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COMPLEX HESSIAN-TYPE EQUATIONS IN THE WEIGHTED m-SUBHARMONIC CLASS

КОМПЛЕКСНІ РІВНЯННЯ ТИПУ ГЕССЕ У ЗВАЖЕНОМУ m-СУБГАРМОНІЧНОМУ КЛАСІ

We study the existence of a solution to a general type of complex Hessian equation on some Cegrell classes. For a given measure μ defined on an m-hyperconvex domain $\Omega \subset \mathbb{C}^n$, under suitable conditions, we prove that the equation $\chi(.)H_m(.) = \mu$ has a solution that belongs to the class $\mathcal{E}_{m,\chi}(\Omega)$.

Досліджено існування розв'язку для комплексного рівняння Гессе загального типу на деяких класах Сегрелля. Для заданої міри μ, що визначена на m-гіперопуклій області $\Omega \subset \mathbb{C}^n$, доведено, що за відповідних умов рівняння $\chi(.)H_m(.) = \mu$ має розв'язок, який належить класу $\mathcal{E}_{m,\chi}(\Omega)$.

1. Introduction. The complex Hessian equations on m-hyperconvex domain $\Omega \subset \mathbb{C}^n$ have been the object of several research works not only because they are second-order versions of PDEs which generalize the complex Monge – Ampère equation (when $m = n$) but also because they play an important role in various problems related to Khälerian geometry and pluripotential theory. In the particular case $m = n$, Cegrell [4, 5] has introduced the classes of plurisubharmonic functions $\mathcal{E}(\Omega)$ and $\mathcal{F}(\Omega)$ that represent the admissible solutions of these equations. In the general case $1 \leq m \leq n$ those classes were extended by Lu [14] who introduced the classes $\mathcal{F}_m(\Omega)$, $\mathcal{N}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$. He proved that the Hessian operator is well defined on those classes. The later are constituted by addmissible solutions for the associated Hessian equation. In [2], Benelkourchi, Guedj and Zeriahi introduced and investigated the weighted pluricomplex energy classes $\mathcal{E}_\chi(\Omega)$ for a given increasing negative function χ defined on $\mathbb{R}^-$. So it was accurate to consider the associated Hessian equation to those classes. In this paper we study the existence of a solution for the equation

$$-\chi(.)H_m(.) = \mu, \tag{1}$$

where H_m is the complex Hessian operator and μ is a given nonnegative measure defined on Ω. The equation (1) was studied by several researchers in the case $m = n$ (see [7, 9]). Firstly, R. Czyz [7] proved the existence of a solution u to equation (1) such that $u \in \mathcal{E}_\chi(\Omega)$. One of the most important results in this case is made by L. M. Hai, P. H. Hiep and N. X. Hong [9] who developed the findings in [7] with more suitable conditions.

For the case of m-subharmonic functions, Lu [14] solved the degenerate Hessian equation (when $\chi \equiv -1$) under the assumption that the Radon measure μ is vanishing on all m-polar set. Recently V. V. Hung and N. V. Phu [12] dealt with this issue when the right-hand side in (1) is a Radon

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finite measure and under the assumption that there exists a subsolution to the given equation. They proved that there exists a function $u \in \mathcal{E}_m(\Omega)$ solution of (1). Note that all of the cited works were established for the particular case $\chi \equiv -1$. This paper is devoted to study the general case when the function χ is not necessarity equal to -1 for $1 \le m \le n$. Specifically, we prove that the equation (1) has a unique solution even if μ has no mass on all m-polar sets. We aim further to show that the existence of a solution for the given equation is equivalent to the existence of a local solution for the same equation.

2. Preliminaries. In this section, we recall some elementary notions in the pluripotential theory. To simplify we use the following notation $d := \partial + \overline{\partial}$, $d^c := i(\overline{\partial} - \partial)$ and $\beta := dd^c|z|^2$.

Definition 2.1. *Let $u : \Omega \to \mathbb{R} \cup \{-\infty\}$. We say that u is m-subharmonic (m-sh for short) if and only if the following conditions are satisfied*:

(1) *the function u is subharmonic*;

(2) *for all m-positive $(1,1)$-forms $\gamma_1, \ldots, \gamma_{m-1}$ one has*

$$dd^c u \wedge \beta^{n-m} \wedge \gamma_1 \wedge \ldots \wedge \gamma_{m-1} \ge 0.$$

The cone of m-sh functions will be denoted by $\mathcal{SH}_m(\Omega)$.

***Remark* 2.1.** If $m = n$ in the above definition, then

$$\mathcal{SH}_n(\Omega) = PSH(\Omega),$$

where $PSH(\Omega)$ is the set of all plurisubharmonic functions in Ω.

For more details on m-sh function, the reader can refer to [3, 13, 14, 16].

For a given locally bounded m-sh function u, Błocki [3] defined, by induction, the following positive closed current:

$$dd^c u_1 \wedge \ldots \wedge dd^c u_k \wedge \beta^{n-m} := dd^c(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_k \wedge \beta^{n-m}),$$

where $u_1, \ldots, u_k \in \mathcal{SH}_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$. In particular, one can associate to $u \in \mathcal{SH}_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ a positive measure called the Hessian measure of u and defined by $H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$.

Definition 2.2. 1. *A bounded domain Ω in $\mathbb{C}^n$ is said to be m-hyperconvex if the following property holds for some continuous m-sh functions $\rho \colon \Omega \to \mathbb{R}^-$: $\{\rho < c\} \Subset \Omega$ for every $c < 0$.*

2. *A set $M \subset \Omega$ is called m-polar if there exists $u \in \mathcal{SH}_m(\Omega)$ such that $M \subset \{u = -\infty\}$.*

Throughout the rest of the paper, we denote by Ω a m-hyperconvex domain of $\mathbb{C}^n$. To study the Hessian operator, Lu [14, 15] introduced the following classes of m-sh functions to generalize Cegrell classes. Those classes are defined as follows.

