \|\phi\| = 1\big\}$.

Proof. From the definition of the quaternionic Davis – Wielandt shell, we get

$$\begin{aligned}\mathcal{DW}_{\mathbb{H}}(S + T) &= \Big\{\big(\langle S + T\phi, \phi\rangle_{\mathbb{H}}, \langle S + T\phi, S + T\phi\rangle_{\mathbb{H}}\big) : \phi \in V_{\mathbb{H}}^R, \|\phi\| = 1\Big\} \\ &= \Big\{\big(\langle S\phi, \phi\rangle_{\mathbb{H}}, \langle S\phi, S\phi\rangle_{\mathbb{H}}\big) + \big(\langle T\phi, \phi\rangle_{\mathbb{H}}, \langle T\phi, T\phi\rangle_{\mathbb{H}}\big) \\ &\qquad + \big(0, \langle(S^*T + T^*S)\phi, \phi\rangle_{\mathbb{H}}\big) : \phi \in V_{\mathbb{H}}^R, \|\phi\| = 1\Big\}.\end{aligned}$$

Hence,

$$\mathcal{DW}_{\mathbb{H}}(S + T) \subseteq \mathcal{DW}_{\mathbb{H}}(S) + \mathcal{DW}_{\mathbb{H}}(T) + \mathcal{O},$$

where $\mathcal{O} = \big\{(0, \langle(S^*T + T^*S)\phi, \phi\rangle_{\mathbb{H}}) : \phi \in V_{\mathbb{H}}^R, \|\phi\| = 1\big\}$. This implies the required inequality of the theorem.

***Corollary* 2.1.** *Let* $S, T \in \mathcal{B}(V_{\mathbb{H}}^R)$ *such that* $S^*T + T^*S = 0$. *Then*

$$\mathcal{DW}_{\mathbb{H}}(S + T) \subseteq \mathcal{DW}_{\mathbb{H}}(S) + \mathcal{DW}_{\mathbb{H}}(T).$$

Some algebraic properties of the numerical ranges are worth noticing.

Theorem 2.4. *Let* $T \in \mathcal{B}(V_{\mathbb{H}}^R)$. *Then*:

(i) *for every unitary operator* $U \in \mathcal{B}(V_{\mathbb{H}}^R)$, *we have*

$$\mathcal{DW}_{\mathbb{H}}(U^*TU) = \mathcal{DW}_{\mathbb{H}}(T);$$

(ii) *for every* $\alpha, \beta \in \mathbb{H}$, *we get*

$$\mathcal{DW}_{\mathbb{H}}\big(\alpha T + \beta \mathbb{I}_{\mathbb{H}}^{R}\big) = \Big\{\big(\overline{\alpha}\mathbf{q} + \overline{\beta}, |\alpha|^2\mathbf{s} + 2\,\mathrm{Re}(\overline{\alpha}\mathbf{q}\beta) + |\beta|^2\big) : (\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T)\Big\};$$

(iii) $\mathcal{DW}_{\mathbb{H}}(T^*) = \big\{(\overline{\mathbf{q}}, \mathbf{s}) : (\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T)\big\}$.

Proof. (i) Let $(\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(U^*TU)$. Then there exists unit vector $\phi \in V_{\mathbb{H}}^{R}$ such that

$$\mathbf{q} = \langle U^*TU\phi, \phi\rangle_{\mathbb{H}} \quad \text{and} \quad \mathbf{s} = \langle U^*TU\phi, U^*TU\phi\rangle_{\mathbb{H}}.$$

Let $\psi = U\phi$, then

$$\mathbf{q} = \langle T\psi, \psi\rangle_{\mathbb{H}} \quad \text{and} \quad \mathbf{s} = \langle T\psi, T\psi\rangle_{\mathbb{H}}.$$

Hence,

$$(\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T).$$

(ii) Let $(\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T)$. Then there exists unit vector $\phi \in V_{\mathbb{H}}^{R}$ such that

$$\mathbf{q} = \langle T\phi, \phi\rangle_{\mathbb{H}} \quad \text{and} \quad \mathbf{s} = \langle T\phi, T\phi\rangle_{\mathbb{H}}.$$

