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PERIODIC AND WEAKLY PERIODIC GROUND STATES CORRESPONDING TO THE SUBGROUPS OF INDEX THREE FOR THE ISING MODEL ON THE CAYLEY TREE OF ORDER THREE

ПЕРІОДИЧНІ ТА СЛАБКО ПЕРІОДИЧНІ ОСНОВНІ СТАНИ, ЩО ВІДПОВІДАЮТЬ ПІДГРУПАМ ІНДЕКСУ ТРИ, ДЛЯ МОДЕЛІ ІЗІНГА НА ДЕРЕВІ КЕЙЛІ ТРЕТЬОГО ПОРЯДКУ

We determine periodic and weakly periodic ground states with subgroups of index three for the Ising model on the Cayley tree of order three.

Знайдено періодичні та слабко періодичні основні стани з підгрупами індексу три для моделі Ізінга на дереві Кейлі третього порядку.

1. Introduction. The Ising model, with two values of spin ± 1 was considered in [11, 14] and became actively researched in the 1990's and afterwards (see, for example, [1 – 7, 10, 12]).

Each Gibbs measure is associated with a single phase of a physical system. The existence of two or more Gibbs measures means that phase transitions exist. One of fundamental problems is to describe the extreme Gibbs measures corresponding to a given Hamiltonian. As is known, the phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of this Hamiltonian. At low temperatures, a periodic ground state corresponds to a periodic Gibbs measure, see [13, 17]. The problem naturally arises on description of periodic and weakly periodic ground states. For the Ising model with competing interactions on the Cayley tree, translation-invariant and periodic ground states correspond to normal subgroups of even indices are studied in [1, 16]. As usual, more simple and interesting ground states are periodic ones. On the other hand, it is necessary to find weakly periodic ground states for some parameters which a periodic ground state does not exist.

Main concepts and notations of weakly periodic ground states are introduced in [18]. For the Ising model with competing interactions, weakly periodic ground states correspond to normal subgroups of indices two and four are described in [18, 20]. For the Potts model, such states for normal subgroups of index 2 are studied in [21, 22]. Also, in [23] for the Potts model, periodic and weakly periodic ground states for normal subgroups of index 4 are studied.

A full description of normal subgroups of indices $2i,\ i = \overline{1,5}$, for the group representation of the Cayley tree is given in [8, 9, 19]. Also, in [15] the existence of all subgroups of finite index for the group is proved and a full description of (not normal) subgroups of index 3 is given. Note that there are some papers which devoted to periodic and weakly periodic ground states for normal groups of finite index. In this paper, for the first time we study periodic and weakly periodic ground states for (not normal) subgroups of index 3. Note that periodic and weakly periodic ground states depend on the subgroups (in particular, normal subgroups). Moreover, the invariance properties do not hold for

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the subgroup (not normal subgroup). Besides, this problem is more difficult than to study periodic and weakly periodic ground states constructed by the normal subgroups. Thus, it is interesting to study the subgroups of index 3, naturally.

This paper is organized as follows. In Section 2, we recall the main definitions and known facts. In Section 3, we describe periodic and weakly periodic ground states.

2. Main definitions and known facts. ***The Cayley tree.*** The Cayley tree Γ^k (see [2]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighboring vertices*, and we write $l = \langle x, y\rangle$. The distance on this tree is defined as the number of nearest neighbour pairs of the minimal path between the vertices x and y (where path is a collection of nearest neighbour pairs, two consecutive pairs sharing at least a given vertex) and denoted by $d(x, y)$.

For the fixed $x^0 \in V$ (as usual, x^0 is called a root of the tree) we set

$$W_n = \{x \in V \mid d(x, x^0) = n\},$$

$$V_n = \{x \in V \mid d(x, x^0) \leq n\}, \qquad L_n = \{l = \langle x, y\rangle \in L \mid x, y \in V_n\}.$$

We write $x < y$ if the path from x^0 to y goes through x and $|x| = d(x, x^0)$, $x \in V$.

It is known (see [6]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k+1$ cyclic groups $\{e, a_i\}$, $i = 1, \ldots, k+1$, of the second order (i.e., $a_i^2 = e$, $a_i \neq e$) with generators $a_1, a_2, \ldots, a_{k+1}$.

