DOI: 10.37863/umzh.v75i7.7155

UDC 517.9

Ravshan Ashurov[1] (Institute of Mathematics, Uzbekistan Academy of Science, Tashkent and School of Engineering, Central Asian University, Uzbekistan),
Marjona Shakarova (National University of Uzbekistan named after Mirzo Ulugbek, Tashkent)

TIME-DEPENDENT SOURCE IDENTIFICATION PROBLEM FOR A FRACTIONAL SCHRÖDINGER EQUATION WITH THE RIEMANN – LIOUVILLE DERIVATIVE

ЗАЛЕЖНА ВІД ЧАСУ ЗАДАЧА ІДЕНТИФІКАЦІЇ ДЖЕРЕЛА ДЛЯ ДРОБОВОГО РІВНЯННЯ ШРЕДІНГЕРА З ПОХІДНОЮ РІМАНА – ЛІУВІЛЛЯ

We consider a Schrödinger equation $i\partial_t^\rho u(x,t) - u_{xx}(x,t) = p(t)q(x) + f(x,t),\ 0 < t \le T,\ 0 < \rho < 1$, with the Riemann – Liouville derivative. An inverse problem is investigated in which, parallel with $u(x,t)$, a time-dependent factor $p(t)$ of the source function is also unknown. To solve this inverse problem, we use an additional condition $B[u(\cdot,t)] = \psi(t)$ with an arbitrary bounded linear functional B. The existence and uniqueness theorem for the solution to the problem under consideration is proved. The stability inequalities are obtained. The applied method make it possible to study a similar problem by taking, instead of d^2/dx^2, an arbitrary elliptic differential operator $A(x,D)$ with compact inverse.

Розглянуто рівняння Шредінгера $i\partial_t^\rho u(x,t) - u_{xx}(x,t) = p(t)q(x) + f(x,t),\ 0 < t \le T,\ 0 < \rho < 1$, з похідною Рімана – Ліувілля. Досліджено обернену задачу, в якій крім $u(x,t)$ також невідомий залежний від часу множник $p(t)$ функції джерела. Для розв'язання оберненої задачі введено додаткову умову $B[u(\cdot,t)] = \psi(t)$ для довільного обмеженого лінійного функціонала B. Доведено теорему існування та єдиності розв'язку задачі, що розглядається. Отримано нерівності щодо стійкості. Застосований метод дає змогу дослідити аналогічну задачу, в якій замість d^2/dx^2 фігурує довільний еліптичний диференціальний оператор $A(x,D)$, що має компактний обернений оператор.

1. Introduction. The fractional integration of order $\sigma < 0$ of function $h(t)$ defined on $[0,\infty)$ has the form (see, e.g., [1, p. 14; 2, Chapter 3])

$$J_t^\sigma h(t) = \frac{1}{\Gamma(-\sigma)} \int\limits_0^t \frac{h(\xi)}{(t-\xi)^{\sigma+1}} d\xi, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\sigma)$ is Euler's gamma function. Using this definition one can define the Riemann – Liouville fractional derivative of order ρ:

$$\partial_t^\rho h(t) = \frac{d}{dt} J_t^{\rho-1} h(t).$$

Note that if $\rho = 1$, then the fractional derivative coincides with the ordinary classical derivative of the first order: $\partial_t h(t) = (d/dt)h(t)$.

Let $\rho \in (0,1)$ be a fixed number and $\Omega = (0,\pi) \times (0,T]$. Consider the following initial-boundary value problem for the Shrödinger equation:

[1] Corresponding author, e-mail: ashurovr@gmail.com.

$$\begin{aligned} & i\partial_t^\rho u(x,t) - u_{xx}(x,t) = p(t)q(x) + f(x,t), \quad (x,t) \in \Omega, \\ & u(0,t) = u(\pi,t) = 0, \quad 0 \le t \le T, \\ & \lim_{t\to 0} J_t^{\rho-1} u(x,t) = \varphi(x), \quad 0 \le x \le \pi, \end{aligned} \tag{1.1}$$

where $t^{1-\rho}p(t)$, $t^{1-\rho}f(x,t)$ and $\varphi(x)$, $q(x)$ are continuous functions in the closed domain $\overline{\Omega}$. This problem is also called the *forward problem*.

If $p(t)$ is a known function, then under certain conditions on the given functions a solution to problem (1.1) exists and it is unique (see, e.g., [3]).

We note the following property of the Riemann – Liouville integrals, which simplifies the verification of the initial condition in problem (1.1) (see, e.g., [1, p. 104]):

$$\lim_{t\to +0} J_t^{\alpha-1} h(t) = \Gamma(\alpha) \lim_{t\to +0} t^{1-\alpha} h(t). \tag{1.2}$$

From here, in particular, it follows that the solution of the forward problem can have a singularity at zero $t = 0$ of order $t^{\rho-1}$.

Let $C[0,l]$ be the set of continuous functions defined on $[0,l]$ with the standard max-norm $\|\cdot\|_{C[0,l]}$. The purpose of this paper is not only to find a solution $u(x,t)$, but also to determine the time-dependent part $p(t)$ of the source function. To solve this time-dependent source identification problem one needs an extra condition. Following the papers of A. Ashyralyev et al. [4 – 6], we consider the additional condition in a rather general form:

$$B[u(\cdot,t)] = \psi(t), \quad 0 \le t \le T, \tag{1.3}$$

where $B\colon C[0,\pi] \to R$ is a given bounded linear functional: $\big\|B[h(\cdot,t)]\big\|_{C[0,T]} \le b\big\|h(x,t)\big\|_{C(\overline{\Omega})}$, and $\psi(t)$ is a given continuous function. For example, as the functional B one can take $B\big[u(\cdot,t)\big] = u(x_0,t)$, $x_0 \in [0,\pi]$, or $B\big[u(\cdot,t)\big] = \int_0^\pi u(x,t)dx$, or a linear combination of these two functionals.

We call the initial-boundary value problem (1.1) together with additional condition (1.3) *the inverse problem*.

