DOI: 10.37863/umzh.v75i7.7214

UDC 517.5

Hasan Şahin[1], **İsmet Yildiz** (Department of Mathematics, Faculty of Arts and Sciences, Duzce University, Turkey)

DETERMINATION OF SOME PROPERTIES OF STARLIKE AND CLOSE-TO-CONVEX FUNCTIONS ACCORDING TO SUBORDINATE CONDITIONS WITH CONVEXITY OF A CERTAIN ANALYTIC FUNCTION

ВИЗНАЧЕННЯ ДЕЯКИХ ВЛАСТИВОСТЕЙ ЗІРКОПОДІБНИХ І БЛИЗЬКИХ ДО ОПУКЛИХ ФУНКЦІЙ ЗА ПІДПОРЯДКОВАНИМИ УМОВАМИ З ОПУКЛІСТЮ ПЕВНОЇ АНАЛІТИЧНОЇ ФУНКЦІЇ

Investigation of the theory of complex functions is one of the most fascinating aspects of theory of complex analytic functions of one variable. It has a huge impact on all areas of mathematics. Many mathematical concepts are explained when viewed through the theory of complex functions. Let $f(z) \in A,\ f(z) = z + \sum_{n\geq 2}^{\infty} a_n z^n$, be an analytic function in the open unit disc $U = \{z : |z| < 1,\ z \in \mathbb{C}\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. For close-to-convex and starlike functions, new and different conditions are obtained by using subordination properties, where r is a positive integer of order 2^{-r} $\left(0 < 2^{-r} \leq \dfrac{1}{2}\right)$. By using subordination, we propose a criterion for $f(z) \in S^*[a^r, b^r]$. The relations for starlike and close-to-convex functions are investigated under certain conditions according to their subordination properties. At the same time, we analyze the convexity of some analytic functions and study their regional transformations. In addition, the properties of convexity for $f(z) \in A$ are examined.

Дослідження з теорії комплексних функцій є одним із найцікавіших аспектів теорії комплексних аналітичних функцій однієї змінної. Вони мають величезний вплив на всі області математики. Багато математичних понять пояснюються з точки зору теорії комплексних функцій. Нехай $f(z) \in A,\ f(z) = z + \sum_{n\geq 2}^{\infty} a_n z^n$, —- аналітична функція на відкритому одиничному диску $U = \{z : |z| < 1,\ z \in \mathbb{C}\}$, що нормована таким чином: $f(0) = 0,\ f'(0) = 1$. Для зірчастих функцій та функцій, що близькі до опуклих, за допомогою властивостей підпорядкування отримано нові та відмінні умови, де r — натуральне число порядку 2^{-r} $\left(0 < 2^{-r} \leq \dfrac{1}{2}\right)$. Використовуючи підпорядкованість, ми пропонуємо деякий критерій для $f(z) \in S^*[a^r, b^r]$. Досліджено співвідношення між зірчастими функціями та функціями, що близькі до опуклих, за певних умов згідно з властивостями підпорядкування. Також досліджено опуклість деяких аналітичних функцій і вивчено їх локальні перетворення. Крім того, досліджено властивості опуклості для $f(z) \in A$.

1. Introduction and definition. Close to convexity of the analytical functions in the unit disk U has an important impact on the theory of geometric functions. Let A denote the class of functions of the form

$$f(z) = z + \sum_{n\geq 2} a_n z^n,$$

which are normalized analytic functions by the conditions $f(0) = f'(0) - 1 = 0$ in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. The class of starlike, convex and close to convex of order α are defined respectively

[1] Corresponding author, e-mail: hasansahin13@gmail.com.

$$S^*(\alpha)\colon \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad f(z)\in A \quad \text{and} \quad 0\le\alpha<1,$$

$$K(\alpha)\colon \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, \quad f(z)\in A \quad \text{and} \quad 0\le\alpha<1,$$

$$K^*(\alpha)\colon \mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right) > \alpha, \quad f(z)\in A,\, g(z)\in K \quad \text{and} \quad 0\le\alpha<1.$$

We say that every starlike function is close to convex function (see [2, 3, 10] for more details).

The subordinate relationship between these function order 2^{-r} $\left(0<\dfrac{1}{2^r}\le\dfrac{1}{2}\right)$, which we defined, was focused on and r is a positive integer. Let $f(z)\in A$ be an analytic function in the open unit disk $U=\{z\colon |z|<1,\ z\in\mathbb{C}\}$ normalized by

$$f(z)=z+\sum_{n\ge2}^{\infty}a_nz^n \quad \text{for} \quad f(0)=0, \quad f'(0)=1,$$

S denotes the class of functions $f(z)$ in class A, which $f(z)$ is a univalent function. If the class of starlike functions $f(z)\in A$ is also defined by S^*, then functions $f(z)\in A$ belong to U as starlike of order 2^{-r} $\left(0<\dfrac{1}{2^r}\le\dfrac{1}{2}\right)$, i.e., $f(z)\in S^*\left(\dfrac{1}{r^r}\right)$. If the function $f(z)\in A$ is, respectively, the starlike function degree $\dfrac{1}{2^r}$ and the convex function degree $\dfrac{1}{2^r}$, then [4, 5, 10]

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)\ge\frac{1}{2^r} \qquad \text{and} \qquad \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\ge\frac{1}{2^r} \quad \text{for} \quad z\in U.$$

Now we will examine the exact properties convex and starlike functions of order 2^{-r} in this paper. According to this selection, we can give the above definitions as follows: we say that $f(z)\in A$ is, respectively, the starlike function of order 2^{-r} and the convex function of order 2^{-r} $\left(0<\dfrac{1}{2^r}\le\dfrac{1}{2}\right)$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)\le 2^{-r} \qquad \text{and} \qquad \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\le 2^{-r} \quad \text{for} \quad z\in U.$$