Definition 2.3. *We denote by*

$$\mathcal{E}^0_m(\Omega) = \left\{ u \in \mathcal{SH}^-_m(\Omega) \cap L^\infty(\Omega); \quad \lim_{z \to \xi} u(z) = 0 \ \forall \xi \in \partial\Omega, \quad \int\limits_\Omega H_m(u) < +\infty \right\},$$

$$\mathcal{F}_m(\Omega) = \left\{ u \in \mathcal{SH}^-_m(\Omega); \quad \exists (u_j) \subset \mathcal{E}^0_m, \ u_j \searrow u \ in \ \Omega \quad \sup_j \int\limits_\Omega H_m(u_j) < +\infty \right\}$$

and

$$\mathcal{E}_m(\Omega) = \{ u \in \mathcal{SH}^-_m(\Omega) : \forall U \Subset \Omega \ \exists \ u_U \in \mathcal{F}_m(\Omega); \ u_U = u \ on \ U \}.$$

Definition 2.4. *A function $u \in \mathcal{SH}_m(\Omega)$ is said to be m-maximal, if for every $v \in \mathcal{SH}_m(\Omega)$ such that if $v \leq u$ outside a compact subset of Ω, then $v \leq u$ in Ω.*

The family of m-maximal functions in $\mathcal{SH}_m(\Omega)$ will be denoted as $\mathcal{MSH}_m(\Omega)$.

Definition 2.5. *A sequence $(\Omega_j)_j$ of strictly m-pseudoconvex subsets of Ω is called the fundamental increasing sequence associated to Ω if and only if $\Omega_j \Subset \Omega_{j+1}$, $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$ and, for every j, there exists a smooth strictly m-sh function φ in a neighborhood V of Ω_j such that $\Omega_j := \{z \in V/\varphi(z) < 0\}$.*

Definition 2.6. *Let $u \in \mathcal{SH}_m^-(\Omega)$ and $(\Omega_j)_j$ be the sequence defined above. Take the function u^j defined by*

$$u^j = \sup\Big\{\psi \in \mathcal{SH}_m(\Omega): \ \psi_{|\Omega\setminus\Omega_j} \leq u\Big\} \in \mathcal{MSH}_m(\Omega)$$

and define $\widetilde{u} := (\lim_{j\to+\infty} u^j)^* \in \mathcal{SH}_m(\Omega)$.

If $u \in \mathcal{E}_m(\Omega)$ then by [3, 15] $\widetilde{u} \in \mathcal{E}_m(\Omega) \cap \mathcal{SH}_m(\Omega)$.

In [17], author introduced a new Cegrell class $\mathcal{N}_m(\Omega) := \{u \in \mathcal{E}_m : \ \widetilde{u} = 0\}$. It is easy to check that $\mathcal{N}_m(\Omega)$ is a convex cone satisfying

$$\mathcal{E}_m^0(\Omega) \subset \mathcal{F}_m(\Omega) \subset \mathcal{N}_m(\Omega) \subset \mathcal{E}_m(\Omega).$$

Definition 2.7. *Let $\mathcal{L}_m \in \{\mathcal{E}_m^0, \mathcal{F}_m, \mathcal{N}_m, \mathcal{E}_m\}$ and $H \in \mathcal{E}_m(\Omega) \cap \mathcal{MSH}_m(\Omega)$. A function $u \in \mathcal{SH}_m(\Omega)$ belongs to $\mathcal{L}_m(\Omega, H)$ ($\mathcal{L}_m(H)$ for short) if there exists $\psi \in \mathcal{L}_m$ satisfying $\psi + H \leq u \leq H$. We define*

$$\mathcal{N}_m^a(\Omega) := \{u \in \mathcal{N}_m : \ H_m(u)(M) = 0 \ \textit{for } m\textit{-polar set } M\}.$$

Definition 2.8. 1. *Let E be a Radon subset of Ω. The Cap_s-capacity of a E with respect to Ω is expressed as follows*:

$$\mathrm{Cap}_s(E) = \mathrm{Cap}_s(E, \Omega) = \sup\left\{\int\limits_E H_s(u) \ , \ u \in \mathcal{SH}_m(\Omega), -1 \leq u \leq 0\right\},$$

where $1 \leq s \leq m$.

2. *We say that a sequence $(u_j)_j$, of real-valued Radon measurable functions defined on Ω, converges to u in Cap_s-capacity, when $j \to +\infty$ if, for every compact subset K of Ω and $\varepsilon > 0$, the following limit holds*:

$$\lim_{j\to+\infty} \mathrm{Cap}_s(\{z \in K : |u_j(z) - u(z)| > \varepsilon\}) = 0.$$

For a given increasing function $\chi : \mathbb{R}^- \to \mathbb{R}^-$, Benelkourchi, Guedj and Zeriahi [2] introduced and investigated the fundamental weighted energy classes which was generalized by [12] as follows.

Definition 2.9. *We say that $u \in \mathcal{E}_{m,\chi}(\Omega)$ if and only if there exists $(u_j)_j \subset \mathcal{E}_m^0(\Omega)$ such that $u_j \searrow u$ in Ω and*

$$\sup_{j\in\mathbb{N}} \int\limits_\Omega -\chi(u_j) H_m(u_j) < +\infty.$$

The class $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_m(\Omega)$ when $\chi \not\equiv 0$ (see [11]).

3. The Hessian-type equation in the classes $\mathcal{E}_{m,F}(H,\Omega)$ and $\mathcal{E}_{m,\chi}(\Omega)$. Throughout this section we consider the function $F\colon \mathbb{R}^- \times \Omega \longrightarrow \mathbb{R}^+$ and μ a measure defined on Ω. This section is devoted to study the existence of the solution to the equation

$$\mathcal{H}_{m,F}(.) = \mu,$$

where $\mathcal{H}_{m,F}(u) := F(u(z),z)H_m(u)$. To simplify notation we set

$$\mathfrak{C}(\mathbb{R}^-) := \{\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^-; \chi \text{ is increasing, continuous and } \chi(t) < 0 \ \forall t < 0\}$$

and

$$\mathfrak{D}(\mathbb{R}^-,\Omega) := \{F : \mathbb{R}^- \times \Omega \longrightarrow \mathbb{R}^+; \text{ for all } z \in \Omega \text{ the function } F(.,z) \text{ is decreasing on } \Omega\}.$$

Definition 3.1. *For every $F \in \mathfrak{D}(\mathbb{R}^-,\Omega)$ and $H \in \mathcal{E}_m(\Omega) \cap \mathcal{MSH}_m(\Omega)$ we define*

$$\mathcal{E}_{m,F}(H,\Omega) := \left\{ \varphi \in \mathcal{N}_m(H) : \exists\, \mathcal{E}^0_m(H) \ni \varphi_j \searrow \varphi, \quad \sup_{j\geq 1} \int\limits_\Omega \mathcal{H}_{m,F}(\varphi_j) < +\infty \right\}.$$

Firstly, we will extend the well-known comparison principle to the operator $\mathcal{H}_{m,F}(.)$. Namely, we prove the following theorem.