When solving the inverse problem, we will investigate the Cauchy and initial-boundary value problems for various differential equations. In this case, by the solution of the problem we mean the classical solution, i.e., we will assume that all derivatives and functions involved in the equation are continuous with respect to the variable x and t in a open set. As an example, let us give the definition of the solution to the inverse problem.

Definition 1.1. *A pair of functions* $\big\{u(x,t), p(t)\big\}$ *with the properties*:

(1) $\partial_t^\rho u(x,t),\ u_{xx}(x,t) \in C(\Omega)$,

(2) $t^{1-\rho}u(x,t) \in C(\overline{\Omega})$,

(3) $t^{1-\rho}p(t) \in C[0,T]$,

and satisfying conditions (1.1), (1.3) *is called* ***the solution*** *of the inverse problem.*

Note that condition (3) in this definition is taken in order to cover a wider class of functions, as function $p(t)$. In this regard, it should be noted that, to the best of our knowledge, the time-dependent source identification problem for equations with the Riemann – Liouville derivative is being studied for the first time.

Taking into account the boundary conditions in problem (1.1), it is convenient for us to introduce the Hölder classes as follows. Let $\omega_g(\delta)$ be the modulus of continuity of function $g(x) \in C[0,\pi]$, i.e.,

$$\omega_g(\delta) = \sup_{|x_1-x_2|\le\delta} \big|g(x_1) - g(x_2)\big|, \quad x_1, x_2 \in [0,\pi].$$

If $\omega_g(\delta) \le C\delta^a$ is true for some $a > 0$, where C does not depend on δ and $g(0) = g(\pi) = 0$, then $g(x)$ is said to belong to the Hölder class $C^a[0,\pi]$. Let us denote the smallest of all such constants C by $\|g\|_{C^a[0,\pi]}$. Similarly, if the continuous function $h(x,t)$ is defined on $[0,\pi]\times[0,T]$, then the value

$$\omega_h(\delta;t) = \sup_{|x_1-x_2|\le\delta} \big|h(x_1,t) - h(x_2,t)\big|, \quad x_1, x_2 \in [0,\pi],$$

is the modulus of continuity of function $h(x,t)$ with respect to the variable x. In case when $\omega_h(\delta;t) \le C\delta^a$, where C does not depend on t and δ and $h(0,t) = h(\pi,t) = 0,\ t \in [0,T]$, we say that $h(x,t)$ belongs to the Hölder class $C_x^a(\overline{\Omega})$. Similarly, we denote the smallest constant C by $\|h\|_{C_x^a(\overline{\Omega})}$.

Let $C_{2,x}^a(\overline{\Omega})$ denote the class of functions $h(x,t)$ such that $h_{xx}(x,t) \in C_x^a(\overline{\Omega})$ and $h(0,t) = h(\pi,t) = 0$, $t \in [0,T]$. Note that condition $h_{xx}(x,t) \in C_x^a(\overline{\Omega})$ implies that $h_{xx}(0,t) = h_{xx}(\pi,t) = 0,\ t \in [0,T]$. For a function of one variable $g(x)$, we introduce classes $C_2^a[0,\pi]$ in a similar way.

Theorem 1.1. *Let* $a > \dfrac{1}{2}$ *and the following conditions be satisfied*:

(1) $t^{1-\rho}f(x,t) \in C_x^a(\overline{\Omega})$,

(2) $\varphi \in C^a[0,\pi]$,

(3) $t^{1-\rho}\psi(t),\ t^{1-\rho}\partial_t^\rho\psi(t) \in C[0,T]$,

(4) $q \in C_2^a[0,\pi],\ B[q(x)] \ne 0$.

Then the inverse problem has a unique solution $\big\{u(x,t), p(t)\big\}$.

Everywhere below we denote by a an arbitrary number greater than $1/2$: $a > 1/2$.

If we additionally require that the initial function $\varphi \in C_2^a[0,\pi]$, then we can establish the following result on the stability of the solution of the inverse problem.

Theorem 1.2. *Let assumptions of Theorem* 1.1 *be satisfied and* $\varphi \in C_2^a[0,\pi]$. *Then the solution to the invest problem obeys the stability estimate*

$$\big\|t^{1-\rho}\partial_t^\rho u\big\|_{C(\overline{\Omega})} + \|t^{1-\rho}u_{xx}\|_{C(\overline{\Omega})} + \|t^{1-\rho}p\|_{C[0,T]}$$
$$\le C_{\rho,q,B}\Big[\|\varphi_{xx}\|_{C^a[0,\pi]} + \|t^{1-\rho}\psi\|_{C[0,T]} + \|t^{1-\rho}\partial_t^\rho\psi\|_{C[0,T]} + \big\|t^{1-\rho}f(x,t)\big\|_{C_x^a(\overline{\Omega})}\Big],$$

where $C_{\rho,q,B}$ *is a constant*, *depending only on* ρ, q *and* B.

It should be noted that the method proposed here, based on the Fourier method, is applicable to the equation in (1.1) with an arbitrary elliptic differential operator $A(x,D)$ instead of d^2/dx^2, if only the corresponding spectral problem has a complete system of orthonormal eigenfunctions in $L_2(G)$, $G \subset R^N$.

The interest in the study of source (right-hand side of the equation $F(x,t)$) identification inverse problems is caused primarily in connection with practical requirements in various branches of mechanics, seismology, medical tomography, and geophysics (see, e.g., the survey paper [7]). The identification of $F(x,t) = h(t)$ is appropriate, for example, in cases of accidents at nuclear power plants, when it can be assumed that the location of the source is known, but the decay of the radiation power over time is unknown and it is important to estimate it. On the other hand, one example of the identification of $F(x,t) = g(x)$ can be the detection of illegal wastewater discharges, which is a serious problem in some countries.

The inverse problem of determining the source function F with the final time observation have been well studied and many theoretical researches have been published for classical partial differential equations (see, e.g., [8, 9]). As for fractional differential equations, it is possible to construct theories parallel to [8, 9], and the work is now ongoing. Let us mention only some of these works (a detailed review can be found in [7]).

It should be noted right away that for the abstract case of the source function $F(x,t)$ there is currently no general closed theory. Known results deal with separated source term $F(x,t) = h(t)g(x)$. The appropriate choice of the overdetermination depends on the choice whether the unknown is $h(t)$ or $g(x)$.