A function $f(z)\in A$ is said to be convex of order 2^{-r} in $|z|<1$ if it satisfies

$$0<\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\le\frac{1}{2^r}.$$

A function $f(z)\in A$ is said to be starlike of order 2^{-r} in $|z|<1$ if it satisfies

$$0<\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)\le\frac{1}{2^r}.$$

K denotes the class of convex functions $f(z)$ in class A, which $f(z)$ is a univalent function. Let S^* be the class starlike functions and K be the class convex functions. A function $f(z)\in A$ is said to be close to convex if there exists a convex function $g(z)$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g'(z)}\right) < 2^{-r}, \quad z \in U.$$

Every convex function is obviously close to convex function. More generally, every starlike function is close to convex function. Let K^* be the class of close to convex functions $f(z) \in A$. Indeed, each $f(z) \in S^*$ has the from $f(z) = zg'(z)$ for some $g(z) \in K$ and

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) = \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < 2^{-r}.$$

Lemma 1 [7, 10]**.** *The convex function $f(z) \in K$ be close to convex function if and only if $zf'(z) \in S^*$ such that starlike function $f(z) \in S^*$ must be present.*

Proof. Let $f(z) \in K$ be the convex function as $h(z) = zf'(z)$. By taking logarithmic derivatives from both sides of this equation, we have

$$\ln(h(z)) = \ln(zf'(z)),$$

$$\frac{h'(z)}{h(z)} = \frac{1}{z} + \frac{f''(z)}{f'(z)}, \qquad \frac{zh'(z)}{h(z)} = 1 + \frac{zf''(z)}{f'(z)},$$

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) > 0.$$

Since $f(z) \in K$ is a convex function, the subclass of class univalent functions S consisting of the class starlike functions S^* and the class starlike functions S^* consisting of the convex functions.

Theorem 1 (Noshiro – Warschawski theorem) [10]**.** *If $f(z)$ is analytic in a convex function domain $|z| < 1$ and $\operatorname{Re}(f'(z)) > 0$, then $f(z)$ is univalent in $|z| < 1$.*

Theorem 2. *Every close to convex function is univalent. A function $f(z)$ analytic in $|z| < 1$ is said to be close to convex function if there exists convex function $g(z)$ such that*

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) = \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \textit{for all} \quad z \in |z| < 1.$$

Proof. If $f(z)$ is convex function, then there must be a convex function $g(z)$ that provides the following inequality:

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0 \quad \text{for all} \quad z \in U = |z| < 1.$$

Let $z \in |z| < 1$ be the range of $g(z)$ and consider the function $\Im(z)$:

$$\Im(z) = f(g^{-1}(\omega)) \quad \text{for all} \quad z \in |z| < 1.$$

In this case, we have

$$\Im'(z) = \frac{f'(g^{-1}(\omega))}{g'((g^{-1}(\omega))} = \frac{f'(z)}{g'(z)}.$$

So, $\operatorname{Re}(\Im'(z)) > 0$ in $z \in |z| < 1$. This function $\Im(z)$ is univalent or $f(z)$ is the starlike function close-to-convex, so $f(z)$ is the close-to-convex function, where every starlike function is close-to-convex function.

Definition 1 [9, 12]. *If $f(z)$ is analytic and univalent in $U = \{z : |z| < 1,\ z \in \mathbb{C}\}$ and $g(z)$ is a analytic function in $|z| < 1$ with $g(0) = f(0)$ and $f(U) \subset g(U)$, then, for $|f'(0)| \le |g'(0)|$, the function $f(z)$ is called the subordinate of the $g(z)$ and is given by $f(z) \prec g(z)$.*

Lemma 2 [8]. *Let $q(z)$ be univalent in U and $\vartheta(\omega)$ and $\psi(\omega)$ be analytic in a domain in Ucontaining $q(U)$ with $\psi(\omega) \neq 0$ when $\omega \in q(U)$. Let*

$$\wp(z) = zq'(z)\psi(q(z)) \qquad \textit{and} \qquad g(z) = \vartheta(q(z)) + \wp(z)$$

and suppose that

a) *$\wp(z)$ is starlike in $|z| < 1$*

and

b) $\mathrm{Re}\left(\dfrac{zg'(z)}{\wp(z)}\right) = \mathrm{Re}\left(\dfrac{\vartheta'(q(z))}{\psi(q(z)} + \dfrac{z\wp'(z)}{\wp(z)}\right) > 0 \quad for \quad z \in |z| < 1.$

If $p(z)$ is analytic in $|z| < 1$ with $p(0) = f(0)$, $p(U) \subset U$ and

$$\vartheta(p(z)) + zp'(z)\psi(p(z)) \prec \vartheta(q(z)) + zq'(z)\psi(q(z)),$$

then

$$p(z) \prec q(z).$$

Lemma 3 [1]. *Let w(z) be a different from fixed analytic function in $z \in |z| < 1$ with $w(0) = 0$. If $w(z)$ attains value on the circle $|z| = r < 1$ at z_0, then $z_0 w'(z_0) = kw(z_0)$, where $k \ge 1$ is real number.*

Theorem 3. *Let $f(z) \in A$ be a function that provides the conditions*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3(\lambda - 1)(2 - 2^{-r})}{2(\lambda + 1)(2 + 2^{-r})} \quad for \quad 1 < \lambda \le 2 \qquad and \qquad z \in |z| < 1.$$

