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UNIVALENCE CRITERIA FOR LOCALLY UNIVALENT ANALYTIC FUNCTIONS

КРИТЕРІЇ ОДНОЗНАЧНОСТІ ДЛЯ ЛОКАЛЬНО ОДНОЗНАЧНИХ АНАЛІТИЧНИХ ФУНКЦІЙ

Suppose that $p(z) = 1 + z\phi''(z)/\phi'(z)$, where $\phi(z)$ is a locally univalent analytic function in the unit disk $\mathbf{D}$ with $\phi(0) = \phi'(1) - 1 = 0$. We establish the lower and upper bounds for the best constants σ_0 and σ_1 such that $e^{-\sigma_0/2} < |p(z)| < e^{\sigma_0/2}$ and $|p(w)/p(z)| < e^{\sigma_1}$ for $z, w \in \mathbf{D}$, respectively, imply the univalence of $\phi(z)$ in $\mathbf{D}$.

Припустимо, що $p(z) = 1 + z\phi''(z)/\phi'(z)$, де $\phi(z)$ — локально однозначна аналітична функція в одиничному диску $\mathbf{D}$ з $\phi(0) = \phi'(1) - 1 = 0$. Отримано нижню та верхню оцінки для найкращих сталих σ_0 та σ_1, таких що $e^{-\sigma_0/2} < |p(z)| < e^{\sigma_0/2}$ і $|p(w)/p(z)| < e^{\sigma_1}$ для $z, w \in \mathbf{D}$ відповідно означають однозначність $\phi(z)$ в $\mathbf{D}$.

1. Introduction. Let $\mathbf{D}$ be the unit disk in the complex plane $\mathbf{C}$. Suppose that ϕ is a locally univalent analytic function in $\mathbf{D}$, the pre-Schwarzian derivative P_ϕ of ϕ is defined by $P_\phi = \phi''/\phi'$. It is well-known that P_ϕ plays an important role in the study of univalent functions and Teichmüller space (see [10]).

Using P_ϕ, in 1972, Becker [2] stated that if a locally univalent analytic function ϕ in $\mathbf{D}$ satisfies

$$|zP_\phi(z)|(1 - |z|^2) \leq 1, \quad z \in \mathbf{D}, \tag{1.1}$$

then ϕ is univalent in $\mathbf{D}$. In addition to this criterion, there are also some other criteria by Schwarzian derivatives for univalence of locally univalent analytic functions (see [1, 11 – 14]). By the first order derivatives of ϕ and the quantity of $z\phi'/\phi$, John [6], Gevirtz [4, 5], Kim and Sugawa [9] obtained some criteria for univalence of locally univalent analytic functions.

Let $\mathcal{A}$ be the class of locally univalent analytic functions ϕ in $\mathbf{D}$ with $\phi(0) = \phi'(0) - 1 = 0$, and let

$$M(\phi) = \sup_{z\in\mathbf{D}} |\phi'(z)|, \qquad m(\phi) = \inf_{z\in\mathbf{D}} |\phi'(z)|.$$

Using the quantity ϕ', John [6] proved the following univalence criterion.

Theorem A [6]. *There exists a number* $\gamma \in \left[\dfrac{\pi}{2}, \log(97 + 56\sqrt{3})\right]$ *such that if* $\phi \in \mathcal{A}$ *satisfies* $M(\phi) \leq e^\gamma m(\phi)$, *then* ϕ *is univalent in* $\mathbf{D}$.

The largest possible constant γ in Theorem A is called the logarithmic John constant and we denote it by γ_1. Yamashita [17] improved Theorem A and pointed that $\gamma_1 \leq \pi$. Gevirtz [4, 5] further improved that $\gamma_1 \leq \alpha\pi$, where $\alpha \approx 0.627834$ is the root of the equation

$$\frac{\pi}{e^{2\pi\alpha} - 1} = \sum_{n=1}^{\infty} \frac{n}{n^2 + \alpha^2} \exp\Big(-\frac{n\pi}{2\alpha}\Big). \tag{1.2}$$

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Theorem A and the papers [4, 5] imply that there exists a number $\gamma \in \left[\frac{\pi}{2}, \alpha\pi\right)$, where $\alpha \approx 0.627834$ is given by (1.2), such that if $\phi \in \mathcal{A}$ satisfies $e^{-\gamma/2} < |\phi'(z)| < e^{\gamma/2}$ in $\mathbf{D}$, then ϕ is univalent in $\mathbf{D}$.