Relatively fewer works are devoted to the case when the unknown is the function $h(t)$ (see the survey work [7] and [10] for the case of subdiffusion equations, and, for example, [4 – 6] for the classical heat equation).

Uniqueness questions in the inverse problem of finding a function $g(x)$ in fractional diffusion equations with the sourse function $g(x)h(t)$ has been studied in, e.g., [11 – 13].

In many papers, authors have considered an equation, in which $h(t) \equiv 1$ and $g(x)$ is unknown (see, e.g., [14 – 20]). The case of subdiffusion equations whose elliptic part is an ordinary differential expression is considered in [14 – 19]. The authors of the articles [21 – 25] studied subdiffusion equations in which the elliptic part is either a Laplace operator or a second-order selfadjoint operator. The paper [26] studied the inverse problem for the abstract subdiffution equation. In article [26] and most other articles, including [21 – 24], the Caputo derivative is used as a fractional derivative. The subdiffusion equation considered in the recent papers [3, 27] contains the fractional Riemann – Liouville derivative, and the elliptical part is an arbitrary elliptic expression of order m. In [25, 28], the fractional derivative in the subdiffusion equation is a two-parameter generalized Hilfer fractional derivative. Note also that the papers [21, 24, 28] contain a survey of papers dealing with inverse problems of determining the right-hand side of the subdiffusion equation.

In [25, 29, 30], non-self-adjoint differential operators (with nonlocal boundary conditions) were taken as elliptical part of the equation, and solutions to the inverse problem were found in the form of biortagonal series.

In [20], the authors considered an inverse problem for simultaneously determining the order of the Riemann – Liouville fractional derivative and the source function in the subdiffusion equations. Using the classical Fourier method, the authors proved the uniqueness and existence of a solution to this inverse problem.

It should be noted that in all of the listed works, the Cauchy conditions in time are considered (an exception is work [31], where the integral condition is set with respect to the variable t). In the recent paper [32], for the best of our knowledge, an inverse problem for subdiffusion equation with a nonlocal condition in time is considered for the first time.

The papers [33, 34] deal with the inverse problem of determining the order of the fractional derivative in the subdiffusion equation and in the wave equation, respectively.

Time-dependent source identification problem (1.1) for classical Schrödinger type equations (i.e., $\rho = 1$) with additional condition (1.3) was for the first time investigated in papers of [4–6]. To investigate the inverse problem (1.1), (1.3) we borrow some original ideas from these papers.

2. Preliminaries. In this section, we recall some information about Mittag-Leffler functions, differential and integral equations, which we will use in the following sections.

For $0 < \rho < 1$ and an arbitrary complex number μ, by $E_{\rho,\mu}(z)$ we denote the Mittag-Leffler function of complex argument z with two parameters:

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}. \tag{2.1}$$

If the parameter $\mu = 1$, then we have the classical Mittag-Leffler function: $E_\rho(z) = E_{\rho,1}(z)$.

Since $E_{\rho,\mu}(z)$ is an analytic function of z, then it is bounded for $|z| \le 1$. On the other hand, the well-known asymptotic estimate of the Mittag-Leffler function has the following form (see, e.g., [35, p. 133]):

Lemma 2.1. *Let μ be an arbitrary complex number. Further, let α be a fixed number such that $\frac{\pi}{2}\rho < \alpha < \pi\rho$ and $\alpha \le |\arg z| \le \pi$. Then the following asymptotic estimate holds*:

$$E_{\rho,\mu}(z) = -\sum_{k=1}^{2} \frac{z^{-k}}{\Gamma(\rho - k\mu)} + O(|z|^{-3}), \quad |z| > 1.$$

We can choose the parameter α so that the following estimate is valid.

***Corollary* 2.1.** *For any $t \ge 0$ one has*

$$\left|E_{\rho,\mu}(it)\right| \le \frac{C}{1+t},$$

where constant C does not depend on t and μ.

We will also use a coarser estimate with positive number λ and $0 < \varepsilon < 1$:

$$\left|t^{\rho-1}E_{\rho,\rho}(-i\lambda t^\rho)\right| \le \frac{Ct^{\rho-1}}{1+\lambda t^\rho} \le C\lambda^{\varepsilon-1}t^{\varepsilon\rho-1}, \quad t > 0, \tag{2.2}$$

which is easy to verify. Indeed, let $t^\rho\lambda < 1$, then $t < \lambda^{-1/\rho}$ and $t^{\rho-1} = t^{\rho-\varepsilon\rho}t^{\varepsilon\rho-1} < \lambda^{\varepsilon-1}t^{\varepsilon\rho-1}$. If $t^\rho\lambda \ge 1$, then $\lambda^{-1} \le t^\rho$ and $\lambda^{-1}t^{-1} = \lambda^{-1+\varepsilon}\lambda^{-\varepsilon}t^{-1} \le \lambda^{\varepsilon-1}t^{\varepsilon\rho-1}$.

Lemma 2.2. *Let $t^{1-\rho}g(t) \in C[0,T]$. Then the unique solution of the Cauchy problem*

$$\begin{aligned} &i\partial_t^\rho y(t) + \lambda y(t) = g(t), \quad 0 < t \le T, \\ &\lim_{t\to 0} J_t^{\rho-1} y(t) = y_0 \end{aligned} \tag{2.3}$$

has the form

$$y(t) = t^{\rho-1}E_{\rho,\rho}(i\lambda t^\rho)y_0 - i\int_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i\lambda(t-s)^\rho)g(s)ds.$$

Proof. Multiply equation (2.3) by $(-i)$ and then apply formula (7.2.16) of [36, p. 174] (see also [37, 38]).

Let us denote by A the operator $-d^2/dx^2$ with the domain $D(A) = \{v(x) \in W_2^2(0,\pi)\colon v(0) = v(\pi) = 0\}$, where $W_2^2(0,\pi)$ is the standard Sobolev space. Operator A is selfadjoint in $L_2(0,\pi)$ and has the complete in $L_2(0,\pi)$ set of eigenfunctions $\{v_k(x) = \sin kx\}$ and eigenvalues $\lambda_k = k^2,\ k = 1, 2, \ldots$.