Theorem 3.1. *Let $F \in \mathfrak{D}(\mathbb{R}^-,\Omega)$, $u \in \mathcal{N}^a_m(H)$ and $v \in \mathcal{E}_m(H)$. If $\mathcal{H}_{m,F}(u) \leq \mathcal{H}_{m,F}(v)$, then $u \geq v$.*

Proof. Since

$$H_m(u) \leq \frac{\mathcal{H}_{m,F}(v)}{F(u(z),z)} \leq H_m(v)$$

on $\{u < v\}$, then by Theorem 4.7 in [6] we obtain that $u \geq v$.

***Corollary* 3.1.** *Let $F_1, F_2 \in \mathfrak{D}(\mathbb{R}^-,\Omega)$, $u_1 \in \mathcal{N}^a_m(H)$ and $u_2 \in \mathcal{E}_m(H)$. If $\mathcal{H}_{m,F_1}(u_1) \leq \mathcal{H}_{m,F_2}(u_2)$ and $F_1 \leq F_2$, then $u_1 \geq u_2$.*

Proof. Using the hypothesis, it is easy to see that

$$\mathcal{H}_{m,F_1}(u_1) \leq \mathcal{H}_{m,F_2}(u_2) \leq \mathcal{H}_{m,F_1}(u_2).$$

The result follows using Theorem 3.1.

Theorem 3.2. *Assume that the measure μ is nonnegative, finite with no mass on every m-polar subset of Ω and $\inf_{z\in\Omega} F(t,z) > 0$ for all $t < 0$. Then there exists $u \in \mathcal{E}_{m,F}(H,\Omega)$ such that $\mathcal{H}_{m,F}(u) = \mu$. Moreover, the function u is unique.*

Proof. Using Theorem 1.7.1 in [15], there exist $g \in \mathcal{E}^0_m(\Omega)$ and $0 \leq f \in \mathbb{L}^1_{\text{loc}}(H_m(g))$ such that $fH_m(g) = \mu$.

Let $(\Omega_j)_j$ be the sequence defined in Definition 2.5 and take $\mu_j := 1_{\Omega_j} \min(f,j) H_m(g)$. Take $\xi_j \in \mathcal{C}^\infty(\mathbb{R}^- \times \Omega)$ such that $\xi_j \nearrow \dfrac{1}{F}$ and $\xi_j(.,z)$ is increasing for all $z \in \Omega$. Put $\xi := \dfrac{1}{F}$ and $F_j := \dfrac{1}{\xi_j}$. It is easy to see that the sequence $(F_j)_j$ decreases to F. So, by Proposition 3.4 in [1] there exists $u_j \in \mathcal{F}^a_m(H)$ satisfying $H_m(u_j) = \xi_j d\mu_j$.

We deduce that $\mathcal{H}_{m,F}(u_j) = d\mu_j$. It follows, by Corollary 3.1, that $u_j \searrow u$. We prove that $u \in \mathcal{E}_{m,F}(H,\Omega)$. For this it suffices, by definition, to show that $u_j \in \mathcal{E}^0_m(H)$, $u \in \mathcal{N}_m(H)$ and $\sup_{j\geq 1} \displaystyle\int_\Omega \mathcal{H}_{m,F}(u_j) < +\infty$. So, the proof will be computed in three steps.

Step 1. The proof of $u_j \in \mathcal{E}^0_m(H)$.

In this step we have to construct a sequence $v_j \in \mathcal{E}^0_m(\Omega)$ such that $H \geq u_j \geq H + v_j$. By [1], there exists $v_j \in \mathcal{F}^a_m(\Omega)$ such that $H_m(v_j) = \xi_j d\mu_j$. So, $\mathcal{H}_{m,F}(v_j) = d\mu_j$. Since the function $\xi_j(v_j(z), z), z \in \Omega_j$ is bounded from above and

$$H_m(v_j) = \xi_j(v_j(z), z)d\mu_j = \xi_j(v_j(z), z)1_{\Omega_j} \min(f, j)H_m(g),$$

then we deduce, using the comparison principle, that $v_j \in \mathcal{E}^0_m(\Omega)$. On the other hand, since we have by construction that $H + v_j \in \mathcal{F}^a_m(H)$ and

$$\mathcal{H}_{m,F_j}(u_j) = d\mu_j = \mathcal{H}_{m,F_j}(v_j) \leq \mathcal{H}_{m,F_j}(H + v_j),$$

we get by Theorem 3.1 that $u_j \geq H + v_j$. The proof of the first step is done.

Step 2. We prove that $u \in \mathcal{N}_m(H)$.

Set $v := \lim_{j\to\infty} v_j$ where $(v_j)_j$ is the sequence that appears in the first step. We have to prove first that $v \in \mathcal{N}_m(\Omega)$. By hypothesis we get $\inf_{z\in\Omega} F(t, z) > 0$ for all $t < 0$. So, following the same technics as in Theorem 3.2 of [6] it remains to prove that $\sup_{j\geq1} \int_\Omega \mathcal{H}_{m,F}(v_j) < +\infty$. By the same argument as in the first step, we deduce that the sequence $(v_j)_j$ is decreasing and

$$\sup_{j\geq1} \int_\Omega \mathcal{H}_{m,F}(v_j) \leq \sup_{j\geq1} \int_\Omega \mathcal{H}_{m,F_j}(v_j) = \sup_{j\geq1} \int_\Omega d\mu_j = \mu(\Omega).$$

Hence, we get that $v \in \mathcal{N}_m(\Omega)$. Again by the first step, we have $H \geq u_j \geq H + v_j$ for all j. It follows that $H \geq u \geq H + v$ and $u \in \mathcal{N}_m(H)$.