Then

$$\frac{zf'(z)}{f(z)} \prec \frac{\lambda(1 - z)}{\lambda - z}. \tag{1}$$

The result is sharp for the function $f(z) \in A$ given by

$$f(z) = z\left(1 - \frac{z}{\lambda}\right)^{\lambda - 1}. \tag{2}$$

Proof. Let us define the function $f(z) \in A$ by

$$\frac{zf'(z)}{f(z)} = \frac{z[2^{-r} - w(z)]}{\lambda - w(z)}. \tag{3}$$

Then $w(z)$ is analytic in $U = |z| < 1$ with $w(0) = 0$. By logarithmic derivative of both sides of (3), we have

$$\ln z + \ln f'(z) - \ln f(z) = \ln z + \ln\left(2^{-r} - w\right) - \ln(\lambda - w(z)),$$

$$\frac{1}{z}+\frac{f''(z)}{f'(z)}=\frac{f'(z)}{f(z)}+\frac{1}{z}-\frac{w'(z)}{2^{-r}-w(z)}+\frac{w'(z)}{\lambda-w(z)},$$

$$1+\frac{zf''(z)}{f'(z)}=\frac{zf'(z)}{f(z)}+1-\frac{zw'(z)}{2^{-r}-w(z)}+\frac{zw'(z)}{\lambda-w(z)}.$$

Also, if the logarithmic derivative is taken from both sides of the equation (2), then

$$\ln f(z)=\ln z+(\lambda-1)\ln\Big(1-\frac{z}{\lambda}\Big),$$

$$\frac{f'(z)}{f(z)}=\frac{1}{z}-\frac{\lambda-1}{\lambda-z}\quad\text{or}\quad\frac{zf'(z)}{f(z)}=1-\frac{(\lambda-1)z}{\lambda-z}.$$

So we take that $\dfrac{zf'(z)}{f(z)}+1=\dfrac{\lambda(1-z)}{\lambda-z}$. If used from the above, then

$$1+\frac{zf''(z)}{f'(z)}=\frac{\lambda(2^{-r}-w(z))}{\lambda-w(z)}-\frac{zw'(z)}{2^{-r}-w(z)}+\frac{zw'(z)}{\lambda-w(z)},$$

where $w(0)=0$ and $0<2^{-r}\le\dfrac{1}{2}$. Then clearly that

$$\frac{zf'(z)}{f(z)}=\frac{z(2^{-r}-w(z))}{\lambda-w(z)},$$

if we wrote it up above

$$1+\frac{zf''(z)}{f'(z)}=\frac{\lambda(2^{-r}-w(z))}{\lambda-w(z)}-\frac{zw'(z)}{2^{-r}-w(z)}+\frac{zw'(z)}{\lambda-w(z)}.$$

Suppose now that there exists a point $z_0\in|z|<1$ such that $|w(z_0)|=1$ and $|w(z)|<1$ when $|z|<|z_0|$. If we apply Lemmas 1, 2 and 3, we obtain

$$\begin{aligned}\operatorname{Re}\left[1+\frac{z_0f''(z_0)}{f'(z_0)}\right]&=\operatorname{Re}\left[\frac{\lambda\big(2^{-r}-e^{i\theta}\big)}{\lambda-e^{i\theta}}\right]-\operatorname{Re}\left[\frac{ke^{i\theta}}{2^{-r}-e^{i\theta}}\right]+\operatorname{Re}\left[\frac{ke^{i\theta}}{\lambda-e^{i\theta}}\right]\\
&=\operatorname{Re}\left[\frac{\lambda(2^{-r}-\cos\theta-i\sin\theta)}{\lambda-\cos\theta-i\sin\theta}\right]-\operatorname{Re}\left[\frac{k(\cos\theta+i\sin\theta)}{2^{-r}-\cos\theta-i\sin\theta}\right]+\operatorname{Re}\left[\frac{k(\cos\theta+i\sin\theta)}{\lambda-(\cos\theta+i\sin\theta)}\right]\\
&=\frac{\lambda^2 2^{-r}-\lambda^2 2^{-r}\cos\theta-\lambda^2\cos\theta+\lambda}{(\lambda-\cos\theta)^2+\sin^2\theta}+\frac{k2^{-r}\cos\theta-k}{(2^{-r}-\cos\theta)^2+\sin^2\theta}+\frac{\lambda k\cos\theta-k}{(\lambda-\cos\theta)^2+\sin^2\theta}\\
&=\frac{\lambda(1-\lambda\cos\theta)-\lambda 2^{-r}(\lambda-\cos\theta)}{1+\lambda^2-2\cos\theta}+\frac{k2^{-r}\cos\theta-k}{1+2^{-r}-2^{1-r}\cos\theta}+\frac{\lambda k\cos\theta-k}{1+\lambda^2-2\lambda\cos\theta}\\
&=\frac{(1-\lambda\cos\theta)[\lambda(1-2^{-r})-k]}{1+\lambda^2-2\cos\theta}-\frac{k(1-2^{-r}\cos\theta)}{1+2^{-2r}-2^{1-r}\cos\theta}.\end{aligned}$$

According to the above, we can write the inequality

$$\operatorname{Re}\left[1+\frac{z_0f''(z_0)}{f'(z_0)}\right]\ge\frac{3(\lambda-1)(2-2^{-r})}{2(\lambda+1)(2+2^{-r})}.$$