Let $S(x)$ be the set of "direct successors" of $x \in G_k$, i.e.,

$$S(x) = \{y \in W_{n+1} \mid d(y, x) = 1\}, \quad x \in W_n.$$

Also, $S_1(x)$ is the set of all nearest neighboring vertices of $x \in G_k$, i.e.,

$$S_1(x) = \{y \in G_k \colon \langle x, y\rangle\} \qquad \text{and} \qquad \{x_\downarrow\} = S_1(x) \setminus S(x).$$

The Ising model. At first, we give main definitions and facts about the Ising model. We consider models where the spin takes values in the set $\Phi = \{-1, 1\}$. For $A \subseteq V$ a spin *configuration* σ_A on A is defined as a function $x \in A \to \sigma_A(x) \in \Phi$; the set of all configurations is denoted by $\Omega_A = \Phi^A$. Put $\Omega = \Omega_V$, $\sigma = \sigma_V$ and $-\sigma_A = \{-\sigma_A(x),\ x \in A\}$. Define a *periodic configuration* as a configuration $\sigma \in \Omega$ which is invariant under cosets of a subgroup $G_k^* \subset G_k$ of finite index. More precisely, a configuration $\sigma \in \Omega$ is called G_k^*-periodic if $\sigma(yx) = \sigma(x)$ for any $x \in G_k$ and $y \in G_k^*$.

The index of a subgroup is called the *period of the corresponding periodic configuration*. A configuration that is invariant with respect to all cosets is called *translation-invariant*.

Let $G_k/G_k^* = \{H_1, \ldots, H_r\}$ be a family of cosets, where G_k^* is a subgroup of index $r \geq 1$. Configuration $\sigma(x)$, $x \in V$, is called G_k^*-*weakly periodic*, if $\sigma(x) = \sigma_{ij}$ for $x \in H_i$, $x_\downarrow \in H_j$ $\forall x \in G_k$.

The Ising model with competing interactions has the form

$$H(\sigma) = J_1 \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y) + J_2 \sum_{\substack{x,y \in V\colon \\ d(x,y)=2}} \sigma(x)\sigma(y), \tag{1}$$

where $J_1, J_2 \in R$ and $\sigma \in \Omega$.

For a pair of configurations σ and φ that coincide almost everywhere, i.e., everywhere except for a finite number of positions, we consider a relative Hamiltonian $H(\sigma, \varphi)$, the difference between the energies of the configurations σ and φ has the form

$$H(\sigma, \varphi) = J_1 \sum_{\langle x,y\rangle \in L} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)) + J_2 \sum_{\substack{x,y\in V:\\ d(x,y)=2}} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)),$$

where $J = (J_1, J_2) \in R^2$ is an arbitrary fixed parameter.

Let M be the set of unit balls with vertices in V. We call the restriction of a configuration σ to the ball $b \in M$ a *bounded configuration* σ_b.

Define the energy of a ball b for configuration σ by

$$U(\sigma_b) \equiv U(\sigma_b, J) = \frac{1}{2} J_1 \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y) + J_2 \sum_{d(x,y)=2} \sigma(x)\sigma(y), \quad x, y \in b,$$

where $J = (J_1, J_2) \in R^2$.

We shall say that two bounded configurations σ_b and $\sigma'_{b'}$ belong to the same class if $U(\sigma_b) = U(\sigma'_{b'})$ and we write $\sigma'_{b'} \sim \sigma_b$.

Let A be a set, then $|A|$ is the cardinality of A.

Lemma 1 [1]. 1. *For any configuration σ_b we have*

$$U(\sigma_b) \in \{U_0, U_1, \ldots, U_{k+1}\},$$

where

$$U_i = \left(\frac{k+1}{2} - i\right) J_1 + \left(\frac{k(k+1)}{2} + 2i(i-k-1)\right) J_2, \quad i = 0, 1, \ldots, k+1.$$

2. *Let $\mathcal{C}_i = \Omega_i \cup \Omega_i^-$, $i = 0, \ldots, k+1$, where*

$$\Omega_i = \big\{\sigma_b : \sigma_b(c_b) = +1, \ \big|\{x \in b \setminus \{c_b\} : \sigma_b(x) = -1\}\big| = i\big\},$$

$$\Omega_i^- = \big\{-\sigma_b = \{-\sigma_b(x), x \in b\} : \sigma_b \in \Omega_i\big\},$$

and c_b is the center of the ball b. Then for $\sigma_b \in \mathcal{C}_i$ we have $U(\sigma_b) = U_i$.