Consider the operator $E_{\rho,\mu}(itA)$, defined by the spectral theorem of J. von Neumann:

$$E_{\rho,\mu}(itA)h(x,t) = \sum_{k=1}^{\infty} E_{\rho,\mu}(it\lambda_k)h_k(t)v_k(x).$$

Here and everywhere below, by $h_k(t)$ we will denote the Fourier coefficients of a function $h(x,t)$: $h_k(t) = (h(x,t), v_k)$, $(\cdot,\cdot)$ stands for the scalar product in $L_2(0,\pi)$. This series converges in the $L_2(0,\pi)$-norm. But we need to investigate the uniform convergence of this series in Ω. To do this, we recall the following statement.

Lemma 2.3. *Let* $g \in C^a[0,\pi]$. *Then, for any* $\sigma \in [0, a - 1/2)$, *one has*

$$\sum_{k=1}^{\infty} k^{\sigma}|g_k| < \infty.$$

For $\sigma = 0$ this assertion coincides with the well-known theorem of S. N. Bernshtein on the absolute convergence of trigonometric series and is proved in exactly the same way as this theorem. For the convenience of readers, we recall the main points of the proof (see, e.g., [39, p. 384]).

Proof. In Theorem 3.1 of A. Zygmund [39, p. 384], it is proved that for an arbitrary function $g(x) \in C[0,\pi]$, with the properties $g(0) = g(\pi) = 0$, one has the estimate

$$\sum_{k=2^{n-1}+1}^{2^n} |g_k|^2 \le \omega_g^2\left(\frac{1}{2^{n+1}}\right).$$

Therefore, if $\sigma \ge 0$, then by the Cauchy – Bunyakovsky inequality

$$\sum_{k=2^{n-1}+1}^{2^n} k^{\sigma}|g_k| \le \left(\sum_{k=2^{n-1}+1}^{2^n} |g_k|^2\right)^{\frac12}\left(\sum_{k=2^{n-1}+1}^{2^n} k^{2\sigma}\right)^{\frac12} \le C2^{n(\frac12+\sigma)}\omega_g\left(\frac{1}{2^{n+1}}\right),$$

and finally

$$\sum_{k=2}^{\infty} k^{\sigma}|g_k| = \sum_{n=1}^{\infty}\sum_{k=2^{n-1}+1}^{2^n} k^{\sigma}|g_k| \le C\sum_{n=1}^{\infty} 2^{n(\frac12+\sigma)}\omega_g\left(\frac{1}{2^{n+1}}\right).$$

Obviously, if $\omega_g(\delta) \le C\delta^a$, $a > 1/2$ and $0 < \sigma < a - 1/2$, then the last series converges:

$$\sum_{k=2}^{\infty} k^{\sigma}|g_k| \le C\|g\|_{C^a[0,\pi]}.$$

Lemma 2.4. *Let* $h(x,t) \in C_x^a(\overline{\Omega})$. *Then* $E_{\rho,\mu}(itA)h(x,t) \in C(\overline{\Omega})$ *and* $\dfrac{\partial^2}{\partial x^2}E_{\rho,\mu}(itA)h(x,t) \in C\big([0,\pi]\times(0,T]\big)$. *Moreover, the following estimates hold*:

$$\big\|E_{\rho,\mu}(itA)h(x,t)\big\|_{C(\overline{\Omega})} \le C\|h\|_{C_x^a(\overline{\Omega})}, \tag{2.4}$$

$$\left\|\frac{\partial^2}{\partial x^2}E_{\rho,\mu}(itA)h(x,t)\right\|_{C[0,\pi]} \le Ct^{-1}\|h\|_{C_x^a(\overline{\Omega})}, \quad t>0. \tag{2.5}$$

If $h(x,t)\in C_{2,x}^a(\overline{\Omega})$, *then*

$$\left\|\frac{\partial^2}{\partial x^2}E_{\rho,\mu}(itA)h(x,t)\right\|_{C(\overline{\Omega})} \le C\|h_{xx}\|_{C_x^a(\overline{\Omega})}. \tag{2.6}$$

Proof. By definition one has

$$\big|E_{\rho,\mu}(itA)h(x,t)\big| = \left|\sum_{k=1}^{\infty}E_{\rho,\mu}(it\lambda_k)h_k(t)v_k(x)\right| \le \sum_{k=1}^{\infty}\big|E_{\rho,\mu}(it\lambda_k)h_k(t)\big|.$$

Corollary 2.1 and Lemma 2.3 imply that

$$\big|E_{\rho,\mu}(itA)h(x,t)\big| \le C\sum_{k=1}^{\infty}\left|\frac{h_k(t)}{1+t\lambda_k}\right| \le C\|h\|_{C_x^a(\overline{\Omega})}.$$

On the other hand,

$$\left|\frac{\partial^2}{\partial x^2}E_{\rho,\mu}(itA)h(x,t)\right| \le C\sum_{k=1}^{\infty}\left|\frac{\lambda_k h_k(t)}{1+t\lambda_k}\right| \le Ct^{-1}\|h\|_{C_x^a(\overline{\Omega})}, \quad t>0.$$

If $h(x,t)\in C_{2,x}^a(\overline{\Omega})$, then $h_k(t) = -\lambda_k^{-1}(h_{xx})_k(t)$. Therefore,

$$\left|\frac{\partial^2}{\partial x^2}E_{\rho,\mu}(itA)h(x,t)\right| \le C\|h_{xx}\|_{C_x^a(\overline{\Omega})}, \quad 0\le t\le T.$$

Lemma 2.5. *Let* $t^{1-\rho}g(x,t)\in C_x^a(\overline{\Omega})$. *Then there exists a positive constant* c_1 *such that*

$$\left|t^{1-\rho}\int\limits_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i(t-s)^{\rho}A)g(x,s)ds\right| \le c_1\frac{t^{\rho}}{\rho}\|t^{1-\rho}g\|_{C_x^a(\overline{\Omega})}. \tag{2.7}$$

Proof. Applying estimate (2.4), we get

$$\left|t^{1-\rho}\int\limits_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i(t-s)^{\rho}A)g(x,s)ds\right| \le Ct^{1-\rho}\int\limits_0^t (t-s)^{\rho-1}s^{\rho-1}ds\cdot\|t^{1-\rho}g\|_{C_x^a(\overline{\Omega})}.$$

For the integral one has

$$\int\limits_0^t (t-s)^{\rho-1}s^{\rho-1}ds = \int\limits_0^{\frac{t}{2}}\cdot + \int\limits_{\frac{t}{2}}^{t}\cdot \le \frac{2^{2(1-\rho)}}{\rho}t^{2\rho-1}. \tag{2.8}$$

Denoting $c_1 = 4C$, we obtain the assertion of the lemma.

