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Pascal Lefèvre[1] (University of Artois, Laboratoire de Mathématiques de Lens (LML), Lens, France)

WEIGHTED DISCRETE HARDY'S INEQUALITIES

ЗВАЖЕНІ ДИСКРЕТНІ НЕРІВНОСТІ ГАРДІ

We give a short proof of a weighted version of the discrete Hardy inequality. This includes the known case of classical monomial weights with optimal constant. The proof is based on the ideas of the short direct proof given recently in [P. Lefèvre, Arch. Math. (Basel), **114**, № 2, 195–198 (2020)].

Запропоновано коротке доведення зваженої версії дискретної нерівності Гарді, яке включає відомий випадок класичних мономіальних ваг з оптимальною сталою. Доведення спирається на ідеї короткого прямого доведення, що було нещодавно наведено в роботі [P. Lefèvre, Arch. Math. (Basel), **114**, № 2, 195 – 198 (2020)].

In the sequel, we work with $p > 1$ and $p' = \dfrac{p}{p-1}$ denotes its conjugate exponent.

The notation $\mathbb{N}_0$ stands for the set of nonnegative integral numbers: $0, 1, 2, \ldots$.

For $y \in \mathbb{R}^+$, we write $[y] = \max\{k \in \mathbb{N}_0 \mid k \leq y\}$ its integral part.

As usual, given a sequence $(w_n)_{n\geq 0}$ of positive numbers, $\ell^p(w)$ is the space of sequences of complex numbers $a = (a_n)_{n\geq 0}$ such that $\sum_{n\geq 0} |a_n|^p w_n < \infty$, equipped with the norm

$$\|a\|_{\ell^p(w)} = \left(\sum_{n=0}^{+\infty} |a_n|^p w_n\right)^{\frac{1}{p}}.$$

When $w_n = 1$ for every $n \in \mathbb{N}_0$, we simply write ℓ^p.

Given a sequence $a = (a_k)_{k\geq 0}$ of complex numbers, we associate the sequence

$$A_n = \frac{1}{n+1}\sum_{k=0}^{n} a_k.$$

We recall the **discrete Hardy inequality**.

Let $p > 1$. *For every* $a \in \ell^p$, *the sequence* $A = (A_n)_{n\geq 0}$ *belongs to* ℓ^p *and* $\|A\|_{\ell^p} \leq p'\|a\|_{\ell^p}$, *i.e.*,

$$\left(\sum_{n=0}^{+\infty}\left|\frac{1}{n+1}\sum_{k=0}^{n} a_k\right|^p\right)^{\frac{1}{p}} \leq p'\left(\sum_{n=0}^{+\infty}|a_n|^p\right)^{\frac{1}{p}}.$$

This inequality is equivalent to the boundedness of the Cesáro operator defined by $\Gamma(a) = (A_n)_{n\in\mathbb{N}}$, with $\|\Gamma\| \leq p'$, viewed as an operator on ℓ^p. Actually the constant p' is optimal and $\|\Gamma\| = p'$. See [3] for the original result, [7] for a very interesting historical survey on the subject, [6] for a very recent nice extension, and [8] for a short proof.

[1] E-mail: pascal.lefevre@univ-artois.fr.

The aim of this note is to give a short proof of the boundedness of Γ as an operator on weighted $\ell^p(w)$ spaces, with a sharp norm, under an homogeneity type assumption, following the same ideas than [8].

In particular, this includes the case of classical weights $\big(n^{\alpha p}\big)_{n\geq 0}$ (with exact norm), so that we recover easily some results of [1] (see Corollary 8 (iii), applied with $1-\alpha$ and $\beta=\alpha$, but where the norm is not given) and [5] (see Theorem 6.2, applied with $p\alpha$ instead of α). For instance, this inequality is one of the ingredient of the nice paper [2] on the spectrum of the discrete Cesáro operator.

Main theorem. *Let $w=(w_n)_{n\geq 0}$ and $w'=(w'_n)_{n\geq 0}$ be sequences of nonnegative real numbers and $(w_n)_{n\geq 0}$ is nondecreasing.*

We assume that there exists a measurable positive function f on $(0,1)$ such that

1) *we have a subhomogeneity property: $w'_n\leq f(s)w_{[(n+1)s]}$, where $s\in(0,1)$ and $n\in\mathbb{N}_0$,*

2) $K=\displaystyle\int_0^1\left(\frac{f(s)}{s}\right)^{\frac{1}{p}}ds<\infty.$

Then Γ is bounded from $\ell^p(w)$ to $\ell^p\big(w'\big)$ with $\|\Gamma\|\leq K$:

$$\left(\sum_{n=0}^{+\infty}w'_n\left|\frac{1}{n+1}\sum_{k=0}^{n}x_k\right|^p\right)^{\frac{1}{p}}\leq K\left(\sum_{n=0}^{+\infty}|x_n|^p w_n\right)^{\frac{1}{p}}.$$

Point out that the subhomogeneity property can be formulated in the following way: for every $n\in(\lambda m,\lambda(m+1))\cap\mathbb{N}$, $w'_{n-1}\leq f(1/\lambda)\,w_m$, where $\lambda\geq 1$ and $m\in\mathbb{N}_0$.