Next, let $\alpha, \beta \in \mathbb{H}$, then we have

$$\big\langle(\alpha T + \beta \mathbb{I}_{\mathbb{H}}^{R})\phi, \phi\big\rangle_{\mathbb{H}} = \langle \alpha T\phi, \phi\rangle_{\mathbb{H}} + \langle \beta \mathbb{I}_{\mathbb{H}}^{R}\phi, \phi\rangle_{\mathbb{H}} = \overline{\alpha}\mathbf{q} + \overline{\beta},$$

$$\big\langle(\alpha T + \beta \mathbb{I}_{\mathbb{H}}^{R}\phi, \alpha T + \beta \mathbb{I}_{\mathbb{H}}^{R}\phi\big\rangle_{\mathbb{H}} = \langle \alpha T\phi, \alpha T\phi\rangle_{\mathbb{H}} + 2\,\mathrm{Re}\left(\overline{\alpha}\langle T\phi, \phi\rangle_{\mathbb{H}}\beta\right) + \langle \beta \mathbb{I}_{\mathbb{H}}^{R}\phi, \beta \mathbb{I}_{\mathbb{H}}^{R}\phi\rangle_{\mathbb{H}}$$
$$= |\alpha|^2\mathbf{s} + 2\,\mathrm{Re}(\overline{\alpha}\mathbf{q}\beta) + |\beta|^2.$$

Hence,

$$\mathcal{DW}_{\mathbb{H}}(\alpha T + \beta \mathbb{I}_{\mathbb{H}}^{R}) = \Big\{\big(\overline{\alpha}\mathbf{q} + \overline{\beta}, |\alpha|^2\mathbf{s} + 2\,\mathrm{Re}(\overline{\alpha}\mathbf{q}\beta) + |\beta|^2\big) : (\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T)\Big\}.$$

(iii) Let $(\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T)$. Then there exists unit vector $\phi \in V_{\mathbb{H}}^{R}$ such that

$$\mathbf{q} = \langle T\phi, \phi\rangle_{\mathbb{H}} \quad \text{and} \quad \mathbf{s} = \langle T\phi, T\phi\rangle_{\mathbb{H}}.$$

Since

$$\langle T^*\phi, \phi\rangle_{\mathbb{H}} = \overline{\langle T\phi, \phi\rangle_{\mathbb{H}}} = \overline{\mathbf{q}},$$
$$\langle T^*\phi, T^*\phi\rangle_{\mathbb{H}} = \|T^*\phi\|^2 = \|T\phi\|^2 = \langle T\phi, T\phi\rangle_{\mathbb{H}} = \mathbf{s},$$

then

$$\mathcal{DW}_{\mathbb{H}}(T^*) = \big\{(\overline{\mathbf{q}}, \mathbf{s}) : (\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T)\big\}.$$

***Example* 2.3.** Let $\mathbf{q} \in \mathbb{H}$, $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ with $A, B \in \mathcal{B}\big(V_{\mathbb{H}}^{R}\big)$ and $U = \begin{pmatrix} \mathbb{I}_{\mathbb{H}}^{R} & 0 \\ 0 & e^{i\frac{\theta}{2}}\mathbb{I}_{\mathbb{H}}^{R} \end{pmatrix}$ for every $\theta \in \mathbb{R}$. Then

$$\mathcal{DW}_{\mathbb{H}}(U^*TU) = \mathcal{DW}_{\mathbb{H}}(T).$$

By using Theorem 2.4 (i), we get

$$\mathcal{DW}_{\mathbb{H}}\left(\begin{pmatrix} 0 & A \\ e^{i\theta}B & 0 \end{pmatrix}\right) = \mathcal{DW}_{\mathbb{H}}\left(U^*\begin{pmatrix} 0 & A \\ e^{i\theta}B & 0 \end{pmatrix}U\right) = \mathcal{DW}_{\mathbb{H}}\left(\begin{pmatrix} 0 & e^{i\frac{\theta}{2}}A \\ e^{i\frac{\theta}{2}}B & 0 \end{pmatrix}\right)$$

$$= \mathcal{DW}_{\mathbb{H}}\left(\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right).$$

Theorem 2.5. *Let* $T \in \mathcal{B}(V_{\mathbb{H}}^R)$. *Then*:

(i) $(\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T)$ *if and only if there exists an orthonormal pairs of vectors* $\phi, \varphi \in V_{\mathbb{H}}^R$ *such that*

$$T\phi = \mathbf{q}\phi + \sqrt{\mathbf{s} - |\mathbf{q}|^2}\varphi;$$

(ii) *the set* $\mathcal{DW}_{\mathbb{H}}(T)$ *is bounded*; *in particular*, $\mathcal{DW}_{\mathbb{H}}(T) \subseteq \mathcal{S}(T)$ *with*

$$\mathcal{S}(T) = \left\{(\mathbf{q}, \mathbf{s}) \in \mathbb{H} \times \mathbb{R}^+ : |\mathbf{q}|^2 \leq \mathbf{s} \leq \|T\|^2\right\}.$$