3. *The class $\mathcal{C}_i$ contains $\dfrac{2(k+1)!}{i!(k-i+1)!}$ configurations.*

Definition 1. *A configuration φ is called a ground state for the Hamiltonian* (1) *if it satisfies the condition*

$$U(\varphi_b) = \min\{U_0, U_1, \ldots, U_{k+1}\} \quad \textit{for any} \quad b \in M.$$

Denote

$$U_i(J) = U(\sigma_b, J), \quad \text{if} \quad \sigma_b \in \mathcal{C}_i, \quad i = 0, 1, \ldots, k+1.$$

The quantity $U_i(J)$ is a linear function of the parameter $J \in R^2$. For every fixed $m = 0, 1, \ldots, k+1$ we denote

$$A_m = \{J \in R^2 : U_m(J) = \min\{U_0(J), U_1(J), \dots, U_{k+1}(J)\}\}. \tag{2}$$

It is easy to check that

$$A_0 = \{J \in R^2 : J_1 \le 0,\ J_1 + 2kJ_2 \le 0\},$$

$$A_m = \big\{J \in R^2 : J_2 \ge 0,\ 2(2m-k-2)J_2 \le J_1 \le 2(2m-k)J_2\big\}, \quad m = 1, 2, \dots, k,$$

$$A_{k+1} = \{J \in R^2 : J_1 \ge 0,\ J_1 - 2kJ_2 \ge 0\} \qquad \text{and} \qquad R^2 = \bigcup_{i=0}^{k+1} A_i.$$

3. Periodic and weakly periodic ground states. In this section, we study periodic and weakly periodic ground states. It is known that ground states depend on choosing subgroups for given Hamiltonian. According to this reason we give how to choose a subgroup with index of three of the group G_k.

Let G_k be a free product of $k+1$ cyclic groups of the second order with generators $a_1, a_2, \dots, a_{k+1}$, respectively. Then from Theorem 1 in [16], it is known that:

the group G_k does not have normal subgroups of odd index ($\neq 1$);

the group G_k has normal subgroups of arbitrary even index.

Now, we give a construction of subgroups of index 3 of the group G_k (for more details, see [15]).

Let $N_k = \{1, 2, \dots, k+1\}$ and $B_0 \subset N_k$, $0 \le |B_0| \le k-1$. (B_1, B_2) be a partition of the set $N_k \setminus B_0$. Put m_j be a minimal element of B_j, $j \in \{1, 2\}$. Then we consider the homomorphism $u_{B_1B_2} : \langle e, a_1, a_2, \dots, a_{k+1}\rangle \to \langle e, a_{m_1}, a_{m_2}\rangle$ (where e is identity element) given by

$$u_{B_1B_2}(x) = \begin{cases} e, & \text{if} \quad x = a_i, \quad i \in N_k \backslash (B_1 \cup B_2), \\ a_{m_j}, & \text{if} \quad x = a_i, \quad i \in B_j, \quad j = 1, 2. \end{cases} \tag{3}$$

Let $l(x)$ be the length of x. For $1 \le q \le s$, we define $\gamma_s : \langle e, a_{m_1}, a_{m_2}\rangle \to \{e, a_{m_1}, a_{m_2}\}$ by the formula

$$\gamma_s(x) = \begin{cases} e, \quad \text{if} \quad x = e, \\ a_{m_1}a_{m_2}a_{m_1}\dots a_{m_j}, \quad \text{if} \quad x \in \Big\{ \underbrace{a_{m_1}a_{m_2}a_{m_1}\dots a_{m_j}}_{q}, \underbrace{a_{m_2}a_{m_1}a_{m_2}\dots a_{m_{3-j}}}_{2s+1-q} \Big\}, \\ a_{m_2}a_{m_1}a_{m_2}\dots a_{m_j}, \quad \text{if} \quad x \in \Big\{ \underbrace{a_{m_2}a_{m_1}a_{m_2}\dots a_{m_j}}_{q}, \underbrace{a_{m_1}a_{m_2}a_{m_1}\dots a_{m_{3-j}}}_{2s+1-q} \Big\}, \\ \gamma_s\Big(a_{m_j}\dots\gamma_s(\underbrace{a_{m_j}a_{m_{3-j}}\dots a_{m_{3-j}}}_{2s})\Big), \quad \text{if} \quad x = a_{m_j}a_{m_{3-j}}\dots a_{m_{3-j}}, \quad l(x) > 2s, \\ \gamma_s\Big(a_{m_j}\dots\gamma_s(\underbrace{a_{m_{3-j}}a_{m_j}\dots a_{m_j}}_{2s})\Big), \quad \text{if} \quad x = a_{m_j}a_{m_{3-j}}\dots a_{m_{3-j}}, \quad l(x) > 2s. \end{cases} \tag{4}$$

Denote

$$\Im^s_{B_1B_2}(G_k) = \{x \in G_k \mid \gamma_s(u_{B_1B_2}(x)) = e\}.$$

Lemma 2 [15]. *Let* (B_1, B_2) *be a partition of the set* $N_k \setminus B_0,\ 0 \leq |B_0| \leq k-1$. *Then* $x \in \Im^s_{B_1B_2}(G_k)$ *if and only if the number* $l(u_{B_1B_2}(x))$ *is divisible by* $2s+1$.