In particular, it applies to $w'_n=w_n=(n+1)^{\alpha p}$ for $n\in\mathbb{N}_0$ with $f(s)=s^{-\alpha p}$ (for $s\in(0,1)$). We get immediately the following theorem.

Theorem 1. *Let $\alpha\in[0,1/p')$.*

Then Γ is bounded on $\ell^p(w)$ with $\|\Gamma\|=\left(\dfrac{1}{p'}-\alpha\right)^{-1}$:

$$\left(\sum_{n=0}^{+\infty}\left|\frac{1}{(n+1)^{1-\alpha}}\sum_{k=0}^{n}x_k\right|^p\right)^{\frac{1}{p}}\leq\left(\frac{1}{p'}-\alpha\right)^{-1}\left(\sum_{n=0}^{+\infty}|x_n|^p(n+1)^{\alpha p}\right)^{\frac{1}{p}}.$$

It is easy to see that this bound is sharp. Indeed, for $\varepsilon\in\big(0,1/p'-\alpha\big)$, just test the sequence $a_n=(n+1)^{\frac{1}{p'}-\alpha-\varepsilon}-n^{\frac{1}{p'}-\alpha-\varepsilon}$.

Before giving the proof of the main theorem, let us mention a general remark about monotone rearrangements of sequences.

We recall that we can define the monotone rearrangement of a vanishing sequence of nonnegative numbers $(b_k)_{k\geq 0}$ as the following nonincreasing sequence:

$$b^*_N=\inf_{|E|=N}\sup_{n\notin E}b_n\quad\forall N\in\mathbb{N}_0.$$

The following consequence of the Abel transform principle is well-known:

Let $(c_k)_{k\geq 0}$ be a nonincreasing sequence of nonnegative numbers. Let $(u_k)_{k\geq 0}$ and $\big(u'_k\big)_{k\geq 0}$ be two sequence such that, for every $n\geq 0$, $\sum_{k=0}^{n}u'_k\geq\sum_{k=0}^{n}u_k$.

Then $\sum_{n=0}^{N} c_n u_n' \geq \sum_{n=0}^{N} c_n u_n$ for every $N \geq 0$.

Indeed write $U_n' = \sum_{k=0}^{n} u_k'$ and $U_n = \sum_{k=0}^{n} u_k$ and defines for convenience $U_{-1}' = U_{-1} = 0$. A simple Abel transform gives, for every $N \geq 0$,

$$\sum_{n=0}^{N} c_n u_n' = \sum_{n=0}^{N} c_n \big(U_n' - U_{n-1}'\big) = c_{N+1} U_N' + \sum_{n=0}^{N} (c_n - c_{n+1}) U_n' \geq c_{N+1} U_N + \sum_{n=0}^{N} (c_n - c_{n+1}) U_n$$

and another Abel transform gives the result.

In particular, we have the following simple fact.

Fact. *Let $(c_k)_{k\geq 0}$ be a nonincreasing sequence of nonnegative real numbers. Let $(u_k)_{k\geq 0}$ be a vanishing sequence of nonnegative numbers and $\big(u_k^*\big)_{k\geq 0}$ its monotone rearrangement.*

Then

$$\sum_{n=0}^{+\infty} c_n u_n \leq \sum_{n=0}^{+\infty} c_n u_n^*.$$

Indeed we just point out that, by definition, for every $n \geq 0$, we have $\sum_{k=0}^{n} u_k^* \geq \sum_{k=0}^{n} u_k$.

Proof of the main theorem. Let $a \in \ell^p(w)$. We assume first that $\big(|a_k|^p w_k\big)_k$ is nonincreasing.