Step 3. The proof of $\sup_{j\geq1} \int_\Omega \mathcal{H}_{m,F}(u_j) < +\infty$.

Following the same reason as in the second step we get that

$$\sup_{j\geq1} \int_\Omega \mathcal{H}_{m,F}(u_j) \leq \sup_{j\geq1} \int_\Omega \mathcal{H}_{m,F_j}(u_j) = \sup_{j\geq1} \int_\Omega d\mu_j = \mu(\Omega).$$

The proof of the third step is done, and we deduce finally that $u \in \mathcal{E}_{m,F}(H, \Omega)$. To finish the proof of the theorem we observe that

$$\mathcal{H}_{m,F}(u) = \lim_{j\to\infty} \mathcal{H}_{m,F_j}(u_j) = \lim_{j\to\infty} d\mu_j = d\mu.$$

Now by Theorem 3.1 we get the uniqueness of u and the desired result follows.

Theorem 3.2 is proved.

Lemma 3.1. *Let* $u,\ v \in \mathcal{E}_m(\Omega)$ *and* $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^-$ *be an increasing continuous function with* $\chi(-\infty) > -\infty$. *If the nonnegative Radon measure* μ *has no mass on all* m*-polar subsets with* $-\chi(u)H_m(u) \geq \mu$ *and* $-\chi(v)H_m(v) \geq \mu$, *then*

$$-\chi(\max(u, v))H_m(\max(u, v)) \geq \mu.$$

Proof. Without loss of generality we may choose $\delta_j \searrow 0$ such that $\mu(\{u = v - \delta_j\}) = 0$ for all $j \geq 1$. By hypothesis, the function χ is an increasing function, so using Theorem 3.6 in [13] we get, for all $j \geq 1$, one has

$$\begin{aligned}\mu &= 1_{\{u>v-\delta_j\}}\mu + 1_{\{u<v-\delta_j\}}\mu \\ &\leq -1_{\{u>v-\delta_j\}}\chi(u)H_m(u) - 1_{\{u<v-\delta_j\}}\chi(v)H_m(v) \\ &\leq -1_{\{u>v-\delta_j\}}\chi(u)H_m(u) - 1_{\{u<v-\delta_j\}}\chi(v-\delta_j)H_m(v) \\ &\leq -\chi(\max(u, v-\delta_j))H_m(\max(u, v-\delta_j)).\end{aligned}$$

The result follows by letting $j \to \infty$ and using Theorem 4.11 in [11].

Proposition 3.1. *Let* $v \in \mathcal{F}_m(\Omega)$, $\chi \in \mathfrak{C}(\mathbb{R}^-)$ *with* $\chi(-\infty) > -\infty$. *Take* $\mathcal{A}(\sigma, v) = \{\varphi \in \mathcal{E}_m(\Omega) : \sigma \leq -\chi(\varphi)H_m(\varphi)\ ,\ \varphi \leq v\}$ *and a finite Radon measure* σ *which vanishes on* m*-polar sets of* Ω *such that*:

(1) $\operatorname{supp}\sigma \Subset \Omega$,

(2) $\operatorname{supp} H_m(v) \Subset \Omega$ *and* $H_m(v)$ *is carried by a* m*-polar set.*

Then the function u *defined by* $u := (\sup\{\varphi : \varphi \in \mathcal{A}(\sigma, v)\})^\star$ *belongs to* $\mathcal{F}_m(\Omega)$. *Moreover,* $-\chi(u)H_m(u) = \sigma + H_m(v)$.

Proof. Without loss of generality we can assume that $\chi(-\infty) = -1$. We prove first that $u \in \mathcal{F}_m(\Omega)$. Theorem 3.2 implies the existence of $f \in \mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{N}_m(\Omega)$ satisfying $-\chi(f)H_m(f) = \sigma$.

By hypothesis we have $\operatorname{supp} H_m(f) = \operatorname{supp} \dfrac{\sigma}{-\chi(f)} \Subset \Omega$, so $\displaystyle\int_\Omega H_m(f) < +\infty$. It follows that $f \in \mathcal{F}^a_m(\Omega)$. Since $\sigma \leq -\chi(f+v)H_m(f+v)$, then the function $(f+v) \in \mathcal{A}(\sigma, v)$ and $f + v \leq u \leq v$. Finally, we obtain that $u \in \mathcal{F}_m(\Omega)$ and the proof of the first assertion of the theorem is completed.

Now we prove that $-\chi(u)H_m(u) = \sigma + H_m(v)$. For this we prove first that $\sigma + H_m(v) \leq -\chi(u)H_m(u)$. Using Lemma 3.1, we deduce that, for every $\varphi, \psi \in \mathcal{A}(\sigma, v)$ one has $\max(\varphi, \psi) \in \mathcal{A}(\sigma, v)$. Using the Choquet lemma we deduce the existence of a sequence $(u_j) \subset \mathcal{A}(\sigma, v)$ satisfying $u = (\sup_{\in \mathbb{N}^*} u_j)^*$. Take $\tilde{u}_j = \max\{u_1, ..., u_j\} \in \mathcal{A}(\sigma, v)$. We get that $\tilde{u}_j \nearrow u$ almost everywhere. By Theorem 4.11 in [11] we obtain the weak convergence of $-\chi(\tilde{u}_j)H_m(\tilde{u}_j)$ toward $-\chi(u)H_m(u)$. So

$$\sigma \leq -\chi(u)H_m(u) \tag{2}$$

and $u \in \mathcal{A}(\sigma, v)$.

On the other hand, we have

$$\begin{aligned}-\chi(u)H_m(u) &= -1_{\{u=-\infty\}}\chi(u)H_m(u) - 1_{\{u>-\infty\}}\chi(u)H_m(u) \\ &= 1_{\{u=-\infty\}}H_m(u) - 1_{\{u>-\infty\}}\chi(u)H_m(u).\end{aligned}$$

Using Proposition 5.2 in [13] we

$$H_m(v) = 1_{\{v=-\infty\}}H_m(v) \leq 1_{\{u=-\infty\}}H_m(u) \leq -\chi(u)H_m(u). \tag{3}$$

By combining (2) and (3), we get $\sigma + H_m(v) \leq -\chi(u)H_m(u)$. It remain to prove the converse inequality. Namely, we have to prove that $-\chi(u)H_m(u) \leq \sigma + H_m(v)$.