In fact, in addition to the quantity ϕ', there are other quantities $z\phi'/\phi$ and $1 + z\phi''/\phi'$, which can be used to give criterion for univalence of locally univalent analytic function. As we know that for $\phi \in \mathcal{A}$, if $\Re\{z\phi'/\phi\} > 0$ and $\Re\{1 + z\phi''/\phi'\} > 0$, then ϕ is, respectively, starlike and convex in $\mathbf{D}$ (see [16]). For $\phi \in \mathcal{A}$, let h represent one of the quantities ϕ', $z\phi'/\phi$ and $1 + z\phi''/\phi'$. If $h \in \left\{w \in \mathbf{C} \colon \left|\frac{w-1}{w+1}\right| \le k < 1\right\}$, then ϕ is univalent in $\mathbf{D}$ and has a quasiconformal extension to $\mathbf{C}$ (see [3, 7, 15] for more details).

Let

$$L(\phi) = \sup_{z\in\mathbf{D}} \left|z\phi'(z)/\phi(z)\right|, \qquad l(\phi) = \inf_{z\in\mathbf{D}} \left|z\phi'(z)/\phi(z)\right|.$$

Instead of ϕ' by $z\phi'/\phi$, Kim and Sugawa [9] obtained a similar result to Theorem A as follows.

Theorem B [9]. 1. *There exists a number $\delta > 0$ such that if $\phi \in \mathcal{A}$ satisfies $e^{-\delta/2} < |z\phi'(z)/\phi(z)| < e^{\delta/2}$ in $\mathbf{D}$, then ϕ is univalent in $\mathbf{D}$. Let δ_1 denote the largest possible numbers δ, then $\frac{\pi}{3} < \delta_1 < \frac{5\pi}{7}$.*

2. *There exists a number $\delta > 0$ such that if $\phi \in \mathcal{A}$ satisfies $L(\phi) \le e^{\delta} l(\phi)$, then ϕ is univalent in $\mathbf{D}$. Let δ_0 denote the largest possible numbers δ, then $\frac{7\pi}{25} < \delta_0 < \frac{5\pi}{7}$.*

In light of these results, naturally, a question arises: can we substitute $1 + z\phi''/\phi'$ for ϕ' in Theorem A or $z\phi'/\phi$ in Theorem B? Theorem 1.1 will give an affirmative answer to this question.

Let $\phi \in \mathcal{A}$, we set

$$S(\phi) = \sup_{z\in\mathbf{D}} \left|1 + z\phi''(z)/\phi'(z)\right|, \qquad s(\phi) = \inf_{z\in\mathbf{D}} \left|1 + z\phi''(z)/\phi'(z)\right|.$$

Here $0 \le s(\phi) \le 1 \le S(\phi) \le +\infty$ since $1 + z\phi''(z)/\phi'(z) = 1$, when $\phi \in \mathcal{A}$ and $z = 0$. Now, we state our result as follows.

Theorem 1.1. 1. *There exists a number $\sigma > 0$ such that if $\phi \in \mathcal{A}$ satisfies $e^{-\sigma/2} < |1 + z\phi''(z)/\phi'(z)| < e^{\sigma/2}$ in $\mathbf{D}$, then ϕ is univalent in $\mathbf{D}$. Let σ_0 denotes the largest possible numbers σ, then $\sigma_l \le \sigma_0 < \frac{8}{5}\pi$, where $\sigma_l \approx 1.586795$ is given by*

$$\frac{2\sigma}{\pi} e^{\sigma/2} \frac{1 - x_0^2}{2x_0} = 1$$

and $x_0 \approx 0.647918$ is given by

$$1 - 2x \operatorname{arctanh}(x) = 0 \tag{1.3}$$

in $x \in (0, 1)$.

2. *There exists a number $\sigma > 0$ such that if $\phi \in \mathcal{A}$ satisfies $S(\phi) \le e^{\sigma} s(\phi)$, then ϕ is univalent in $\mathbf{D}$. Let σ_1 denote the largest possible numbers σ, then $\sigma_L \le \sigma_1 < \frac{8}{5}\pi$, where $\sigma_L \approx 1.131536$ is given by*

$$\frac{2\sigma}{\pi}e^{\sigma}\frac{1-x_0^2}{2x_0}=1$$

and x_0 *is given by* (1.3).

This paper is organized as follows. In Section 2, we give some preliminaries, which include some definitions and lemmas. Using the idea of the proof in [9], we shall prove Theorem 1.1 in Section 3.