Thus, according to the last inequality above, we can come to the conclusion that

$$f(z) = z\Big(1 - \frac{z}{\lambda}\Big)^{\lambda-1} \qquad \text{and} \qquad \ln z = \ln z + (\lambda - 1)\ln\Big(1 - \frac{z}{\lambda}\Big),$$

$$\frac{zf'(z)}{f(z)} = l - \frac{\lambda - 1}{\lambda - z} \prec \frac{\lambda(1 - z)}{\lambda - z},$$

which implies the subordination (1). In this way, the proof is completed. If $-1 \le b < a < 1$, then an important class is defined by

$$S^*[1, -1] = \Big\{f(z) : \frac{zf'(z)}{f(z)} \prec \frac{1 + az}{1 + bz}\Big\}.$$

Geometrically, this means that the image of $|z| < 1$ by $\dfrac{zf'(z)}{f(z)}$ is inside the open disk centered on the real axis with diameter end points $\dfrac{1 - a}{1 - b}$ and $\dfrac{1 + a}{1 + b}$. Special selection of a and b lead us to the following: $S^*[(1 - 2^{-r}), -1] \equiv S^*$ is the class of starlike functions. $S^*[(1 - 2^{-r}), -1] \equiv S^*\Big(\dfrac{1}{2^r}\Big)$, $0 < \dfrac{1}{2^r} < 1$, is the class of starlike functions of order $\dfrac{1}{2^r}$, defined by $f(z) \in S^*\Big(\dfrac{1}{2^r}\Big)$. Also, $K\Big(\dfrac{1}{2^r}\Big)$ is the class of starlike functions of order $\dfrac{1}{2^r}$, defined by $f(z) \in K\Big(\dfrac{1}{2^r}\Big)$ if and only if $zf'(z) \in S^*\Big(\dfrac{1}{2^r}\Big)$, i.e., $0 < \operatorname{Re}\Big(1 + \dfrac{zf''(z)}{f'(z)}\Big) \le \dfrac{1}{2^r}$ for $z \in |z| < 1$. We will work on the following class in this study under conditions $0 < \dfrac{1}{2^r} \le \dfrac{1}{2}$ and $\lambda > 0$:

$$\Im(\lambda, 2^{-r}) = \left\{ f(z) \in A : \left| \frac{1 - 2^{-r} + 2^{-r}\dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} - (1 - 2^{-r}) \right| < \lambda \right\} \quad \text{for} \quad z \in U = \{z : |z| < 1\},$$

$0 < \dfrac{1}{2^r} \le \dfrac{1}{2}$ and give sufficient conditions that embed it into the classes $S^*[a, b]$. In this we will first use Lemma 2 for subordination.

2. Main theorem for subordination.

Theorem 4 (main result 1). *If* $f(z) \in A, -1 \le b < a < 1$, $\dfrac{2 + |a^r|}{5 + |a^r|} < \dfrac{1}{2^r} < 1$ *and*

$$\frac{1 - 2^{-r} + \dfrac{z.2^{-r} f''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} \prec 2^{-r} + (1 - 2^{1-r})\frac{1 + b^r z}{1 + a^r z} + \frac{z.2^{-r}(a^r - b^r)}{(1 + a^r z)^2} \equiv g(z),$$

then $f(z) \in S^*[a^r, b^r]$.

Proof. Let us define the following functions:

$$p(z) = \frac{f(z)}{zf'(z)}, \qquad q(z) = \frac{1 + b^r z}{1 + a^r z}, \qquad \vartheta(\omega) = (1 - 2^{1-r})\omega + 2^{-r} \qquad \text{and} \qquad \psi(\omega) = -2^{-r}.$$

Then $q(z)$ is a convex function. Realy,

$$q(z)=\frac{1+b^rz}{1+a^rz} \qquad \text{and} \qquad q'(z)=\frac{b^r-a^r}{(1+a^rz)^2},$$

$$zq'(z)=\frac{(b^r-a^r)z}{(1+a^rz)^2} \qquad \text{and} \qquad \ln(z)+\ln(q'(z))=\ln z-2\ln(1+a^rz),$$

$$\frac{1}{z}+\frac{zq''(z)}{q'(z)}=\frac{1}{z}-\frac{2a^r}{(1+a^rz)} \qquad \text{and} \qquad 1+\frac{zq''(z)}{q'(z)}=1-\frac{2a^rz}{(1+a^rz)}.$$

Then we have

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right)=\operatorname{Re}\left(1-\frac{2a^rz}{1+a^rz}\right)>0.$$

Thus, from Lemma 2, $q(z)$ is convex and univalent. On the other hand, from Lemma 2 we obtain

$$\wp(z)=zq'(z)\psi(q(z)) \qquad \text{and} \qquad g(z)=\vartheta(q(z))+\wp(z),$$

$$q(z)=\frac{1+b^rz}{1+a^rz} \qquad \text{and} \qquad q'(z)=\frac{b^r-a^r}{(1+a^rz)^2}. \tag{4}$$

(4) values are written in their places above and if the necessary operation are performed. Then, according to the following equations, we get

$$\wp(z)=zq'(z)\psi(q(z))=\frac{z.2^{-r}(a^r-b^r)}{(1+a^rz)^2} \qquad \text{and} \qquad \ln(\wp(z))=\ln z-2\ln(1+a^rz),$$

$$\frac{z\wp'(z)}{\wp(z)}=1-\frac{2a^rz}{1+a^rz}=\frac{1-a^rz}{1+a^rz} \qquad \text{and} \qquad \operatorname{Re}\left(\frac{z\wp'(z)}{\wp(z)}\right)=\operatorname{Re}\left(1-\frac{2a^rz}{1+a^rz}\right)>0,$$