Take Ω_1 a m-hyperconvex domain and $(v_j) \subset \mathcal{E}_m^0(\Omega)$ such that $\operatorname{supp}\sigma \cup \operatorname{supp} H_m(v) \Subset \Omega_1 \Subset \Omega$, $v_j \searrow v$ in Ω_1 and $\operatorname{supp} H_m(v_j) \subset \overline{\Omega}_1$. As

$$\int\limits_{\Omega} \sigma - \chi(v_j)H_m(v_j) \leq \int\limits_{\Omega} \sigma + H_m(v_j) < +\infty,$$

so Theorem 3.2 ensures the existence and the uniqueness of $w_j \in \mathcal{E}_{m,\chi}(\Omega)$ satisfying $\sigma - \chi(v_j)H_m(v_j) = -\chi(w_j)H_m(w_j)$. It follows that $-\chi(v_j)H_m(v_j) \leq -\chi(w_j)H_m(w_j) \leq -\chi(f + v_j)H_m(f + v_j)$. By Corollary 3.1 we get that $f + v_j \leq w_j \leq v_j$. So we deduce that $w_j \in \mathcal{A}(\sigma, v_j)$ and $w_j \in \mathcal{F}_m(\Omega)$. If we set that

$$u_j = (\operatorname{supp}\{\varphi \colon \varphi \in \mathcal{A}(\sigma, v_j)\})^*,$$

then using the same argument as above we get that $u_j \in \mathcal{F}_m(\Omega)$, and by definition of the class $\mathcal{A}(\sigma, v_j)$ we deduce that, for all $j \geq 1$, $u_j \geq w_j$. Moreover, $u_j \searrow u$ when $j \to +\infty$. We claim that $w_j \to u$ in Cap_{m-1}-capacity. To prove the claim it suffices to show that $u_j - w_j \to 0$ in Cap_{m-1}-capacity.

Let $g \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}^\infty(\Omega)$ be a strictly m-sh function. For $\delta > 0$ and $j_0 \geq 1$, by Proposition 5.3 in [13], we have

$$\begin{aligned}
&\operatorname{Cap}_{m-1}(\{u_j - w_j > \delta\}) \\
&\quad = \sup\left\{ \int\limits_{\{u_j - w_j > \delta\}} dd^c g \wedge (dd^c\phi)^{m-1} \wedge \beta^{n-m} : \phi \in \mathcal{SH}_m(\Omega), -1 \leq \phi \leq 0 \right\} \\
&\quad \leq \sup\left\{ \frac{1}{\delta^m} \int\limits_{\{u_j - w_j > \delta\}} (u_j - w_j)^m dd^c g \wedge (dd^c\phi)^{m-1} \wedge \beta^{n-m} : \phi \in \mathcal{SH}_m(\Omega), -1 \leq \phi \leq 0 \right\} \\
&\quad \leq \sup\left\{ \frac{1}{\delta^m} \int\limits_{\Omega} (u_j - w_j)^m dd^c g \wedge (dd^c\phi)^{m-1} \wedge \beta^{n-m} : \phi \in \mathcal{SH}_m(\Omega), -1 \leq \phi \leq 0 \right\} \\
&\quad \leq \frac{m!}{\delta^m} \int\limits_{\Omega} -g(H_m(w_j) - H_m(u_j)) \leq \frac{m!}{\delta^m} \int\limits_{\Omega} -gH_m(w_j) + \frac{m!}{\delta^m} \int\limits_{\Omega} gH_m(u) \\
&\quad \leq \frac{m!}{\delta^m} \int\limits_{\Omega} -g\frac{-\chi(w_j)H_m(w_j)}{-\chi(u_j)} + \frac{m!}{\delta^m} \int\limits_{\Omega} gH_m(u) \\
&\quad \leq \frac{m!}{\delta^m} \int\limits_{\Omega} -g\frac{\sigma + H_m(v_j)}{-\chi(u_j)} + \frac{m!}{\delta^m} \int\limits_{\Omega} gH_m(u) \\
&\quad \leq \frac{m!}{\delta^m} \int\limits_{\Omega} -g\frac{\sigma + H_m(v_j)}{-\chi(u_{j_0})} + \frac{m!}{\delta^m} \int\limits_{\Omega} gH_m(u).
\end{aligned}$$

Since $\operatorname{supp}\sigma \cup \operatorname{supp} H_m(v_j) \Subset \overline{\Omega}_1$, we get that

$$\begin{aligned}
&\limsup_{j\to\infty} \operatorname{Cap}_{m-1}(\{u_j - w_j > \delta\}) \\
&\qquad \le \frac{m!}{\delta^m} \int\limits_\Omega -g \frac{\sigma + H_m(v)}{-\chi(u_{j_0})} + \frac{m!}{\delta^m} \int\limits_\Omega g H_m(u) \\
&\qquad \le \limsup_{j_0\to\infty} \frac{m!}{\delta^m} \int\limits_\Omega -g \frac{\sigma + H_m(v)}{-\chi(u_{j_0})} + \frac{m!}{\delta^m} \int\limits_\Omega g H_m(u) \\
&\qquad = \frac{m!}{\delta^m} \int\limits_\Omega -g \left(\frac{\sigma + H_m(v)}{-\chi(u)} - H_m(u) \right) \le 0.
\end{aligned}$$

Note that in the last inequality we used the Lebesgue monotone convergence theorem. This proves the claim. Now using Theorem 4.11 in [11] we finally obtain

$$-\chi(u) H_m(u) \le \liminf_{j\to\infty} -\chi(w_j) H_m(w_j).$$

It follows that $-\chi(u) H_m(u) \le \sigma + H_m(v)$. In conclusion we get

$$-\chi(u) H_m(u) = \sigma + H_m(v).$$

Proposition 3.1 is proved.

The following theorem is the main result in this section. We prove the existence of a solution for the Hessian equation with respect to the operator $-\chi(.)H_m(.)$. This result is an extension of Theorem 5.9 in [13], it suffices to take $\chi \equiv -1$ to recover it.