2. Preliminaries. In this section, we give some preliminaries for proving our result. We firstly recall the basic hyperbolic geometry of the unit disc $\mathbf{D}$. The hyperbolic distance between two points $z_1, z_2 \in \mathbf{D}$ is defined by

$$d(z_1,z_2)=\inf_{\iota}\int_{\iota}\frac{|dz|}{1-|z|^2},$$

where the infimum is taken over all rectifiable paths ι joining z_1 and z_2 in $\mathbf{D}$. The Schwarz–Pick lemma asserts that

$$\frac{|\omega'(z)|}{1-|\omega(z)|^2}\le\frac{1}{1-|z|^2},\qquad z\in\mathbf{D},$$

for any analytic map $\omega\colon \mathbf{D}\to\mathbf{D}$. Particularly, any analytic automorphism T of $\mathbf{D}$ satisfies

$$\frac{|T'(z)|}{1-|T(z)|^2}=\frac{1}{1-|z|^2},$$

and, therefore, $d(T(z_1),T(z_2))=d(z_1,z_2)$ for $z_1,z_2\in\mathbf{D}$.

To prove Theorem 1.1, we need the following lemmas.

Lemma 2.1. *Let* $\phi\in\mathcal{A}$. *If* $S(\phi)/s(\phi)<+\infty$ *and*

$$\frac{2}{\pi}\log\frac{S(\phi)}{s(\phi)}S(\phi)(1-|z|^2)\operatorname{arctanh}|z|\le 1$$

holds for all $z\in\mathbf{D}$, *then* ϕ *is univalent in* $\mathbf{D}$.

Proof. It is easy to see that the function $\arctan z=\dfrac{1}{2i}\log\left(\dfrac{1+iz}{1-iz}\right)$ maps the unit disk conformally onto the vertical parallel strip $|\mathrm{Re}w|<\dfrac{\pi}{4}$. For a constant $a>0$, the function

$$Q_a(z)=\exp(2a\arctan z)=\left(\frac{1-iz}{1+iz}\right)^{ai} \tag{2.1}$$

is the universal covering projection of $\mathbf{D}$ onto the annulus $e^{-\pi a/2}<|w|<e^{\pi a/2}$. Noting that $Q_a(0)=1$ and

$$\frac{Q_a'(z)}{Q_a(z)}=\frac{2a}{1+z^2}. \tag{2.2}$$

Let $p(z)=1+z\phi''(z)/\phi'(z)$. If p is a constant, then $\phi=z$ and is univalent in $\mathbf{D}$. Without loss of generality, we assume that p is not a constant so that $s(\phi)<1<S(\phi)$. Let

$$\sigma=\log\frac{S(\phi)}{s(\phi)}<\infty,\quad m=\sqrt{S(\phi)s(\phi)}.$$

We consider the universal covering map $Q=mQ_a$ of $\mathbf{D}$ onto the annulus

$$W = \{w\colon s(\phi) < |w| < S(\phi)\} = \{w\colon me^{-\sigma/2} < |w| < me^{\sigma/2}\},$$

where $a = \sigma/\pi$. It is obvious that $p(\mathbf{D}) \subset W$. Since the interval $(-1, 1)$ is mapped onto $(s(\phi), S(\phi))$ by Q, we can choose an $\alpha \in (-1, 1)$, so that $Q(\alpha) = 1$. Then $P = Q \circ T$ is an universal covering map, where $T(z) = (z + \alpha)/(1 + \alpha z)$. Since $P\colon \mathbf{D} \to W$ is a covering map, we can take a lift ω of p with $\omega(0) = 0$ and $p = P \circ \omega$. Then

$$\frac{z\phi''(z)}{\phi'(z)} = p(z) - 1 = P(\omega(z)) - 1. \tag{2.3}$$

Let ι be the image of the line segment $(0, \omega(z))$ under T. Then by (2.2), we have

$$P(\omega(z)) - 1 = \int\limits_0^{\omega(z)} P'(t)dt = \int\limits_0^{\omega(z)} Q'(T(t))T'(t)dt$$

$$= \int\limits_\iota Q'(u)du = \int\limits_\iota \frac{2aQ(u)}{1+u^2}du.$$

The last equality holds since $\dfrac{Q'(u)}{Q(u)} = \dfrac{Q_a'(u)}{Q_a(u)} = \dfrac{2a}{1+u^2}$. By $|Q(u)| \le S(\phi)$, we obtain

$$|P(\omega(z)) - 1| \le 2aS(\phi) \int\limits_\iota \frac{|du|}{1 - |u|^2}$$

$$= 2aS(\phi) \int\limits_0^{\omega(z)} \frac{|du|}{1 - |u|^2} = 2aS(\phi)d(0, \omega(z)) \le 2aS(\phi) \operatorname{arctanh}|z|. \tag{2.4}$$