$\wp(z)$ is starlike in $U=|z|<1$. From the hypothesis of Theorem 3 and Lemma 2,

$$g(z)=\vartheta(q(z))+\wp(z)=2^{-r}+(1-2^{1-r})\frac{1+b^rz}{1+a^rz}+\frac{z.2^{-r}(a^r-b^r)}{(1+a^rz)^2}$$

and

$$\wp(z)=\frac{z.2^{-r}(a^r-b^r)}{(1+a^rz)^2}.$$

If necessary operations are performed in the above equation, then

$$g'(z)=\frac{(1-2^{1-r})(b^r-a^r)(1+a^rz)+(a^r-b^r)(1-a^rz)}{(1+a^rz)^3}$$

$$=\frac{(b^r-a^r)(1-3.2^{-r}+a^rz-a^rz.2^{-r})}{(1+a^rz)^3},$$

$$zg'(z)=\frac{z(b^r-a^r)(1-3.2^{-r}+a^rz-a^rz.2^{-r})}{(1+a^rz)^3},$$

$$\frac{zg'(z)}{\wp(z)}=\frac{z(b^r-a^r)(1-3.2^{-r}+a^rz-a^rz.2^{-r})}{(1+a^rz)^3}\,\frac{(1+a^rz)^2}{z.2^{-r}((a^r-b^r)}$$

$$= 1 - \frac{1}{2^{-r}} + \frac{2}{1 + a^r z}, \quad z \in |z| < 1,$$

$$\left(\frac{zg'(z)}{\wp(z)}\right) = \mathrm{Re}\left(1 - \frac{1}{2^{-r}} + \frac{2}{1 + |a^r|}\right) > 1 - \frac{1}{2^{-r}} + \frac{2}{1 + |a^r|},$$

that is necessary and sufficient to get the greatest value or equal to the value of 2^{-r} must be

$$\frac{1}{2^r} > \frac{2 + |a^r|}{5 + |a^r|}.$$

According to Lemma 2 $p(z) \prec q(z)$. So from here $f(z) \in S^*[a^r, b^r]$. According to the definition of subordination, Theorem 3 means that $g(U)$ has the largest region in the complex plane. In this the following condition must be met:

$$\frac{1 - 2^{-r} + 2^{-r}\dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} \in g(U)$$

for all $z \in |z| < 1$. Then $f(z) \in S^*[a^r, b^r]$, and, thus, we have that

$$-1 < \frac{2 + |a^r|}{5 + |a^r|} < \frac{1}{2^r} < 1 \quad \text{for} \quad -1 \leq b < a < 1.$$

Lemma 4 [6, 11]. *Suppose that $w(z)$ is a nonconstant analytic function in a given domain D with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in D$, then*

$$z_0 w'(z_0) = k w(z_0),$$

where k is a real number satisfying $k \geq 1$.

Lemma 5 [10]. *Let us take two analytical functions named as $f(z)$ and $g(z)$ in the unit disk. If we have an analytic function φ providing the conditions $\varphi(0) = 0$ and $|\varphi(z)| < 1$ such that $f(z) = g(\varphi(z))$ for $z \in D$, then we state that f is subordinate to g written as $f \prec g$.*

3. Main theorem for convexity.

Theorem 5. *If the function $f(z)$ provides the following inequality for close to convex function $f(z) \in A$:*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{2^r - 1}{2^r}$$

for $r = 1, 2, 3, \ldots,$ then

$$\mathrm{Re}\, f'(z) > \frac{1}{4}.$$

Proof. $f'(z) = \dfrac{1 + 2^{-r} w(z)}{1 + w(z)}$ is defined for $w(z)$, where $w(z) \neq -1,\ z \in U,\ 0 < 2^{-r} \leq \dfrac{1}{2}$, Clearly, $w(z)$ is analytic in U with $w(0) = 0$. If necessary operations are made by taking the derivative from both sides of the equality

$$zf'(z) = \frac{z(1 + 2^{-r} w(z))}{1 + w(z)},$$

we get

$$f'(z) + zf''(z) = \frac{[(1 + 2^{-r}w(z)) + z2^{-r}w'(z)](1 + w(z)) - zw'(z)(1 + 2^{-r}w(z))}{[1 + w(z)]^2}$$

$$= \frac{(1 + w(z))(1 + 2^{-r}w(z)) + 2^{-r}zw'(1 + w(z)) - zw'(1 + 2^{-r}w(z))}{(1 + w(z))^2},$$

$$1 + \frac{zf''(z)}{f'(z)} = \frac{(1 + w(z))^2(1 + 2^{-r}w(z)) + z2^{-r}w'(z)(1 + w(z)) - zw'(1 + w(z))(1 + 2^{-r}w(z))}{(1 + 2^{-r}w(z))(1 + w(z))^2},$$

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{z2^{-r}w'(z)}{1 + 2^{-r}w(z)} - \frac{zw'(z)}{1 + w(z)}.$$

Assume that there exists a point $z_0 \in U$ with $|z| < |z_0|$ and such that $|w(z_0)| = 1$ and $|w(z)| < 1$. By applying Lemma 4, we find $z_0w'(z_0) = kw(z_0)$ for $k \geq 1$ and $w(z_0) = e^{i\theta}$, $\theta \in \mathrm{R}$ and $r = 1, 2, 3, \ldots$. Thus, we have