Theorem 3.3. *Let $\chi \in \mathfrak{C}(\mathbb{R}^-)$ and μ be a Radon measure. Assume that*

(1) *there exists $w \in \mathcal{E}_{m,\chi}(\Omega)$ such that $\mu \le -\chi(w) H_m(w)$,*

(2) $\mu(\Omega) < +\infty$.

Then there exists $u \in \mathcal{E}_{m,\chi}(\Omega)$ such that $-\chi(u) H_m(u) = \mu$. Moreover, $u \ge w$.

Proof. Assume first that $\chi(-\infty) = -\infty$. So by Proposition 4.4 in [11] we deduce that $w \in \mathcal{E}_m^a(\Omega)$. Hence, the measure μ has no mass on all m-polar sets of Ω. So Theorem 3.2 guarantees the existence of $u \in \mathcal{E}_{m,\chi}(\Omega)$ such that $-\chi(u) H_m(u) = \mu$. The fact that $u \ge w$ follows directly using Corollary 3.1 and Corollary 3.3 in [6]. The proof is completed when $\chi(-\infty) = -\infty$.

In the general case $\chi(-\infty) > -\infty$, using Theorem 3.5 in [8] the measure μ can be written as follows: $\mu = \sigma + \nu$, where σ and ν are Radon measures defined on Ω such that σ vanishes on all m-polar sets and ν is carried by a m-polar set. By hypothesis we have $\nu \le -\chi(w) H_m(w) \le H_m(w)$, so using Theorem 4.7 in [8] there exists $v \in \mathcal{N}_m(\Omega)$ such that $H_m(v) = \nu$, $v \ge w$ and $H_m(v)$ is carried by the m-polar set $\{v = -\infty\}$.

Let $(\Omega_j)_j$ be an increasing sequence $\Omega_j \Subset \Omega$ and $\Omega_j \nearrow \Omega$ when $j \nearrow \infty$. Using case 1 of the proof of Proposition 5.17 in [13], there exists a decreasing sequence $v_j \in \mathcal{F}_m(\Omega)$, $v_j \ge v$ and $H_m(v_j) = 1_{\Omega_j} H_m(v) = 1_{\Omega_j}\nu$. Now if we take $\sigma_j := 1_{\Omega_j}\sigma$ and $u_j := \sup\{\varphi : \varphi \in \mathcal{A}(\sigma_j, v_j)\}$, then by Proposition 3.1 we have $-\chi(u_j) H_m(u_j) = \sigma_j + H_m(v_j)$. We deduce that $w \in \mathcal{A}(\sigma_j, v_j)$ so $u_j \ge w$ for every j. It follows that $u_j \searrow u \ge w$. Now as $\sigma_j + H_m(v_j) \to \sigma + H_m(v)$ weakly so by Theorem 4.11 in [11] we get

$$-\chi(u)H_m(u) = \sigma + H_m(v) = \mu.$$

Theorem 3.3 is proved.

***Corollary* 3.2.** *Let $\chi \in \mathfrak{C}(\mathbb{R}^-)$, $\Omega_1 \Subset \Omega_2 \subset \Omega$ be bounded m-hyperconvex domains and $u \in \mathcal{E}_{m,\chi}(\Omega_2)$. Then there exists $\tilde{u} \in \mathcal{E}_{m,\chi}(\Omega)$ such that $-\chi(\tilde{u})H_m(\tilde{u}) = 1_{\Omega_1}(-\chi(u))H_m(u)$ on Ω.*

Proof. Assume first that $\chi(-\infty) = -\infty$ so by Proposition 4.4 in [11] one has that $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_m^a(\Omega)$. It follows that the nonnegative measure $\mu = -1_{\Omega_1}\chi(u)H_m(u)$ has no mass on all m-polar sets of Ω and Theorem 3.2 guarantee the existence of $\tilde{u} \in \mathcal{E}_{m,\chi}(\Omega)$ satisfying $-\chi(\tilde{u})H_m(\tilde{u}) = -1_{\Omega_1}\chi(u)H_m(u)$. The proof of the theorem is done in the case $\chi(-\infty) = -\infty$.

In the general case $\chi(-\infty) > -\infty$. Using Corollary 3.3 in [6], we get that $\mathcal{E}_{m,\chi}(\Omega_2) \subset \mathcal{N}_m(\Omega_2) \subset \mathcal{E}_m(\Omega_2)$, hence, there is exists $u_1 \in \mathcal{F}_m(\Omega_2)$ such that $u = u_1$ on Ω_2. If we take

$$u_2 = \sup\{\psi \in \mathcal{SH}_m^-(\Omega) : \psi \le u_1 \text{ on } \Omega_2\},$$

then $u_2 \in \mathcal{F}_m(\Omega)$ and Lemma 3.2 in [10] implies that $H_m(u_2) \le 1_{\Omega_2}H_m(u_1)$ on Ω. Now as $u_2 \le u_1$ in Ω_2, then by Proposition 5.2 in [13] one can obtain that

$$1_{\{u_1=-\infty\}}H_m(u_1)) \le 1_{\{u_2=-\infty\}}H_m(u_2 \text{ on } \Omega_2.$$

It follows that

$$1_{\Omega_2}1_{\{u_1=-\infty\}}H_m(u_1) = 1_{\{u_2=-\infty\}}H_m(u_2) \text{ on } \Omega.$$

On the other hand, $-1_{\Omega_1\cap\{u>-\infty\}}\chi(u)H_m(u)(M) = 0$ for every m-polar set $M \subset \Omega$ and $\int_\Omega -1_{\Omega_1\cap\{u>-\infty\}}\chi(u)H_m(u) \le \int_{\Omega_2}(-\chi(u))H_m(u) < +\infty$, so Theorem 3.2 ensures the existence of $w \in \mathcal{E}_{m,\chi}(\Omega)$ with

$$-\chi(w)H_m(w) = -1_{\Omega_1\cap\{u>-\infty\}}\chi(u)H_m(u).$$

Now if we set $\tilde{w} = \sup\{\psi \in \mathcal{SH}_m^-(\Omega) : \psi \le w \text{ on } \Omega_2\}$, then $\tilde{w} \in \mathcal{F}_m(\Omega)$, $\tilde{w} \ge w$ and $w = \tilde{w}$ in Ω_2. As