Proposition 1 [15]. *For the group* G_k *the following equality holds*:

$$\{K \mid K \text{ is a subgroup of } G_k \text{ of index } 3\} = \left\{\Im^1_{B_1B_2} \mid B_1, B_2 \text{ is a partition of } N_k \setminus B_0\right\}.$$

We consider periodic and weakly periodic ground states on the Cayley tree of order three, i.e., $k=3$. Now we consider all cases of subgroups of index 3 of the group G_3.

1. Let $B_0 = \{3,4\}$, $B_d = \{d\}$, $d \in \{1,2\}$, i.e., $m_i = i, i \in \{1,2\}$. We consider homomorphism $u^{(1)}_{B_1B_2} : \langle e, a_1, a_2, a_3, a_4\rangle \to \langle e, a_1, a_2\rangle$ (3) and $\gamma^{(1)} : \langle e, a_1, a_2\rangle \to \{e, a_1, a_2\}$ (4):

$$u^{(1)}_{B_1B_2}(x) = \begin{cases} e, & \text{if} \quad x \in \{e, a_3, a_4\}, \\ a_i & \text{if} \quad x = a_i, \quad i = 1, 2, \end{cases}$$

$$\gamma^{(1)}(x) = \begin{cases} e, & \text{if} \quad x = e, \\ a_1, & \text{if} \quad x \in \{a_1, a_2a_1\}, \\ a_2, & \text{if} \quad x \in \{a_2, a_1a_2\}, \\ \gamma^{(1)}(a_ia_{3-i}\ldots\gamma^{(1)}(a_ia_{3-i})), & \text{if} \quad x = a_ia_{3-i}\ldots a_{3-i}, \quad l(x) \geq 3, \quad i = 1, 2, \\ \gamma^{(1)}(a_ia_{3-i}\ldots\gamma^{(1)}(a_{3-i}a_i)), & \text{if} \quad x = a_ia_{3-i}\ldots a_i, \quad l(x) \geq 3, \quad i = 1, 2. \end{cases}$$

Let $H^{(1)}_1 := \Im^1_{B_1B_2}(G_3)$. Then

$$H^{(1)}_1 = \left\{x \in G_3 \mid \gamma^{(1)}(u^{(1)}_{B_1B_2}(x)) = e\right\}.$$

Since $H^{(1)}_1$ is a subgroup of index 3 of the group G_3, we define a family of cosets:

$$G_3/H^{(1)}_1 = \left\{H^{(1)}_1, H^{(1)}_2, H^{(1)}_3\right\},$$

where

$$H^{(1)}_2 = \left\{x \in G_3 \mid \gamma^{(1)}(u^{(1)}_{B_1B_2}(x)) = a_1\right\}, \qquad H^{(1)}_3 = \left\{x \in G_3 \mid \gamma^{(1)}(u^{(1)}_{B_1B_2}(x)) = a_2\right\}.$$

2. Let $B_0 = \{1\}$, $B_1 = \{2,3\}$, $B_2 = \{4\}$, i.e., $m_1 = 2, m_2 = 4$. We consider homomorphism $u^{(2)}_{B_1B_2} : \langle e, a_1, a_2, a_3, a_4\rangle \to \langle e, a_2, a_4\rangle$ (3) and $\gamma^{(2)} : \langle e, a_2, a_4\rangle \to \{e, a_2, a_4\}$ (4):

$$u^{(2)}_{B_1B_2}(x) = \begin{cases} e, & \text{if} \quad x \in \{e, a_1\}, \\ a_2, & \text{if} \quad x \in \{a_2, a_3\}, \\ a_4, & \text{if} \quad x = a_4, \end{cases}$$

$$\gamma^{(2)}(x) = \begin{cases} e, & \text{if} \quad x = e, \\ a_2, & \text{if} \quad x \in \{a_2, a_4a_2\}, \\ a_4, & \text{if} \quad x \in \{a_4, a_2a_4\}, \\ \gamma^{(2)}\big(a_ia_{6-i}\dots\gamma^{(2)}(a_ia_{6-i})\big), & \text{if} \quad x = a_ia_{6-i}\dots a_{6-i}, \quad l(x) \geq 3, \quad i \in \{2;4\}, \\ \gamma^{(2)}\big(a_ia_{6-i}\dots\gamma^{(2)}(a_{6-i}a_i)\big), & \text{if} \quad x = a_ia_{6-i}\dots a_i, \quad l(x) \geq 3, \quad i \in \{2;4\}. \end{cases}$$

Let $H_1^{(2)} := \Im^1_{B_1B_2}(G_3)$. Then

$$H_1^{(2)} = \Big\{x \in G_3 \mid \gamma^{(2)}\Big(u^{(2)}_{B_1B_2}(x)\Big) = e\Big\}.$$