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] = 1 + \mathrm{Re}\left(\frac{k2^{-r}e^{i\theta}}{1 + 2^{-r}e^{i\theta}}\right) - \mathrm{Re}\left(\frac{ke^{i\theta}}{1 + e^{i\theta}}\right)$$

$$= 1 + \mathrm{Re}\left[k\frac{2^{-r}(\cos\theta + i2^{-r}\sin\theta)(1 + 2^{-r}\cos\theta - i2^{-r}\sin\theta)}{(1 + 2^{-r}\cos\theta)^2 + (2^{-r}\sin\theta)^2}\right]$$

$$- \mathrm{Re}\left[k\frac{(\cos\theta + i\sin\theta)(1 + \cos\theta - i\sin\theta)}{(1 + \cos\theta)^2 + (\sin\theta)^2}\right]$$

$$= 1 + \mathrm{Re}\left[k\frac{2^{-r}\cos\theta + (2^{-r}\cos\theta)^2 + (2^{-r}\sin\theta)^2}{1 + 2^{1-r}\cos\theta + 2^{-2r}}\right]$$

$$- \mathrm{Re}\left[k\frac{\cos\theta + \cos^2\theta + \sin^2\theta}{1 + \cos^2\theta + \sin^2\theta + 2\cos\theta}\right]$$

$$= 1 + \mathrm{Re}\left[k\frac{2^{-r}\cos\theta + 2^{-2r}}{1 + 2^{-2r} + 2^{1-r}\cos\theta}\right] - \mathrm{Re}\left[k\frac{\cos\theta + 1}{2(1 + \cos\theta)}\right]$$

$$= 1 + \frac{k.2^{-r}(\cos\theta + 2^{-r})}{1 + 2^{-2r} + 2^{1-r}\cos\theta} - \frac{k}{2} \leq \frac{2^r - 1}{2^r}$$

for $z_0 \in U,\ 0 < 2^{-r} \leq \dfrac{1}{2}$. It obviously contradicts our hypothesis. That is,

$$\left|\frac{1 - f'(z)}{2^{-r} + f'(z)}\right| < 1,$$

$$\left|1 - f'(z)\right| < \left|2^{-r} + f'(z)\right| \Rightarrow 2f'(z) > 1 - 2^{-r} \Rightarrow f'(z) > \frac{1 - 2^{-r}}{2},$$

$$\mathrm{Re}\, f'(z) > \frac{1 - 2^r}{1 + 2^r}.$$

Theorem 6. *If $f(z) \in A$ provides the inequality*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \leq \frac{3+2^{1-r}}{1+2^{-r}}$$

for $z_0 \in U,\ 0 < 2^{-r} \leq \dfrac{1}{2}$, then

$$\left|f'(z)-1\right| < 1+2^{-r}.$$

Proof. If $w(z)$ is given by $f'(z) = (1+2^{-r})w(z)+1$ for $z \in U$ and $0 < 2^{-r} \leq \dfrac{1}{2}$, then $zf'(z) = (1+2^{-r})zw(z)+z$,

$$f'(z)+zf''(z) = 1+\left(1+2^{-r}\right)w(z)+\left(1+2^{-r}\right)zw'(z),$$

$$1+\frac{zf''(z)}{f'(z)} = \frac{1+(1+2^{-r})w(z)+(1+2^{-r})zw'(z)}{1+(1+2^{-r})w(z)}$$

$$= 1+\frac{(1+2^{-r})zw'(z)}{1+(1+2^{-r})w(z)} = 1+\frac{(1+2^{-r})zw'(z)}{1+(1+2^{-r})w(z)} = 1+\frac{(1+2^{-r})ke^{i\theta}}{1+(1+2^{-r})e^{i\theta}}.$$

By using Lemma 4, we have

$$1+\frac{zf''(z)}{f'(z)} = 1+\frac{k(1+2^{-r})(\cos\theta+i\sin\theta)}{1+(1+2^{-r})(\cos\theta+i\sin\theta)}$$

$$= 1+\frac{k(1+2^{-r})(\cos\theta+i\sin\theta)[1+(1+2^{-r})\cos\theta-i(1+2^{-r})\sin\theta]}{[1+(1+2^{-r})\cos\theta]^2+[(1+2^{-r})\sin\theta]^2},$$

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) = 1+\operatorname{Re}\left[\frac{k(1+2^{-r})\cos\theta+(1+2^{-r})^2}{1+(1+2^{-r})^2+(1+2^{-r})\cos\theta}\right]$$

$$= 1+\frac{(1+2^{-r})[k\cos\theta+(1+2^{-r})]}{1+(1+2^{-r})[\cos\theta+(1+2^{-r})]} \leq \frac{3+2^{1-r}}{1+2^{-r}}.$$

Otherwise,

$$f'(z) = \left(1+2^{-r}\right)w(z)+1 \Rightarrow f'(z)-1 = \left(1+2^{-r}\right)w(z),$$

$$\left|f'(z)-1\right| = \left(1+2^{-r}\right),$$

that is, $f(z) \in K^*$.

Theorem 7 (main result 2)**.** *If $f(z) \in A$ is a function satisfying the inequality*

$$\left|f'(z)-1\right|^a \left|zf''(z)\right|^b < \frac{(1+2^{-r})^{a+b}}{2^{a+2b}}, \quad z \in D, \quad a,b \geq 0,$$

and $r > 0$, then

$$\operatorname{Re} f'(z) \geq \frac{1+2^r}{2^{r+3}}.$$

Proof. The function $w(z)$ is defined as

$$f'(z) = \frac{(1+2^{-r})w(z)}{1+w(z)}$$

for $w(z) \neq -1,\ z \in U$. Then $w(z)$ is analytic in $U = |z| < 1$ with $w(0) = 0$. Then

$$f'(z) - 1 = \frac{(1+2^{-r})w(z)}{1+w(z)} - 1 = \frac{(1+2^{-r})w(z) - 1 - w(z)}{1+w(z)} = \frac{(2^{-r}-1)w(z)}{1+w(z)} = \frac{(2^{-r}-1)e^{i\theta}}{1+e^{i\theta}}.$$