$$-\chi(w)H_m(w) = -1_{\Omega_1}\chi(w)H_m(w) = -1_{\Omega_1}\chi(\tilde{w})H_m(\tilde{w}) \le -\chi(\tilde{w})H_m(\tilde{w}),$$

then by Theorem 3.1 we obtain that $w \ge \tilde{w}$. It follows that $w = \tilde{w}$. Now, since $u_1 = u$ on Ω_1, then

$$\begin{aligned}
-1_{\Omega_1}\chi(u)H_m(u) &= 1_{\Omega_1\cap\{u>-\infty\}}(-\chi(u))H_m(u) + 1_{\Omega_1\cap\{u=-\infty\}}(-\chi(u))H_m(u) \\
&= 1_{\Omega_1\cap\{u>-\infty\}}(-\chi(u))H_m(u) + 1_{\Omega_1\cap\{u_1=-\infty\}}(-\chi(u_1))H_m(u_1) \\
&\le -\chi(w)H_m(w) - \chi(u_2)H_m(u_2) \\
&\le -\chi(w+u_2)(H_m(w) + H_m(u_2)) \\
&\le -\chi(w+u_2)H_m(w+u_2).
\end{aligned}$$

As $w,\ u_2 \in \mathcal{F}_m(\Omega)$, then $w + u_2 \in \mathcal{F}_m(\Omega)$. It follows that $w + u_2 \in \mathcal{E}_{m,\chi}(\Omega)$ and Theorem 3.3 gives the existence of $\tilde{u} \in \mathcal{E}_{m,\chi}(\Omega)$ satisfying $-\chi(\tilde{u})H_m(\tilde{u}) = 1_{\Omega_1}(-\chi(u))H_m(u)$ on Ω.

Corollary 3.2 is proved.

***Corollary* 3.3.** *Let $v \in \mathcal{F}_m(\Omega)$, $f \in L^1_{\text{loc}}(H_m(v))$ with $f \ge 0$ and $\chi \in \mathfrak{C}(\mathbb{R}^-)$. If $\chi(-\infty) > -\infty$, then there exists a decreasing sequence $u_j \in \mathcal{F}_m(\Omega)$ such that $\operatorname{supp}(H_m(u_j)) \Subset \Omega$ and $-\chi(u_j)H_m(u_j) \nearrow fH_m(v)$ as $j \to +\infty$.*

Proof. Let $(\Omega_j)_j$ be an increasing sequence satisfying $\Omega_j \to \Omega$ when $j \nearrow \infty$ and $\Omega_j \Subset \Omega$ for every $j \geq 1$. For every $j \in \mathbb{N}^*$, take $\sigma_j := 1_{\Omega_j \cap \{v>-\infty\}} \min(f,j) H_m(v)$, $u_j = \sup\{\varphi\colon \varphi \in \mathcal{A}(\sigma_j, v^{g_j})\}^*$ and $g_j := 1_{\Omega_j \cap \{v=-\infty\}} \min(f,j)$. Using [8], we have that $v^{g_j} \in \mathcal{F}_m(\Omega)$.

So using Proposition 3.1, we obtain that $u_j \in \mathcal{F}_m(\Omega)$ and

$$-\chi(u_j) H_m(u_j) = \sigma_j + H_m(v^{g_j}) = 1_{\Omega_j \cap \{v=-\infty\}} \min(f,j) H_m(v)$$

$$+ 1_{\Omega_j \cap \{v>-\infty\}} \min(f,j) H_m(v) = 1_{\Omega_j} \min(f,j) H_m(v). \tag{4}$$

Hence, $\displaystyle\int_\Omega -\chi(u_j) H_m(u_j) < +\infty$, and we deduce that $u_j \in \mathcal{E}_{m,\chi}(\Omega)$. To obtain the desired result it suffices to prove that (u_j) is a decreasing sequence. Observe by [8] that the sequence (v^{g_j}) is decreasing so $u_{j+1} \leq v^{g_{j+1}} \leq v^{g_j}$. Moreover,

$$\sigma_j = 1_{\Omega_j \cap \{v>-\infty\}} \min(f,j) H_m(v)$$

$$\leq 1_{\Omega_{j+1} \cap \{v>-\infty\}} \min(f,j+1) H_m(v) = \sigma_{j+1} \leq -\chi(u_{j+1}) H_m(u_{j+1}).$$

We deduce that $u_{j+1} \in \mathcal{A}(\sigma_j, v^{g_j})$ and hence $u_{j+1} \leq u_j$. We obtain finally that (u_j) is a decreasing sequence. The result follows using (4) since we get that $\operatorname{supp}(H_m(u_j)) \Subset \Omega$ and $-\chi(u_j) H_m(u_j) \nearrow f H_m(v)$, as $j \to +\infty$.

Corollary 3.3 is proved.

4. Local subsolution problem for the Hessian equation. In this section μ be nonnegative measure defined on Ω.

Proposition 4.1. *Assume that, for every $z \in \Omega$, there exists $u_z \in \mathcal{E}_m(U_z)$ for some neighborhood U_z of z and satisfying $\mu \leq H_m(u_z)$ in U_z. Then there exist $g \in \mathcal{F}_m(\Omega)$ and $0 \leq f \in \mathbb{L}^1_{\text{loc}}(H_m(g))$ such that $f H_m(g) = \mu$.*

Proof. Fix $z \in \Omega$, and choose m-hyperconvex domains O_z and G_z such that $z \in O_z \Subset G_z \Subset U_z$. Take $w_z \in \mathcal{F}_m(U_z)$ satisfying $w_z = u_z$ in O_z. By Corollary 3.2 in the case when $\chi(t) \equiv -1$, there exists $v_z \in \mathcal{F}_m(\Omega)$ such that $\mu \leq H_m(v_z) = H_m(w_z) = H_m(u_z)$ on O_z.

Consider $(\Omega_j)_j$ the sequence of subsets as in Definition 2.5. Since the subsets $\overline{\Omega}_j$ are compact then by the construction done before, one can find $g_j \in \mathcal{F}_m(\Omega)$ satisfying $H_m(g_j) \geq \mu_{|\overline{\Omega}_j}$. Take $a_j := \dfrac{\varphi_j}{2^j \displaystyle\int_\Omega H_m(g_j)}$ and set g as follows: $g = \sum_{j=1}^{+\infty} a_j g_j$. By the proof of Theorem 5.12 in [13] we get that $g \in \mathcal{F}_m(\Omega)$ and, hence, $\mu << H_m(g)$. It follows that there exists $0 \leq f \in \mathbb{L}^1_{\text{loc}}(H_m(g))$ satisfying $\mu = f H_m(g)$.