***Corollary* 2.2.** *If function $g(x,t)$ can be represented in the form $g_1(x)g_2(t)$, then the right-hand side of estimate* (2.7) *has the form*

$$c_1\frac{t^\rho}{\rho}\|g_1\|_{C^a[0,\pi]}\|t^{1-\rho}g_2\|_{C[0,T]}.$$

Lemma 2.6. *Let $t^{1-\rho}g(x,t)\in C_x^a(\overline{\Omega})$. Then*

$$\left\|\int_0^t (t-s)^{\rho-1}\frac{\partial^2}{\partial x^2}E_{\rho,\rho}(i(t-s)^\rho A)g(x,s)ds\right\|_{C(\overline{\Omega})}\leq C\|t^{1-\rho}g\|_{C_x^a(\overline{\Omega})}.$$

Proof. Let

$$S_j(x,t)=\sum_{k=1}^j\left[\int_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i\lambda_k(t-s)^\rho)g_k(s)ds\right]\lambda_k v_k(x).$$

Choosing ε so that $0<\varepsilon<a-1/2$ and applying the inequality (2.2), we get

$$|S_j(t)|\leq C\sum_{k=1}^j\int_0^t (t-s)^{\varepsilon\rho-1}s^{\rho-1}\lambda_k^\varepsilon|s^{1-\rho}g_k(s)|ds.$$

By Lemma 2.3 we have

$$|S_j(t)|\leq C\|t^{1-\rho}g\|_{C_x^a(\overline{\Omega})},$$

and since

$$\int_0^t (t-s)^{\rho-1}\frac{\partial^2}{\partial x^2}E_{\rho,\rho}\big(i(t-s)^\rho A\big)h(s)ds=\sum_{j=1}^\infty S_j(t),$$

the last inequality implies the assertion of the lemma.

Lemma 2.7. *Let $t^{1-\rho}G(x,t)\in C_x^a(\overline{\Omega})$ and $\varphi\in C^a[0,\pi]$. Then the unique solution of the following initial-boundary value problem*:

$$i\partial_t^\rho w(x,t)-w_{xx}(x,t)=G(x,t),\quad 0<t\leq T,$$

$$w(0,t)=w(\pi,t)=0,\quad 0<t\leq T,$$

$$\lim_{t\to 0}J_t^{\rho-1}w(x,t)=\varphi(x),\quad 0\leq x\leq\pi,$$

has the form

$$w(x,t)=t^{\rho-1}E_\rho(it^\rho A)\varphi(x)-i\int_0^t (t-s)^{\rho-1}E_{\rho,\rho}\big(i(t-s)^\rho A\big)G(x,s)ds.$$

Proof. According to the Fourier method, we will seek the solution to this problem in the form

$$w(x,t)=\sum_{k=1}^\infty T_k(t)v_k(x),$$

where $T_k(t)$ are the unique solutions of the problems

$$i\partial_t^\rho T_k + \lambda_k T_k(t) = G_k(t), \quad 0 < t \le T,$$
$$\lim_{t\to 0} J_t^{\rho-1} T_k(t) = \varphi_k.$$

Lemma 2.2 implies that

$$T_k(t) = t^{\rho-1}E_\rho(i\lambda_k t^\rho)\varphi_k - i\int_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i\lambda_k(t-s)^\rho)G_k(s)ds.$$

Hence, the solution to problem (3.1) has the form

$$w(x,t) = t^{\rho-1}E_\rho(it^\rho A)\varphi(x) - i\int_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i(t-s)^\rho A)G(x,s)ds.$$

Note that the existence of the first term follows from estimate (2.4), and the existence of the second term follows from Lemma 2.5.

By Lemma 2.6 and estimate (2.5), we obtain that $w_{xx}(x,t) \in C(\Omega)$. Since $i\partial_t^\rho w(x,t) = -w_{xx}(x,t) + G(x,t)$, then $\partial_t^\rho w(x,t) \in C(\Omega)$.

The uniqueness of the solution can be proved by the standard technique based on completeness of the set of eigenfunctions $\{v_k(x)\}$ in $L_2(0,\pi)$ (see, e.g., [3]).

Let $t^{1-\rho}F(x,t) \in C(\overline{\Omega})$ and $g(x) \in C^a[0,\pi]$. Consider the Volterra integral equation

$$w(x,t) = F(x,t) + \int_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i(t-s)^\rho A)g(x)B\big[w(\cdot,s)\big]ds. \tag{2.9}$$

Lemma 2.8. *There exists a unique solution $t^{1-\rho}w \in C(\overline{\Omega})$ to the integral equation* (2.9).

Proof. Equation (2.9) is similar to the equations considered in the book [40, p. 199] (Eq. (3.5.4)) and it is solved in essentially the same way. Let us remind the main points.