From Lemma 4, we have

$$\left|f'(z) - 1\right| = \left|\frac{(2^{-r}-1)e^{i\theta}}{1+e^{i\theta}}\right| = \frac{1-2^{-r}}{1+\cos\theta} = \frac{1-2^{-r}}{2}$$

for $\theta = 0$ and

$$\left|f'(z) - 1\right|^a = \left(\frac{1-2^{-r}}{2}\right)^a. \tag{5}$$

Otherwise,

$$zf'(z) = \frac{z + 2^{-r}w(z)z}{1+w(z)} \qquad \text{and} \qquad zf''(z) = \frac{(2^{-r}-1)zw'(z)}{[1+w(z)]^2},$$

$$f'(z) + zf''(z) = \frac{1 + w(z) + 2^{-r}w(z) + 2^{-r}w(z)^2 + 2^{-r}zw'(z) - zw'(z)}{[1+w(z)]^2}.$$

Applying Lemma 4 to the last equation, we get

$$zf''(z) = \frac{(2^{-r}-1)kw'(z)}{[1+w(z)]^2} = \frac{(2^{-r}-1)ke^{i\theta}}{[1+e^{i\theta}]^2} \Rightarrow \left|zf''(z)\right| = \frac{(1-2^{-r})k}{|1+e^{i\theta}|^2}.$$

Suppose there exists a point $z_0 \in |z| < 1$ with $|z| < |z_0|$ and such that $|w(z_0)| = 1$ and $|w(z)| < 1$,

$$\left|zf''(z)\right| = \frac{(1-2^{-r})k}{2^2} \qquad \text{or} \qquad \left|zf''(z)\right|^b = \frac{(1-2^{-r})^b k^b}{2^{2b}}. \tag{6}$$

From (5) and (6), we obtain

$$\left|f'(z_0) - 1\right|^a \left|z_0 f''(z_0)\right|^b = \frac{(1-2^{-r})^{a+b}k^b}{2^{a+2b}} \leq \frac{(1-2^{-r})^{a+b}}{2^{a+2b}},$$

where $z_0 \in U$ and $0 < 2^{-r} \leq \dfrac{1}{2}$. Our hypothesis is clearly contradicted by the last result. Therefore, we obtain $|w(z)| < 1$ for all $z \in U$. It suggests that

$$\left|\frac{f'(z) - 1}{f'(z) - 2^{-r}}\right| \leq 1.$$

Clearly, we have $|f'(z) - 1| \leq \left|\dfrac{1-2^{-r}}{2}\right|$ or $|f'(z) - 1| \leq \dfrac{1-2^{-r}}{2}$. On the other hand, from Lemma 1,

$$f'(z)=\frac{1+2^{-r}w(z)}{1+w(z)}=\frac{1+2^{-r}e^{i\theta}}{1+e^{i\theta}},$$

$$\operatorname{Re} f'(z)=\frac{(1+\cos\theta)+2^{-r}(1+\cos\theta)}{2(1+\cos\theta)}=\frac{1+2^{-r}}{4}$$

for $\theta=0$. By using the above equality, we get

$$\operatorname{Re} f'(z)=\frac{1+2^{-r}}{8.2^{-r}}.$$

Theorem 8 (main result 3). *Let $f(z)\in A$ be a function which provides the conditions*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)<\frac{3(\lambda-1)(2-2^{-r})}{2(\lambda+1)(2+2^{-r})}$$

for $1<\lambda\neq 2$ and $z\in U$. Then

$$\frac{zf'(z)}{f(z)}\prec\frac{\lambda(1-z)}{\lambda-z}, \tag{7}$$

$f(z)\in A$ defined by

$$f(z)=z\left(1-\frac{z}{\lambda}\right)^{\lambda-1}. \tag{8}$$

The result is sharp.

Proof. $f(z)\in A$ is given as follows:

$$\frac{zf'(z)}{f(z)}=\frac{z[2^{-r}-w(z)]}{\lambda-w(z)}. \tag{9}$$

Lemmas 4 and 5 are used for the proof. Here, we say that $w(z)$ is analytic in a domain $U=|z|<1$ with $w(0)=0$. When logarithmic differentiation is used on both sides of the equation (9), we get

$$\ln z+\ln f'(z)-\ln f(z)=\ln z+\ln\big(2^{-r}-w\big)-\ln(\lambda-w(z)),$$

$$\frac{1}{z}+\frac{f''(z)}{f'(z)}=\frac{f'(z)}{f(z)}+\frac{1}{z}-\frac{w'(z)}{2^{-r}-w(z)}+\frac{w'(z)}{\lambda-w(z)},$$

$$1+\frac{zf''(z)}{f'(z)}=\frac{zf'(z)}{f(z)}+1-\frac{zw'(z)}{2^{-r}-w(z)}+\frac{zw'(z)}{\lambda-w(z)}. \tag{10}$$

By logarithmically differentiating from both sides of the equation (8), we easily arrive at:

$$\ln f(z)=\ln z+(\lambda-1)\ln\left(1-\frac{z}{\lambda}\right),$$

$$\frac{zf'(z)}{f(z)}=\frac{1}{z}-\frac{\lambda-1}{\lambda-z}\quad\text{or}\quad\frac{zf'(z)}{f(z)}=1-\frac{(\lambda-1)z}{\lambda-z}.$$