Proposition 4.2. *Let χ be an increasing convex function such that $\chi(-\infty) > -\infty$ and $\chi(t) < 0$ for all $t < 0$. If $\mu(\Omega) < +\infty$, then the following assertion are equivalent*:

(i) *for every $z \in \Omega$ there exist a neighborhood U_z of z and $v_z \in \mathcal{E}_m(U_z)$ such that $\mu \leq H_m(v_z)$ in U_z,*

(ii) *there exists $u \in \mathcal{E}_{m,\chi}(\Omega)$ such that $-\chi(u) H_m(u) = \mu$.*

Proof. The proof of (ii)⇒(i) is obvious.

Now we prove (i)⇒(ii). By combining Proposition 4.1 and Corollary 3.3, we obtain the existence of a decreasing sequence $(u_j)_j \subset \mathcal{F}_m(\Omega)$ such that $-\chi(u_j) H_m(u_j) \nearrow \mu$ when $j \to +\infty$. Set $u := \lim_{j\to+\infty} u_j$. Using Theorem 1.7.1 [15] one can construct a sequence $(v_j)_j \subset \mathcal{E}_0^m(\Omega) \cap \mathcal{C}(\Omega)$ that decreases to u and $w_j := \max(v_j, u_j)$. It easy to check that $w_j \in \mathcal{E}_0^m(\Omega)$ and w_j decreases u.

Now by Lemma 2.7 in [8], we have

$$\int_\Omega -\chi(w_j)H_m(w_j) \le \int_\Omega -\chi(w_j)H_m(u_j) \le \int_\Omega -\chi(u_j)H_m(u_j) \le \mu(\Omega).$$

It follows that $u \in \mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_m(\Omega)$. Moreover, applying Theorem 4.11 in [11], we deduce the weak convergence of $-\chi(u_j)H_m(u_j)$ to $-\chi(u)H_m(u)$ and, hence, $-\chi(u)H_m(u) = \mu$.

Now we solve the Dirichlet problem in the class $\mathcal{E}_m(\Omega)$. Namely, we have the following theorem.

Theorem 4.1. *Assume that the following conditions hold*:

(1) *there exists* $\varphi \in \mathcal{E}_0^m(\Omega)$ *such that* $\int_\Omega -\varphi d\mu < +\infty$,

(2) *for every* $z \in \Omega$ *there exist a neighborhood* U_z *of* z *and* $v_z \in \mathcal{E}_m(U_z)$ *such that* $\mu \le H_m(v_z)$ *in* U_z.

Then there exists a function $u \in \mathcal{N}_m(\Omega)$ *such that* $H_m(u) = \mu$.

Proof. Using Proposition 4.1 and Corollary 3.3 we get the existence of a decreasing sequence $(u_j)_j \subset \mathcal{F}_m(\Omega)$ such that the measure $H_m(u_j) \nearrow \mu$ when $j \to +\infty$. Set $u := \lim_{j\to+\infty} u_j$ and take $O \Subset G \Subset \Omega$. If we consider

$$v_j := \sup\{h \in \mathcal{SH}_m^-(\Omega) \colon h \le u_j \text{ on } O\} \in \mathcal{F}_m(\Omega),$$

then $H_m(v_j) = 0$ on $\Omega \setminus \overline{0}$, and by Lemma 2.7 in [8] we have

$$\int_\Omega -\varphi H_m(v_j) \le \int_\Omega -\varphi H_m(u_j) \le \int_\Omega -\varphi d\mu < +\infty.$$

It follows that for $j \ge 1$ one has

$$\int_\Omega H_m(v_j) < +\infty.$$

Hence by [15], we obtain that $v = \lim_{j\to+\infty} v_j \in \mathcal{F}_m(\Omega)$. Now as $u = v$ on O so $u \in \mathcal{E}_m(\Omega)$ and $H_m(u) = \mu$. To prove the desired result, it remains to show that $u \in \mathcal{N}_m(\Omega)$. Without loss of generality one can assume that φ is a strictly m-sh function with $-1 \le \varphi < 0$. Take $(\Omega_k)_k$ as in Definition 2.5 and

$$u_j^k := \sup\{h \in \mathcal{SH}_m^-(\Omega) \colon h \le u_j \text{ on } \Omega \setminus \overline{\Omega}_k\}.$$

Using the fact that $u_j^k \searrow u^k$ when $j \to +\infty$ and $u^k \nearrow \widetilde{u}$ as $k \to +\infty$, one can find a sequence $j_k \to +\infty$ such that $u_{j_k}^k$ converges a.e. to $\widetilde{u}$. If we denote by

$$\varphi^k = \sup\{h \in \mathcal{SH}_m^-(\Omega) \colon h \le \varphi \text{ on } \Omega \setminus \overline{\Omega}_k\},$$

then by Proposition 5.3 in [13] we get

$$\int_\Omega (-u_{j_k}^k)^m H_m(\varphi) \le m! \int_\Omega -\varphi H_m(u_{j_k}^k) = m! \int_\Omega -\varphi^{k-1} H_m(u_{j_k}^k).$$

If we combine the previous inequality with the fact that $u_{j_k}^k \ge u_{j_k}$, then by Lemma 2.7 in [8] we deduce that

$$\int\limits_{\Omega}(-u_{j_k}^k)^m H_m(\varphi) \leq m!\int\limits_{\Omega} -\varphi^{k-1}H_m(u_{j_k}) \leq m!\int\limits_{\Omega} -\varphi^{k-1}H_m(u).$$

Finally, if $k \to +\infty$, then, by the Lebesgue convergence theorem, we infer that

$$\int\limits_{\Omega}(-\widetilde{u})^m H_m(\varphi) = 0.$$

So $\widetilde{u} = 0$ and, hence, $u \in \mathcal{N}_m(\Omega)$.

Theorem 4.1 is proved.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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