Proof. The “only if” parts are obvious. For the "if" part of (i), let $(\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T)$. Then there exists unit vector $\phi \in V_{\mathbb{H}}^R$ such that

$$\mathbf{q} = \langle T\phi, \phi\rangle_{\mathbb{H}} \quad \text{and} \quad \mathbf{s} = \langle T\phi, T\phi\rangle_{\mathbb{H}}.$$

Thus, $T\phi = \mathbf{q}\phi + \omega\varphi$ for some unit vector $\varphi \in V_{\mathbb{H}}^R$ with $\langle \phi, \varphi\rangle_{\mathbb{H}} = 0$ and $\omega \geq 0$. Now, we have

$$\mathbf{s} = \langle T\phi, T\phi\rangle_{\mathbb{H}} = \|T\phi\|^2 = |\mathbf{q}|^2 + \omega^2,$$

therefore,

$$T\phi = \mathbf{q}\phi + \sqrt{\mathbf{s} - |\mathbf{q}|^2}\varphi.$$

(ii) Let $(\mathbf{q}, \mathbf{s}) \in \mathcal{DW}_{\mathbb{H}}(T)$. From (i), there exists unit vector $\phi \in V_{\mathbb{H}}^R$ such that

$$\mathbf{q} = \langle T\phi, \phi\rangle_{\mathbb{H}} \quad \text{and} \quad \mathbf{s} = \langle T\phi, T\phi\rangle_{\mathbb{H}} = \|T\phi\|^2 = |\mathbf{q}|^2 + \omega^2.$$

Then $\mathbf{s} \leq \|T\|^2$. Also, we have $|\mathbf{q}|^2 \leq \mathbf{s}$. Thus, $\mathcal{DW}_{\mathbb{H}}(T) \subseteq \mathcal{S}(T)$.

Theorem 2.6. *Let* $T \in \mathcal{B}(V_{\mathbb{H}}^R)$. *Then*:

(i) $\mathcal{DW}_{\mathbb{H}}(T) = \left\{(\overline{\mathbf{q}}, |\mathbf{q}|^2) : \mathbf{q} \in \mathbb{H}\right\}$ *if and only if* $T = \mathbf{q}\mathbb{I}_{\mathbb{H}}^R$;

(ii) T *is self-adjoint if and only if* $\mathcal{DW}_{\mathbb{H}}(T) \subseteq \mathbb{R} \times \mathbb{R}$;

(iii) T *is an isometry if and only if* $\mathcal{DW}_{\mathbb{H}}(T) = \left\{(\mathbf{q}, 1) : \mathbf{q} \in \mathcal{W}_{\mathbb{H}}(T)\right\}$.

Proof. The statement (i) is trivial.

(ii) Using the fact that T is self-adjoint, then $\langle T\phi, \phi\rangle_{\mathbb{H}} \in \mathbb{R}$. Hence, our claim follows.

(iii) Let $T \in \mathcal{B}\big(V_{\mathbb{H}}^R\big)$ is an isometry. The first assertion follows from the fact that the projection of $\mathcal{DW}_{\mathbb{H}}(T)$ to its first coordinate equals $\mathcal{W}_{\mathbb{H}}(T)$ and $\|T\phi\| = \|\phi\| = 1$ for all unit vectors $\phi \in V_{\mathbb{H}}^R$.

If $T\phi = \phi\mathbf{q}$ for some $q \in \mathbb{H}$ and $\phi \in V_{\mathbb{H}}^R \backslash \{0\}$, then ϕ is called an eigenvector of T with right eigenvalue q. Then, the set of all eigenvalues of T coincides with $\sigma_{\mathbf{q}}^S(T)$.

***Corollary* 2.2.** *Let* $T \in \mathcal{B}(V_{\mathbb{H}}^R)$. *Then*

$$\mathcal{DW}_{\mathbb{H}}(T) = \left\{(\overline{\mathbf{q}}, |\mathbf{q}|^2) : \mathbf{q} \in \sigma_{\mathbf{q}}^S(T)\right\}.$$

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