Since $H_1^{(2)}$ is a subgroup of index 3 of the group G_3, we define a family of cosets:

$$G_3/H_1^{(2)} = \Big\{H_1^{(2)}, H_2^{(2)}, H_3^{(2)}\Big\},$$

where

$$H_2^{(2)} = \Big\{x \in G_3 \mid \gamma^{(2)}\Big(u^{(2)}_{B_1B_2}(x)\Big) = a_2\Big\}, \qquad H_3^{(2)} = \Big\{x \in G_3 \mid \gamma^{(2)}\Big(u^{(2)}_{B_1B_2}(x)\Big) = a_4\Big\}.$$

3. Let $B_0 = \{\varnothing\}$, $B_1 = \{1\}$, $B_2 = \{2, 3, 4\}$, i.e., $m_1 = 1$, $m_2 = 2$. We consider homomorphism $u^{(3)}_{B_1B_2} : \langle e, a_1, a_2, a_3, a_4\rangle \to \langle e, a_1, a_2\rangle$ (3) and $\gamma^{(3)} : \langle e, a_1, a_2\rangle \to \{e, a_1, a_2\}$ (4):

$$u^{(3)}_{B_1B_2}(x) = \begin{cases} e, & \text{if} \quad x = e, \\ a_1, & \text{if} \quad x = a_1, \\ a_2, & \text{if} \quad x = a_i, \quad i = \overline{2,4}, \end{cases}$$

$$\gamma^{(3)}(x) = \begin{cases} e, & \text{if} \quad x = e, \\ a_1, & \text{if} \quad x \in \{a_1, a_2a_1\}, \\ a_2, & \text{if} \quad x \in \{a_2, a_1a_2\}, \\ \gamma^{(3)}(a_ia_{3-i}\dots\gamma^{(3)}(a_ia_{3-i})), & \text{if} \quad x = a_ia_{3-i}\dots a_{3-i}, \quad l(x) \geq 3, \quad i \in \{1;2\}, \\ \gamma^{(3)}(a_ia_{3-i}\dots\gamma^{(3)}(a_{3-i}a_i)), & \text{if} \quad x = a_ia_{3-i}\dots a_i, \quad l(x) \geq 3, \quad i \in \{1;2\}. \end{cases}$$

Let $H_1^{(3)} := \Im^1_{B_1B_2}(G_3)$. Then

$$H_1^{(3)} = \Big\{x \in G_3 \mid \gamma^{(3)}\Big(u^{(3)}_{B_1B_2}(x)\Big) = e\Big\}.$$

Because $H_1^{(3)}$ is a subgroup of index 3 of the group G_3, we define a family of cosets:

$$G_3/H_1^{(3)} = \Big\{H_1^{(3)}, H_2^{(3)}, H_3^{(3)}\Big\},$$

where

$$H_2^{(3)} = \Big\{x \in G_3 \mid \gamma^{(3)}\Big(u^{(3)}_{B_1B_2}(x)\Big) = a_1\Big\}, \qquad H_3^{(3)} = \Big\{x \in G_3 \mid \gamma^{(3)}\Big(u^{(3)}_{B_1B_2}(x)\Big) = a_2\Big\}.$$

4. Let $B_0 = \{\varnothing\}$, $B_1 = \{1,2\}$, $B_2 = \{3,4\}$, i.e., $m_1 = 1$, $m_2 = 3$. We consider homomorphism $u^{(4)}_{B_1B_2} : \langle e, a_1, a_2, a_3, a_4\rangle \to \langle e, a_1, a_3\rangle$ (3) and $\gamma^{(4)} : \langle e, a_1, a_3\rangle \to \{e, a_1, a_3\}$ (4):

$$u^{(4)}_{B_1B_2}(x) = \begin{cases} e, & \text{if} \quad x = e, \\ a_1, & \text{if} \quad x = a_i, \quad i = 1, 2, \\ a_3, & \text{if} \quad x = a_i, \quad i = 3, 4, \end{cases}$$

$$\gamma^{(4)}(x) = \begin{cases} e, & \text{if} \quad x = e, \\ a_1, & \text{if} \quad x \in \{a_1, a_3a_1\}, \\ a_3, & \text{if} \quad x \in \{a_3, a_1a_3\}, \\ \gamma^{(4)}\big(a_ia_{4-i}\ldots\gamma^{(4)}(a_ia_{4-i})\big), & \text{if} \quad x = a_ia_{4-i}\ldots a_{4-i}, \quad l(x) \geq 3, \quad i \in \{1;3\}, \\ \gamma^{(4)}\big(a_ia_{4-i}\ldots\gamma^{(4)}(a_{4-i}a_i)\big), & \text{if} \quad x = a_ia_{4-i}\ldots a_i, \quad l(x) \geq 3, \quad i \in \{1;3\}. \end{cases}$$