The last inequality holds since $|\omega(z)| \le |z|$. By (2.3) and (2.4), we have

$$(1 - |z|^2)\left|\frac{z\phi''(z)}{\phi'(z)}\right| \le 2aS(\phi)(1 - |z|^2) \operatorname{arctanh}|z|$$

for $z \in \mathbf{D}$. By (1.1), if $S(\phi)/s(\phi) < +\infty$ and

$$\frac{2}{\pi} \log \frac{S(\phi)}{s(\phi)} S(\phi)(1 - |z|^2) \operatorname{arctanh}|z| \le 1, \quad z \in \mathbf{D},$$

holds, ϕ is univalent in $\mathbf{D}$.

Lemma 2.2. *Let* $\sigma > 0$. *If*

$$\frac{2\sigma}{\pi} e^{\sigma/2}(1 - |z|^2) \operatorname{arctanh}|z| \le 1, \quad z \in \mathbf{D}, \tag{2.5}$$

then $\sigma \le \sigma_0$, *where* σ_0 *is stated in Theorem* 1.1.

If

$$\frac{2\sigma}{\pi} e^{\sigma}(1 - |z|^2) \operatorname{arctanh}|z| \le 1, \quad z \in \mathbf{D}, \tag{2.6}$$

then $\sigma \le \sigma_1$, *where* σ_1 *is stated in Theorem* 1.1.

Proof. We assume that (2.5) holds and consider a function $\phi \in \mathcal{A}$ satisfying $e^{-\sigma/2} < \left|1 + \frac{\phi''}{\phi'}\right| < e^{\sigma/2}$. Then $S(\phi) \le e^{\sigma/2}$ and $\log \frac{S(\phi)}{s(\phi)} \le \sigma$, so that

$$\frac{2}{\pi} \log \frac{S(\phi)}{s(\phi)} S(\phi)(1 - |z|^2) \operatorname{arctanh} |z| \le \frac{2\sigma}{\pi} e^{\sigma/2}(1 - |z|^2) \operatorname{arctanh} |z|.$$

By Lemma 2.1, we get that ϕ is univalent in $\mathbf{D}$ if (2.5) holds.

Now, we assume that (2.6) holds and consider a function $\phi \in \mathcal{A}$ satisfying $S(\phi) \le e^{\sigma} s(\phi)$. Since $s(\phi) \le 1$, we have $S(\phi) \le e^{\sigma}$. It follows that

$$\frac{2}{\pi} \log \frac{S(\phi)}{s(\phi)} S(\phi)(1 - |z|^2) \operatorname{arctanh} |z| \le \frac{2\sigma}{\pi} e^{\sigma}(1 - |z|^2) \operatorname{arctanh} |z|.$$

Applying Lemma 2.1, we deduce that ϕ is univalent in $\mathbf{D}$ if (2.6) holds.

To give an upper bound for σ_0 and σ_1, we shall observe its Grunsky coefficients to examine univalence. We borrow some discussions in [9] on the Grunsky coefficients and related results as follows. Suppose that $\phi \in \mathcal{A}$, the Grunsky coefficients $c_{j,k}$ of ϕ are defined by

$$\log \frac{\phi(z) - \phi(w)}{z - w} = - \sum_{j,k=0}^{\infty} c_{j,k} z^j w^k$$

in $|z| < \varepsilon$, $|w| < \varepsilon$ for a small enough $\varepsilon > 0$. It is obvious that $c_{j,k} = c_{k,j}$ holds. Also, $c_{j,0}$ are the logarithmic coefficients of $\phi(z)/z$, i.e.,

$$- \log \frac{\phi(z)}{z} = c_{1,0} z + c_{2,0} z^2 + \ldots.$$

The Grunsky coefficients $c_{j,k}$ of ϕ paly an important role in judging the univalence of ϕ (see [16]). The Grunsky theorem says that $\phi \in \mathcal{A}$ is univalent in $\mathbf{D}$ if and only if

$$\left| \sum_{j,k=1}^{N} c_{j,k} x_j x_k \right| \le \sum_{j=1}^{N} \frac{|x_j|^2}{j}$$