Thus, we obtain

$$\frac{zf'(z)}{f(z)}+1=\frac{\lambda(1-z)}{\lambda-z}. \tag{11}$$

The equality (11) is written instead of at equation (10),

$$1+\frac{zf''(z)}{f'(z)}=\frac{\lambda(2^{-r}-w(z))}{\lambda-w(z)}-\frac{zw'(z)}{2^{-r}-w(z)}+\frac{zw'(z)}{\lambda-w(z)},$$

where $w(0)=0$ and $0<2^{-r}\le\frac{1}{2}$. Then

$$\frac{zf'(z)}{f(z)}=\frac{z(2^{-r}-w(z))}{\lambda-w(z)}.$$

Now assume that there exists a point $z\in U=|z|<1$ with $|z|<|z_0|$ and such that $|w(z_0)|=1$ and $|w(z)|<1$. If we apply Lemmas 4 and 5 for the equality (11), we obtain

$$\operatorname{Re}\left[1+\frac{z_0f''(z_0)}{f'(z_0)}\right]=\operatorname{Re}\left[\frac{\lambda\left(2^{-r}-e^{i\lambda}\right)}{\lambda-e^{i\lambda}}\right]-\operatorname{Re}\left[\frac{ke^{i\lambda}}{2^{-r}-e^{i\lambda}}\right]+\operatorname{Re}\left[\frac{ke^{i\lambda}}{\lambda-e^{i\lambda}}\right]$$

$$=\operatorname{Re}\left[\frac{\lambda(2^{-r}-\cos\theta-i\sin\theta)}{\lambda-\cos\theta-i\sin\theta}\right]-\operatorname{Re}\left[\frac{k(\cos\theta+i\sin\theta)}{2^{-r}-\cos\theta-i\sin\theta}\right]+\operatorname{Re}\left[\frac{k(\cos\theta+i\sin\theta)}{\lambda-(\cos\theta+i\sin\theta)}\right]$$

$$=\frac{\lambda^2 2^{-r}-\lambda^2 2^{-r}\cos\theta-\lambda^2\cos\theta+\lambda}{(\lambda-\cos\theta)^2+\sin^2\theta}+\frac{k2^{-r}\cos\theta-k}{(2^{-r}-\cos\theta)^2+\sin^2\theta}+\frac{\lambda k\cos\theta-k}{(\lambda-\cos\theta)^2+\sin^2\theta}$$

$$=\frac{\lambda(1-\lambda\cos\theta)-\lambda 2^{-r}(\lambda-\cos\theta)}{1+\lambda^2-2\cos\theta}+\frac{k2^{-r}\cos\theta-k}{1+2^{-r}-2^{1-r}\cos\theta}+\frac{\lambda k\cos\theta-k}{1+\lambda^2-2\lambda\cos\theta}$$

$$=\frac{(1-\lambda\cos\theta)[\lambda(1-2^{-r})-k]}{1+\lambda^2-2\cos\theta}-\frac{k(1-2^{-r}\cos\theta)}{1+2^{-2r}-2^{1-r}\cos\theta}.$$

According to the above, we can write the inequality as

$$\operatorname{Re}\left[1+\frac{z_0f''(z_0)}{f'(z_0)}\right]\ge\frac{3(\lambda-1)(2-2^{-r})}{2(\lambda+1)(2+2^{-r})}.$$

Thus, we obtain the next conclusion

$$f(z)=z\left(1-\frac{z}{\lambda}\right)^{\lambda-1},$$

$$\ln z=\ln z+(\lambda-1)\ln\left(1-\frac{z}{\lambda},\right)$$

$$\frac{zf'(z)}{f(z)}=l-\frac{\lambda-1}{\lambda-z}<\frac{\lambda(1-z)}{\lambda-z},$$

which implies the subordination (7).

4. Conclusion. In this paper, the principles of functions close to convex are investigated. In addition, analytical functions and their properties are discussed. Convexity of analytic functions are examined and proofs for 2^{-r} grade cases are obtained. By considering S, K and S^* functions, proofs of these functions are obtained by subordination.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. A. K. Bakhtin, I. V. Denega, *Extremal decomposition of the complex plane with free poles*, J. Math. Sci., **246**, № 1, 1 – 17 (2020).
2. A. W. Goodman, *Univalent functions*, vol. I, Polygonal Publ. House, Washington, New Jersey (1983).
3. H. M. Srivastava, S. Owa (Editors), *Current topics in analytic function theory*, World Sci. Publ. Co., Singapore etc. (1992).
4. H. Liu, *Sufficient conditions for certain analytic functions*, Scholars J. Eng. and Technology (SJET), **2**(2B), 243 – 246 (2014).
5. H. S. Kasana, *Complex variables theory and applications*, Prentice-Hall of İndia Private Limited, New Delhi (2005).
6. I. S. Jack, *Functions starlike and convex of order* α, J. London Math. Soc. (2), **3**, 469 – 474 (1971).
7. M. Nunokawa, M. Aydogan, K. Kuroki, I. Yildiz, S. Owa, *Some properties concerning close-to-convexity of certain analytic functions*, J. Inequal. and Appl., **245**, 1 – 10 (2012).
8. M. Orddovic, S. Owa, *On certain properties for some classes of starlike functions*, J. Math. Anal. and Appl., **145**, 357 – 364 (1990).
9. N. Tuneski, *Some results on starlike and convex function*, Appl. Anal. and Discrete Math., **1**, 293 – 298 (2007).
10. P. L. Duren, *Univalent functions*, Grundlehren math. Wiss., Bd. 259, Springer-Verlag, New York etc. (1983).
11. S. S. Miller, P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. and Appl., **65**, 289 – 305 (1978).
12. S. S. Miller, P. T. Mocanu, *On some classes of first order differential subordinations*, Michigan Math. J., **32**, 185 – 195 (1985).

Received 07.05.22