Let us fix an arbitrary $N \in \mathbb{N}_0$. For every $n \in \mathbb{N}_0$, we write

$$A_n = \sum_{k=0}^{n} \int\limits_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} a_k \, ds = \int\limits_0^1 a_{[(n+1)s]} \, ds.$$

Thanks to the triangular inequality for integrals, we have

$$\left(\sum_{n=0}^{N} |A_n|^p w_n'\right)^{\frac{1}{p}} \leq \int\limits_0^1 \left(\sum_{n=0}^{N} |a_{[(n+1)s]}|^p w_n'\right)^{\frac{1}{p}} ds \leq \int\limits_0^1 \left(f(s) \sum_{n=0}^{+\infty} |a_{[(n+1)s]}|^p w_{[(n+1)s]}\right)^{\frac{1}{p}} ds$$

thanks to the subhomogeneity property.

Here we exploit a variation of the Abel transform argument used in [8], suggested by Y. C. Huang in [4], although the original trick based on the previous remark would work too in our case (see [9]):

$$\sum_{n=0}^{+\infty} |a_{[(n+1)s]}|^p w_{[(n+1)s]} \leq \int\limits_0^{+\infty} |a_{[ts]}|^p w_{[ts]} \, dt$$

because $\big(|a_k|^p w_k\big)_k$ is nonincreasing.

Therefore, we have, for every $s \in (0, 1]$,

$$\sum_{n=0}^{+\infty} |a_{[(n+1)s]}|^p w_{[(n+1)s]} \leq \frac{1}{s} \int\limits_0^{+\infty} |a_{[x]}|^p w_{[x]} \, dx = \frac{1}{s} \sum_{m=0}^{+\infty} |a_m|^p w_m.$$

Integrating with respect to s, we get

$$\left(\sum_{n=0}^{N}|A_n|^p w'_n\right)^{\frac{1}{p}} \leq \int\limits_0^1 \left(\frac{f(s)}{s}\right)^{\frac{1}{p}} \|a\|_{\ell^p(w)}\, ds = K\|a\|_{\ell^p(w)}.$$

Since $N \in \mathbb{N}_0$ is arbitrary, the result is proved in the particular case when $\big(|a_k|^p w_k\big)_k$ is nonincreasing.

Now, in the general case, take $a \in \ell^p(w)$. The sequence $(u_k)_{k\geq 0} = \big(|a_k| w_k^{\frac{1}{p}}\big)_k$ is vanishing, so we can consider its monotone rearrangement $\big(u_k^*\big)_{k\geq 0}$. We define also $c_k = w_k^{-\frac{1}{p}}$ for $k \geq 0$. By asumption, the sequence $\big(c_k\big)_{k\geq 0}$ is nonincreasing.

We point out that for every $N \geq 0$, thanks to the Fact, we have

$$\left|\sum_{n=0}^{N} a_n\right| \leq \sum_{n=0}^{N}|a_n| = \sum_{n=0}^{N} c_n u_n \leq \sum_{n=0}^{N} c_n u_n^*.$$

Applying the first step to the sequence $\big(c_k u_k^*\big)_{k\geq 0}$, we get

$$\|\Gamma(a)\|_{\ell^p(w')} \leq K\left(\sum_{n=0}^{+\infty} w_n |c_n u_n^*|^p\right)^{\frac{1}{p}}$$

but

$$\left(\sum_{n=0}^{+\infty} w_n |c_n u_n^*|^p\right)^{\frac{1}{p}} = \left(\sum_{n=0}^{+\infty}|u_n^*|^p\right)^{\frac{1}{p}} = \left(\sum_{n=0}^{+\infty}|u_n|^p\right)^{\frac{1}{p}} = \|a\|_{\ell^p(w)}.$$

The theorem is proved.

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References

1. G. Bennett, *Some elementary inequalities*, Quart. J. Math. Oxford, **38**, 401 – 425 (1987).
2. G. Curbera, W. Ricker, *Spectrum of the Cesáro operator in ℓ^p*, Arch. Math. (Basel), **100**, № 3, 267 – 271 (2013).
3. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge (1967).
4. Y. C. Huang, *Change of variable and discrete hardy inequality*, preprint.
5. G. J. O. Jameson, R. Lashkaripour, *Norms of certain operators on weighted ℓ^p spaces and Lorentz sequence spaces*, J. Inequal. Pure and Appl. Math., **3**, Issue 1 (2002).
6. F. Fischer, M. Keller, F. Pogorzelski, *An improved discrete p-Hardy inequality*; ArXiv: 1910.03004.
7. A. Kufner, L. Maligranda, L. E. Persson, *The prehistory of the Hardy inequality*, Amer. Math. Monthly, **113**, 715 – 732 (2006).
8. P. Lefèvre, *A short direct proof of the discrete Hardy inequality*, Arch. Math. (Basel), **114**, № 2, 195 – 198 (2020).
9. P. Lefèvre, *Weighted discrete Hardy's inequalities (unpublished)*; https://hal.archives-ouvertes.fr/hal-02528265.

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