Let $H_1^{(4)} := \Im^1_{B_1B_2}(G_3)$. Then

$$H_1^{(4)} = \Big\{x \in G_3 \mid \gamma^{(4)}\Big(u^{(4)}_{B_1B_2}(x)\Big) = e\Big\}.$$

Because $H_1^{(4)}$ is a subgroup of index 3 of the group G_3, we define a family of cosets:

$$G_3/H_1^{(4)} = \big\{H_1^{(4)}, H_2^{(4)}, H_3^{(4)}\big\},$$

where

$$H_2^{(4)} = \Big\{x \in G_3 \mid \gamma^{(4)}\Big(u^{(4)}_{B_1B_2}(x)\Big) = a_1\Big\}, \qquad H_3^{(4)} = \Big\{x \in G_3 \mid \gamma^{(4)}\Big(u^{(4)}_{B_1B_2}(x)\Big) = a_3\Big\}.$$

$H_1^{(j)}$-periodic configurations have the following forms:

$$\sigma(x) = \begin{cases} \sigma_1, & x \in H_1^{(j)}, \\ \sigma_2, & x \in H_2^{(j)}, \\ \sigma_3, & x \in H_3^{(j)}, \end{cases}$$

where $\sigma_i \in \Phi,\ i \in \{1,2,3\},\ j = \overline{1,4}$.

Note that if $\sigma_1 = \sigma_2 = \sigma_3$ then this configuration is *translation-invariant* and the full details of such configuration are given in [16].

Theorem 1. *Let* $k = 3$.

1. *If* $(J_1, J_2) \in A_1 \cap A_2$, *then there exist six* $H_1^{(1)}$*-periodic (except for translation-invariant) ground states which corresponding to the following configurations*:

$$\sigma(x) = \pm \begin{cases} \sigma_1, & \textit{if} \quad x \in H_1^{(1)}, \\ \sigma_2, & \textit{if} \quad x \in H_2^{(1)}, \\ \sigma_3, & \textit{if} \quad x \in H_3^{(1)}, \end{cases}$$

where $(\sigma_1, \sigma_2, \sigma_3) \in \{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$.

2. *If* $(J_1, J_2) \in R^2 \backslash (A_1 \cap A_2)$, *then there exist not* $H_1^{(1)}$*-periodic (except for translation-invariant) ground states.*

Proof. Let $(\sigma_1, \sigma_2, \sigma_3) = (-1, 1, 1)$. Consider the following configuration:

$$\varphi_1(x) = \begin{cases} -1, & \text{if} \quad x \in H_1^{(1)}, \\ 1, & \text{if} \quad x \in H_2^{(1)}, \\ 1, & \text{if} \quad x \in H_3^{(1)}. \end{cases}$$

Denote $A_- = \{x \in S_1(c_b) : \varphi_b(x) = -1\}$, $A_+ = \{x \in S_1(c_b) : \varphi_b(x) = +1\}$ and $\varphi_{i,b} = (\varphi_i)_b$ for any i. If $c_b \in H_1^{(1)}$, then $\varphi_1(c_b) = -1$, $|A_-| = 2$, $|A_+| = 2$ which implies that $\varphi_{1,b} \in C_2$. For this case $c_b \in H_2^{(1)}$, then one gets $\varphi_1(c_b) = 1$, $|A_-| = 1$, $|A_+| = 3$ which implies that $\varphi_{1,b} \in C_1$. Finally, if $c_b \in H_3^{(1)}$, then $\varphi_1(c_b) = 1$, $|A_-| = 1$, $|A_+| = 3$ which implies that $\varphi_{1,b} \in C_1$. Hence, for any $b \in M$ one gets $\varphi_{1,b} \in C_1 \cup C_2$.

From (2) we obtain that $A_1 \cap A_2 = \left\{(J_1, J_2) : J_2 = -\dfrac{1}{2} J_1,\ J_1 \leq 0\right\}$. From Lemma 1 it follows that the periodic configuration φ_1 is $H_1^{(1)}$-periodic ground state on the set $A_1 \cap A_2$. Note that, for any $b \in M$, we have $\varphi_{1,b} \sim -\varphi_{1,b}$, i.e., $-\varphi_{1,b} \in C_1 \cup C_2$ for all $b \in M$. Consequently, the periodic configuration $-\varphi_1$ is $H_1^{(1)}$-periodic ground state on the set $A_1 \cap A_2$.

Similar arguments also apply to the periodic configurations $\pm\varphi_2$ and $\pm\varphi_3$, which corresponding to $(\sigma_1, \sigma_2, \sigma_3) \in \{(1, -1, 1), (1, 1, -1)\}$.