for any positive integer N and any vector $(x_1, \ldots, x_N) \in \mathbf{C}^N$. Later, the Grunsky theorem was strengthened by Pommerenke [16] as follows: if $\phi \in \mathcal{A}$ is univalent in $\mathbf{D}$, then

$$\sum_{j=1}^{\infty} j \left| \sum_{k=1}^{n} c_{j,k} t_k \right|^2 \le \sum_{j=1}^{n} \frac{|t_j|^2}{j} \tag{2.7}$$

holds for all $n \ge 1$ and $t_1, \ldots, t_n \in \mathbf{C}$. The Grunsky coefficients are usually defined for the function $g(\zeta) = \dfrac{1}{\phi\left(\frac{1}{\zeta}\right)}$. This change affects only the coefficients $c_{j,0} = c_{0,j}$, which do not involve the Grunsky inequalities (see [8] for more details). Recently, combing the inequality (2.7), Kim and Sugawa [9] observed the following assertion.

Lemma 2.3 [9]. *A function $\phi \in \mathcal{A}$ is univalent in $\mathbf{D}$ if and only if its Grunsky matrix $G_\phi(n)$ of order n is positive semidefinite for every $n \geq 1$, where $G_\phi(n) = \left[\gamma_{j,k}^{(n)}\right]$ denotes the Hermitian matrix of order n and*

$$\gamma_{j,k}^{(n)} = \frac{\delta_{j,k}}{j} - \sum_{m=1}^{n} m c_{m,j} \overline{c_{m,k}}, \quad 1 \leq j, k \leq n,$$

$$\delta_{j,k} = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$$

To compute the Grunsky coefficients, the following lemma due to Kim and Sugawa [9] is needed.

Lemma 2.4 [9]. *The Grunsky coefficients $c_{j,k}$ of a function $\phi(z) = z + a_2 z^2 + \ldots \in \mathcal{A}$ satisfies*

$$c_{j,k} = \sum_{l=1}^{k-1} \frac{l}{k} a_{k-l} c_{j+1,l} - \sum_{m=1}^{j} a_{m+1} c_{j-m,k} - \frac{a_{j+k+1}}{k}$$

for $j \geq 0$ and $k \geq 1$.

3. Proof of main result. Using the idea of the proofs in [9], we prove Theorem 1.1. Now, we state the outline of the proof of Theorem 1.1 as follows: by Lemmas 2.1 and 2.2, we can get the lower bound for σ_0 and σ_1. Furthermore, by Lemmas 2.3 and 2.4, we give an upper bound for σ_0 and σ_1 by checking nonunivalence of the function $F_a(z) \in \mathcal{A}$ satisfying

$$1 + z\frac{F_a''(z)}{F_a'(z)} = Q_a(z)$$

for a suitably chosen positive constant a, where $Q_a(z)$ is given by (2.1).

***Proof of Theorem* 1.1.** Let $\varphi(x) = (1-x^2)\operatorname{arctanh}(x)$, then we have $\varphi'(x) = 1 - 2x \operatorname{arctanh}(x)$. It follows that

$$\max_{x \in (0,1)} \varphi(x) = (1 - x_0^2)\operatorname{arctanh}(x_0) = \frac{1 - x_0^2}{2x_0},$$

where x_0 is given by (1.3). By Lemm 2.2, we have the lower bound for σ_0 and σ_1.

To give an upper bound for σ_0 and σ_1, we consider the Taylor expansion of the function

$$Q_a(z) = \left(\frac{1 - iz}{1 + iz}\right)^{ai} = 1 + 2az + 2a^2z^2 + \frac{2}{3}a(2a^2 - 1)z^3 + \frac{2}{3}a^2(a^2 - 2)z^4 + \ldots,$$

which means that

$$1 + z\frac{F_a''(z)}{F_a'(z)} = Q_a(z) = 1 + 2az + 2a^2z^2 + \frac{2}{3}a(2a^2 - 1)z^3 + \frac{2}{3}a^2(a^2 - 2)z^4 + \ldots.$$

Elemental computations give

$$\begin{aligned} F_a(z) &= z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots \\ &= z + az^2 + a^2 z^3 + \frac{1}{18}a(17a^2 - 1)z^4 + \frac{1}{45}a^2(38a^2 - 7)z^5 + \ldots. \end{aligned}$$