Equation (2.9) makes sense in any interval $[0,t_1] \in [0,T]$, $0 < t_1 < T$. Choose t_1 such that

$$c_1 b\|g\|_{C^a[0,\pi]}\frac{t_1^\rho}{\rho} < 1 \tag{2.10}$$

and prove the existence of a unique solution $t^{1-\rho}w(x,t) \in C\big([0,\pi]\times[0,t_1]\big)$ to the equation (2.9) on the interval $[0,t_1]$ (here the constant c_1 is taken from estimate (2.7), see Corollary 2.2). For this we use the Banach fixed point theorem for the space $C\big([0,\pi]\times[0,t_1]\big)$ with the weight function $t^{1-\rho}$ (see, e.g., [40, p. 68], Theorem 1.9), where the distance is given by

$$d(w_1,w_2) = \big\|t^{1-\rho}[w_1(x,t) - w_2(x,t)]\big\|_{C([0,\pi]\times[0,t_1])}.$$

Let us denote the right-hand side of equation (2.9) by $Pw(x,t)$, where P is the corresponding linear operator. Applying the Banach fixed point theorem, we have to prove the following:

(a) if $t^{1-\rho}w(x,t) \in C\big([0,\pi]\times[0,t_1]\big)$, then $t^{1-\rho}Pw(x,t) \in C\big([0,\pi]\times[0,t_1]\big)$;

(b) for any $t^{1-\rho}w_1, t^{1-\rho}w_2 \in C\big([0,\pi]\times[0,t_1]\big)$ one has

$$d(Pw_1, Pw_2) \le \delta \cdot d(w_1, w_2), \quad \delta < 1.$$

Lemmas 2.4 and 2.5 imply condition (a). On the other hand, thanks to (2.7) (see Corollary 2.2) we arrive at

$$\left\| t^{1-\rho}\int\limits_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i(t-s)^\rho A)g(x)B[w_1(\cdot,s)-w_2(\cdot,s)]ds \right\|_{C([0,\pi]\times[0,t_1])} \le \delta d(w_1,w_2),$$

where $\delta = c_1 b \|g\|_{C^a[0,\pi]}\dfrac{t_1^\rho}{\rho} < 1$ since condition (2.10).

Hence, by the Banach fixed point theorem, there exists a unique solution $t^{1-\rho}w^\star(x,t) \in C\big([0,\pi]\times[0,t_1]\big)$ to equation (2.9) on the interval $[0,t_1]$, and this solution is a limit of the convergent sequence $w_n(x,t) = P^nF(x,t) = PP^{n-1}F(x,t)$:

$$\lim_{n\to\infty} d\big(w_n(x,t), w^\star(x,t)\big) = 0.$$

Next we consider the interval $[t_1,t_2]$, where $t_2 = t_1 + l_1 < T$, and $l_1 > 0$. Rewrite the equation (2.9) in the form

$$w(x,t) = F_1(x,t) + \int\limits_{t_1}^t (t-s)^{\rho-1}E_{\rho,\rho}(i(t-s)^\rho A)g(x)B\big[w(\cdot,s)\big]ds$$

where

$$F_1(x,t) = F(x,t) + \int\limits_0^{t_1} (t-s)^{\rho-1}E_{\rho,\rho}(i(t-s)^\rho A)g(x)B\big[w(\cdot,s)\big]ds$$

is a known function, since the function $w(x,t)$ is uniquely defined on the interval $[0,t_1]$. Using the same arguments as above, we derive that there exists a unique solution $t^{1-\rho}w^\star(x,t) \in C([0,\pi]\times[t_1,t_2])$ to equation (2.9) on the interval $[t_1,t_2]$. Taking the next interval $[t_2,t_3]$, where $t_3 = t_2 + l_2 < T$ and $l_2 > 0$, and repeating this process (obviously, $l_n > l_0 > 0$), we conclude that there exists a unique solution $t^{1-\rho}w^\star(x,t) \in C\big([0,\pi]\times[0,T]\big)$ to equation (2.9) on the interval $[0,T]$, and this solution is a limit of the convergent sequence $t^{1-\rho}w_n(x,t) \in C\big([0,\pi]\times[0,T]\big)$:

$$\lim_{n\to\infty} \big\|t^{1-\rho}[w_n(x,t) - w^\star(x,t)]\big\|_{C(\overline{\Omega})} = 0,$$

with the choice of certain w_n on each $[0,t_1], \ldots [t_{L-1},T]$.

We need the following kind of Gronwall's inequality:

Lemma 2.9. *Let $0 < \rho < 1$. Assume that the nonnegative function $h(t) \in C[0,T]$ and the positive constants K_0 and K_1 satisfy*

$$h(t) \le K_0 + K_1\int\limits_0^t (t-s)^{\rho-1}s^{\rho-1}h(s)ds$$

for all $t \in [0,T]$. Then there exists a positive constant $C_{\rho,T}$, depending only on ρ, K_2 and T, such that

$$h(t) \le K_0 C_{\rho,T}. \tag{2.11}$$

Usually Gronwall's inequality is formulated with a continuous function $k(s)$ instead of $K_1(t-s)^{\rho-1}s^{\rho-1}$. However, estimate (2.11) is proved in a similar way to the Gronwall inequality. For the convenience of the reader, we present a proof of estimate (2.11).

Proof. Iterating the hypothesis of Gronwall's inequality gives

$$h(t) \le K_0 + K_0K_1 \int_0^t (t-s)^{\rho-1}s^{\rho-1}ds + K_1^2 \int_0^t (t-s)^{\rho-1}s^{\rho-1} \int_0^s (s-\xi)^{\rho-1}\xi^{\rho-1}h(\xi)d\xi ds$$

$$\le K_{\rho,T} + K_1^2 \int_0^t u(\xi)\xi^{\rho-1} \int_\xi^t (t-s)^{\rho-1}(s-\xi)^{\rho-1}s^{\rho-1}dsd\xi,$$

where

$$K_{\rho,T} = K_0 + K_0K_1 \int_0^T (t-s)^{\rho-1}s^{\rho-1}ds.$$

For the inner integral we have (see (2.8))

$$\int_\xi^t (s-\xi)^{\rho-1}(t-s)^{\rho-1}ds = \int_0^{t-\xi} y^{\rho-1}(t-\xi-y)^{\rho-1}dy \le \frac{2^{2(1-\rho)}}{\rho}(t-\xi)^{2\rho-1}.$$

Now the hypothesis is

$$h(t) \le K_0 + K_1^2 \frac{2^{2(1-\rho)}}{\rho} \int_0^t (t-s)^{2\rho-1}h(s)ds.$$

By repeating this process so many times that $k\rho > 1$, we make sure that there is a positive constant $C_\rho = C(\rho, K_2, T) > 0$ such that

$$h(t) \le K_0 + C_\rho \int_0^t h(s)ds$$

or

$$\frac{h(\xi)}{K_0 + C_\rho \int_0^\xi h(s)ds} \le 1.$$

Multiplying this by C_ρ, we get

$$\frac{d}{d\xi} \ln\left(K_0 + C_\rho \int_0^\xi h(s)ds \right) \le C_\rho.$$

Integrating from $\xi = 0$ to $\xi = t$ and exponentiating, we obtain

$$K_0 + C_\rho \int_0^t h(s)ds \le K_0 e^{C_\rho t}.$$

Finally note that the left-hand side is $\ge h(t)$.