Note that there exists nonperiodic (nontranslation-invariant) configuration not mentioned in assertion 1. As above, we prove that those configurations are ground states on the set $A_1 \cap A_2$. Hence, if $(J_1, J_2) \in R^2 \setminus (A_1 \cap A_2)$ there exist not $H_1^{(1)}$-periodic (non translation-invariant) ground states.

Theorem 1 is proved.

***Remark* 1.** $H_1^{(1)}$-periodic ground states which mentioned in Theorem 1 differ from periodic ground states which described in [1]. In addition, in [1] the fact that for a fixed $J = (J_1, J_2)$ maximum number of periodic ground states equals four is proved. In our case, it is equal to six.

In [20, 21] for the normal subgroups of indices two and four, weakly periodic ground states are studied. In [24] we study H_1-weakly periodic ground states on the Cayley tree of order two. Now, we study H_1-weakly periodic ground states, which corresponding to subgroups of index 3 of the group representation of the Cayley tree of order three.

For any element x of G_k, we recall that $x_\downarrow$ on element which satisfies the following condition: $x^{-1} \cdot x_\downarrow \in \{a_i \mid i \in N_k\}$.

Invariance property. For $B_i = \{m_i\}$ and $x, y \in G_k$, if $\gamma(u_{B_1B_2}(x)) = \gamma(u_{B_1B_2}(y))$, $\gamma(u_{B_1B_2}(x_\downarrow)) = \gamma(u_{B_1B_2}(y_\downarrow))$, then

$$\langle\langle \gamma(u_{B_1B_2}(xa_i)) \mid xa_i \in S(x)\rangle\rangle = \langle\langle \gamma(u_{B_1B_2}(ya_i)) \mid ya_i \in S(y)\rangle\rangle,$$

where $\langle\langle \ldots \rangle\rangle$ stands for ordered k-tuples (for more details, see [15]).

In [15] it is given a certain condition on subgroups of the group representation of the Cayley tree such that an invariance property holds. Generally speaking, except for the given condition, the invariance property does not hold. $H_1^{(z)}$-weakly periodic configurations have the following forms:

$$\varphi(x) = \begin{cases} a_{11}, & x_\downarrow \in H_1^{(z)} \quad \text{and} \quad x \in H_1^{(z)}, \\ a_{12}, & x_\downarrow \in H_1^{(z)} \quad \text{and} \quad x \in H_2^{(z)}, \\ a_{13}, & x_\downarrow \in H_1^{(z)} \quad \text{and} \quad x \in H_3^{(z)}, \\ a_{21}, & x_\downarrow \in H_2^{(z)} \quad \text{and} \quad x \in H_1^{(z)}, \\ a_{22}, & x_\downarrow \in H_2^{(z)} \quad \text{and} \quad x \in H_2^{(z)}, \\ a_{23}, & x_\downarrow \in H_2^{(z)} \quad \text{and} \quad x \in H_3^{(z)}, \\ a_{31}, & x_\downarrow \in H_3^{(z)} \quad \text{and} \quad x \in H_1^{(z)}, \\ a_{32}, & x_\downarrow \in H_3^{(z)} \quad \text{and} \quad x \in H_2^{(z)}, \\ a_{33}, & x_\downarrow \in H_3^{(z)} \quad \text{and} \quad x \in H_3^{(z)}, \end{cases}$$

where $a_{ij} \in \Phi,\ i, j \in \{1, 2, 3\},\ z = \overline{1, 2}$. For convenience, we write $\varphi(x) = (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33})$ for such a weakly periodic configuration φ.

Theorem 2. *Let* $k = 3$.

1. *There exists not any* $H_1^{(1)}$*-weakly periodic* (*except for translation-invariant and periodic*) *ground state.*

2. *There exists not any* $H_1^{(l)}$*-periodic and weakly periodic* (*except for translation-invariant*) *ground state, where* $l = 2, 3$.

3. *There exists not any* $H_1^{(4)}$*-periodic* (*except for translation-invariant*) *ground state.*

Proof. 1. Now we prove part 1 of Theorem 2.

Let us consider $\varphi_1 = (-1, -1, 1, -1, 1, 1, -1, 1, 1)$.

1.1. Assume that $c_b \in H_1^{(1)}$. Then all possible cases are:

(a) $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = -1,\ |A_-| = 3,\ |A_+| = 1,\ \varphi_{1,b} \in C_1$,

(b) $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, this case is impossible,

(c) $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, this case is impossible,

(d) $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = -1,\ |A_-| = 2,\ |A_+| = 2,\ \varphi_{1,b} \in C_2$,

(e) $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = -1,\ |A_-| = 3,\ |A_+| = 1,\ \varphi_{1,b} \in C_1$,

(f) $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, this case is impossible.