Moreover, it is easy to see that

$$-\log \frac{F_a(z)}{z} = c_{1,0}z + c_{2,0}z^2 + \ldots,$$

where $c_{1,0} = -a$ and $c_{2,0} = -\dfrac{a^2}{2}$. By Lemma 2.4, we calculate the following:

$$c_{1,1} = -a_2c_{0,1} - a_3 = 0,$$

$$c_{2,1} = c_{1,2} = -a_2c_{1,1} - a_3c_{0,1} - a_4 = \frac{a^3+a}{18},$$

$$c_{3,1} = -a_2c_{2,1} - a_3c_{1,1} - a_4c_{0,1} - a_5 = \frac{2a^4+2a^2}{45},$$

$$c_{2,2} = \frac{c_{3,1}}{2} - a_2c_{1,2} - a_3c_{0,2} - \frac{a_5}{2} = \frac{2a^4+2a^2}{45},$$

it follows that $G_{F_a}(1) = [1]$,

$$G_{F_a}(2) = \begin{bmatrix} \gamma_{1,1}^{(2)} & \gamma_{1,2}^{(2)} \\ \gamma_{2,1}^{(2)} & \gamma_{2,2}^{(2)} \end{bmatrix},$$

where

$$\gamma_{1,1}^{(2)} = 1 - 2c_{2,1}^2 = \frac{162 - a^2 - 2a^4 - a^6}{162},$$

$$\gamma_{1,2}^{(2)} = \gamma_{2,1}^{(2)} = -2c_{2,1}c_{2,2} = -\frac{2a^7+4a^5+2a^3}{405},$$

$$\gamma_{2,2}^{(2)} = \frac{1}{2} - c_{1,2}^2 - 2c_{2,2}^2 = -\frac{8}{2025}a^8 - \frac{89}{8100}a^6 - \frac{41}{4050}a^4 - \frac{1}{324}a^2 + \frac{1}{2}.$$

Next, we consider the function $\psi(a) = \gamma_{1,1}^{(2)}\gamma_{2,2}^{(2)} - \gamma_{1,2}^{(2)}\gamma_{2,1}^{(2)}$, where

$$\begin{aligned}\psi(a) = {} & \frac{162 - a^2 - 2a^4 - a^6}{162}\left[-\frac{8}{2025}a^8 - \frac{89}{8100}a^6 - \frac{41}{4050}a^4 - \frac{1}{324}a^2 + \frac{1}{2}\right] \\ & - \left[\frac{2a^7+4a^5+2a^3}{405}\right]^2 \\ = {} & \frac{1}{52488}a^{12} + \frac{1}{13122}a^{10} - \frac{839}{218700}a^8 - \frac{2296}{164025}a^6 - \frac{21359}{1312200}a^4 \\ & - \frac{1}{162}a^2 + \frac{1}{2}.\end{aligned}$$

A series of calculations show that

$$\psi'(a) = \frac{1}{4374}a^{11} + \frac{5}{6561}a^9 - \frac{1678}{54675}a^7 - \frac{4592}{54675}a^5 - \frac{21359}{328050}a^3 - \frac{1}{81}a$$

and

$$\psi''(a) = \frac{11}{4374}a^{10} + \frac{5}{729}a^8 - \frac{11746}{54675}a^6 - \frac{4592}{10935}a^4 - \frac{64077}{328050}a^2 - \frac{1}{81}.$$

Let $t = a^2$, then we have

$$\psi''(a) := \varphi(t) = \frac{11}{4374}t^5 + \frac{5}{729}t^4 - \frac{11746}{54675}t^3 - \frac{4592}{10935}t^2 - \frac{64077}{328050}t - \frac{1}{81}.$$

It is easy to prove that $\varphi(t) < 0$ for all $t \in \left(0, \frac{64}{25}\right)$. Thus $\psi''(a) < 0$ for all $a \in \left(0, \frac{8}{5}\right)$. Combining $\psi'(0) = 0$ and $\psi'\left(\frac{8}{5}\right) < 0$, we obtain that $\psi(a)$ is decreasing in $a \in \left(0, \frac{8}{5}\right)$. Moreover, a numerical approximation gives

$$\psi\left(\frac{8}{5}\right) \approx -0.008348505466479 < 0,$$

it follows that the determinant of $G_{F_{8/5}}(2)$ is not positive semidefinite. By Lemma 2.3, if a closes enough to $\frac{8}{5}$, we know that $F_a(z)$ is not univalent in $\mathbf{D}$. Therefore, we obtain that $\sigma_0, \sigma_1 < \frac{8\pi}{5}$.

Theorem 1.1 is proved.

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