3. Auxiliary problem and proof of Theorem 1.1. Let us consider the following auxiliary initial-boundary value problem:

$$\begin{aligned} & i\partial_t^\rho \omega(x,t) - \omega_{xx}(x,t) = -i\mu(t)q''(x) + f(x,t), \quad (x,t)\in\Omega, \\ & \omega(0,t) = \omega(\pi,t) = 0, \quad 0\le t\le T, \\ & \lim_{t\to 0} J_t^{\rho-1}\omega(x,t) = \varphi(x), \quad 0\le x\le\pi, \end{aligned} \tag{3.1}$$

where function $\mu(t)$ is the unique solution of the Cauchy problem

$$\begin{aligned} & \partial_t^\rho \mu(t) = p(t), \quad 0 < t \le T, \\ & \lim_{t\to 0} J_t^{\rho-1}\mu(t) = 0. \end{aligned} \tag{3.2}$$

Note that the solution to the Cauchy problem (3.2) has the form (see, e.g., [37])

$$\mu(t) = \frac{1}{\Gamma(\rho)}\int\limits_0^t (t-s)^{\rho-1} p(s)ds.$$

Definition 3.1. *A functions $\omega(x,t)$ with the properties*:
(1) $\partial_t^\rho \omega(x,t), \omega_{xx}(x,t)\in C(\Omega)$,
(2) $t^{1-\rho}\omega_{xx}(x,t)\in C((0,\pi)\times[0,T])$,
(3) $t^{1-\rho}\omega(x,t)\in C(\overline{\Omega})$,
satisfying conditions (3.2), *is called the solution of problem* (3.2).

Lemma 3.1. *Let $\omega(x,t)$ be a solution of problem* (3.1). *Then the unique solution $\{u(x,t),p(t)\}$ to the inverse problem* (1.1), (1.3) *has the form*

$$u(x,t) = \omega(x,t) - i\mu(t)q(x), \tag{3.3}$$

$$p(t) = \frac{i}{B[q(x)]}\big\{\partial_t^\rho\psi(t) - B[\partial_t^\rho\omega(\cdot,t)]\big\}, \tag{3.4}$$

where

$$\mu(t) = \frac{i}{B[q(x)]}\big[\psi(t) - B[\omega(\cdot,t)]\big]. \tag{3.5}$$

Proof. Substitute the function $u(x,t)$, defined by equality (3.3), into the equation in (1.1). Then

$$i\partial_t^\rho\omega(x,t) + \partial_t^\rho\mu(t)q(x) - \omega_{xx}(x,t) + i\mu(t)q''(x) = p(t)q(x) + f(x,t).$$

Since $\partial_t^\rho\mu(t) = p(t)$ (see (3.2)), we obtain equation (3.1), i.e., function $u(x,t)$, defined by (3.3), is a solution of the equation in (1.1). As for the initial condition, again by virtue of (3.2) we get

$$\lim_{t\to 0} J_t^{\rho-1}u(x,t) = \lim_{t\to 0} J_t^{\rho-1}\omega(x,t) - i\lim_{t\to 0} J_t^{\rho-1}\mu(t)q(x) = \lim_{t\to 0} J_t^{\rho-1}\omega(x,t) = \varphi(x).$$

On the other hand, conditions $q(0)=q(\pi)=0$ imply $u(0,t)=u(\pi,t)=0,\ 0\le t\le T$.

From Definition 3.1 of solution $\omega(x,t)$ and the property of the functions $\mu(t)$ and $q(x)$ it immediately follows that the function $u(x,t)$ satisfies the requirements: $\partial_t^\rho u(x,t), u_{xx}(x,t) \in C(\Omega)$, $t^{1-\rho}u(x,t) \in C(\overline{\Omega})$.

Thus, function $u(x,t)$, defined as (3.3), is a solution of the initial-boundary value problem (1.1).

Let us prove equation (3.4). Rewrite (3.3) as

$$iq(x)\mu(t) = \omega(x,t) - u(x,t).$$

Applying (1.3), we obtain

$$i\mu(t)B[q(x)] = B\big[\omega(\cdot,t)\big] - \psi(t),$$

or, since $B[q(x)] \neq 0$, we get (3.5). Finally, by using equality $\partial_t^\rho \mu(t) = p(t)$, we have

$$p(t) = \frac{i}{B[q(x)]}\Big[\partial_t^\rho \psi(t) - B\big[\partial_t^\rho \omega(\cdot,t)\big]\Big],$$

which coincides with (3.4). Moreover, from the definition of solution $\omega(x,t)$ of problem (3.1) and the property of function $\psi(t)$ one has $t^{1-\rho}p(t) \in C[0,T]$.

Thus, to solve the inverse problem (1.1), (1.3), it is sufficient to solve the initial-boundary value problem (3.1).