1.2. Let $c_b \in H_2^{(1)}$, then all possible cases are:
(a) $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = -1$, $|A_-| = 1$, $|A_+| = 3$, $\varphi_{1,b} \in C_3$,
(b) $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$ which is impossible,
(c) $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = 1$, $|A_-| = 2$, $|A_+| = 2$, $\varphi_{1,b} \in C_2$,
(d) $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1$, $|A_-| = 1$, $|A_+| = 3$, $\varphi_{1,b} \in C_1$,
(e) $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1$, $|A_-| = 1$, $|A_+| = 3$, $\varphi_{1,b} \in C_1$,
(f) $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, it is impossible.
1.3. If $c_b \in H_3^{(1)}$, then all possible cases are:
(a) $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = 1$, $|A_-| = 1$, $|A_+| = 3$, $\varphi_{1,b} \in C_1$,
(b) $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, which is impossible,
(c) $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = 1$, $|A_-| = 2$, $|A_+| = 2$, $\varphi_{1,b} \in C_2$,
(d) $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1$, $|A_-| = 1$, $|A_+| = 3$, $\varphi_{1,b} \in C_1$,
(e) $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1$, $|A_-| = 1$, $|A_+| = 3$, $\varphi_{1,b} \in C_1$,
(f) $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, this case is impossible.

Consequently, we prove that $\varphi_{1,b} \in C_1 \cup C_2 \cup C_3$ for all $b \in M$.

From (2) we find that $A_1 \cap A_2 \cap A_3 = \{(J_1, J_2) \in R^2 : J_1 = J_2 = 0\}$. This implies that the configuration φ_1 is not $H_1^{(1)}$-weakly periodic ground state. The same conclusion can be drawn for the remaining configurations. The rest of the proof runs as in the proof of part 1 of Theorem 2 and Theorem 1.

Theorem 2 is proved.

***Remark* 2.** In [24] for the case $k = 2$ periodic (nontranslation-invariant) and weakly periodic (nontranslation-invariant and nonperiodic) ground states are found.

***Remark* 3.** $H_1^{(l)}$-subgroups do not hold invariance property, where $l = 2, 3$.

$H_1^{(m)}$-weakly periodic configurations have the following forms:

$$\varphi(x) = \begin{cases} a_{12}, & x_\downarrow \in H_1^{(m)} \text{ and } x \in H_2^{(m)}, \\ a_{13}, & x_\downarrow \in H_1^{(m)} \text{ and } x \in H_3^{(m)}, \\ a_{21}, & x_\downarrow \in H_2^{(m)} \text{ and } x \in H_1^{(m)}, \\ a_{23}, & x_\downarrow \in H_2^{(m)} \text{ and } x \in H_3^{(m)}, \\ a_{31}, & x_\downarrow \in H_3^{(m)} \text{ and } x \in H_1^{(m)}, \\ a_{32}, & x_\downarrow \in H_3^{(m)} \text{ and } x \in H_2^{(m)}, \end{cases}$$

where $a_{ij} \in \Phi$, $i, j \in \{1, 2, 3\}$, $m = \overline{3, 4}$.

In the sequel, we write $\varphi(x) = (a_{12}, a_{13}, a_{21}, a_{23}, a_{31}, a_{32})$ for such a weakly periodic configuration φ.

Theorem 3. *Let $k = 3$. Then the following assertions hold*:

1(a). *There exist exactly six $H_1^{(4)}$-weakly periodic ground states on* $\left\{J_2 = \frac{1}{2}J_1,\ J_1 \geq 0\right\}$, *which are nonperiodic, having the form* $\varphi_{1,2} = \pm(i, j, i, j, i, j)$, $\varphi_{3,4} = \pm(i, j, i, j, j, i)$, $\varphi_{5,6} = \pm(i, j, j, i, j, i)$, *where* $i \neq j$, $i, j \in \Phi$.

1(b). *There are exactly two* $H_1^{(4)}$*-weakly periodic ground states on* $\left\{J_2 = -\frac{1}{2}J_1,\ J_1 \leq 0\right\}$, *which are nonperiodic*, *having the form* $\varphi_{7,8} = \pm(i,j,j,i,i,j)$ *where* $i \neq j,\ i,j \in \Phi$.

2. *If* $(J_1, J_2) \in R^2 \setminus ((A_1 \cap A_2) \cup (A_2 \cap A_3))$, *there exist not* $H_1^{(4)}$*-weakly periodic* (*except for translation-invariant*) *ground states.*

Proof. The proof follows by the same method as in the proof of part 1 of Theorem 2.

***Remark* 4.** The results of Theorems 1 and 3 do not depend on the choice elements of N_k, however, depend on only power of partition sets of N_k.

***Remark* 5.** Obtained weakly periodic ground states in Theorem 3 are different from weakly periodic ground states which had been found in [20].

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