Theorem 3.1. *Under the assumptions of Theorem* 1.1, *problem* (3.1) *has a unique solution.*

Proof. Let

$$G(x,s) = \frac{i}{B[q(x)]}\Big(B\big[\omega(\cdot,s)\big] - \psi(s)\Big)q''(x) + f(x,s) \tag{3.6}$$

and suppose that $s^{1-\rho}G(x,s) \in C_x^a(\overline{\Omega})$. Then by Lemma 2.7 problem (3.1) is equivalent to the integral equation

$$\omega(x,t) = t^{\rho-1}E_\rho(it^\rho A)\varphi(x) - i\int_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i(t-s)^\rho A)G(x,s)ds.$$

Rewrite this equation as

$$\omega(x,t) = F(x,t) + \int_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i(t-s)^\rho A)\frac{q''(x)}{B[q(x)]}B[\omega(\cdot,s)]ds, \tag{3.7}$$

where

$$F(x,t) = t^{\rho-1}E_\rho(it^\rho A)\varphi(x) - i\int_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i(t-s)^\rho A)\Big[-\frac{iq''(x)}{B[q(x)]}\psi(s) + f(x,s)\Big]ds.$$

In order to apply Lemma 2.8 to equation (3.7), we show that $t^{1-\rho}F(x,t) \in C(\overline{\Omega})$. Indeed, by estimate (2.4) one has $E_\rho(it^\rho A)\varphi(x) \in C(\overline{\Omega})$. According to the conditions of Theorem 1.1 $h(x,s) = s^{1-\rho}\Big[-\frac{iq''(x)}{B[q(x)]}\psi(s) + f(x,s)\Big] \in C_x^a(\overline{\Omega})$. Therefore, by virtue of estimate (2.7), the second term of function $t^{1-\rho}F(x,t)$ also belongs to the class $C(\overline{\Omega})$. Hence, by virtue of Lemma 2.8, the Volterra equation (3.7) has a unique solution $t^{1-\rho}\omega(x,t) \in C(\overline{\Omega})$.

Let us show that $\partial_t^\rho \omega(x,t),\ \omega_{xx}(t) \in C(\Omega)$. First we consider $F_{xx}(x,t)$ and note that, by estimate (2.5), we have $\dfrac{\partial^2}{\partial x^2}E_\rho(it^\rho A)\varphi(x) \in C([0,\pi]\times(0,T])$. Since function h defined above, belongs to the class $C_x^a(\overline{\Omega})$, then, by Lemma 2.6, the second term of function $F_{xx}(x,t)$ belongs to $C(\Omega)$ and satisfies the estimate

$$\left\| t^{1-\rho}\int_0^t (t-s)^{\rho-1}\frac{\partial^2}{\partial x^2}E_{\rho,\rho}(i(t-s)^\rho A)\left[-\frac{iq''(x)}{B[q(x)]}\psi(s)+f(x,s)\right]ds\right\|_{C(\overline{\Omega})}$$

$$\leq C\left[\left\|t^{1-\rho}\frac{q''(x)}{B[q(x)]}\psi(t)\right\|_{C_x^a(\overline{\Omega})}+\left\|t^{1-\rho}f(x,t)\right\|_{C_x^a(\overline{\Omega})}\right]$$

$$\leq C_{a,q,B}\Big[\|t^{1-\rho}\psi\|_{C[0,T]}+\left\|t^{1-\rho}f(x,t)\right\|_{C_x^a(\overline{\Omega})}\Big]. \tag{3.8}$$

We pass to the second term on the right-hand side of equality (3.7). Since $t^{1-\rho}\omega(x,t)\in C(\overline{\Omega})$, the conditions of Theorem 1.1 imply that $s^{1-\rho}\dfrac{q''(x)}{B[q(x)]}B[\omega(\cdot,s)] \in C_x^a(\overline{\Omega})$. Then again by Lemma 2.6, this term belongs to $C(\overline{\Omega})$ and satisfies the estimate

$$\left\| t^{1-\rho}\int_0^t (t-s)^{\rho-1}\frac{\partial^2}{\partial x^2}E_{\rho,\rho}(i(t-s)^\rho A)\frac{q''(x)}{B[q(x)]}B[\omega(\cdot,s)]ds\right\|_{C(H)}$$

$$\leq C\left\|\frac{q''(x)}{B[q(x)]}B[t^{1-\rho}\omega(\cdot,t)]\right\|_{C(\overline{\Omega})}\leq C_{a,q,B}\|t^{1-\rho}\omega(x,t)\|_{C(\overline{\Omega})}. \tag{3.9}$$

Thus, $\omega_{xx}(x,t)\in C((0,T];H)$. On the other hand, by virtue of equation (3.1) and the conditions of Theorem 1.1, we have

$$\partial_t^\rho\omega(x,t)=\omega_{xx}(x,t)-i\mu(t)q''(x)+f(x,t)\in C(\overline{\Omega})).$$

The fact that here $\mu\in C[0,T]$ follows again from the conditions of the Theorem 1.1 and equality (3.5).

It remains to show that $t^{1-\rho}G(x,t)\in C_x^a(\overline{\Omega})$. But this fact follows from the conditions of Theorem 1.1 and the already proven assertion: $t^{1-\rho}\omega(x,t)\in C(\overline{\Omega})$.

As noted above Theorem 1.1 is an immediate consequence of Lemma 3.1 and Theorem 3.1.

4. Proof of Theorem 1.2. First we prove the following statement on the stability of the solution to problem (3.1), (3.2).

Theorem 4.1. *Let assumptions of Theorem* 1.2 *be satisfied. Then the solution to problem* (3.1), (3.2) *obeys the stability estimate*

$$\|t^{1-\rho}\partial_t^\rho\omega\|_{C(\overline{\Omega})}\leq C_{\rho,q,B}\Big[\|\varphi_{xx}\|_{C^a[0,\pi]}+\|t^{1-\rho}\psi\|_{C[0,T]}+\|t^{1-\rho}f(x,t)\|_{C_x^a(\overline{\Omega})}\Big], \tag{4.1}$$

where $C_{\rho,q,B,\epsilon}$ is a constant, depending only on ρ, q and B.

Proof. Let us begin the proof of the inequality (4.1) by establishing an estimate for $\omega_{xx}(x,t)$ and then use it with equation (3.1). To this end we have from (2.6):

$$\left\| \frac{\partial^2}{\partial x^2} E_\rho(it^\rho A)\varphi \right\|_{C(\overline{\Omega})} \le C\|\varphi_{xx}\|_{C^a[0,\pi]}.$$

This estimate together with (3.8) implies

$$\left\|t^{1-\rho}F_{xx}(x,t)\right\|_{C(\overline{\Omega})} \le C\|\varphi_{xx}\|_{C^a[0,\pi]} + C_{a,q,B}\left[\|t^{1-\rho}\psi\|_{C[0,T]} + \|t^{1-\rho}f(x,t)\|_{C_x^a(\overline{\Omega})}\right].$$

Then, by using equality (3.7) and inequality (3.9), we get

$$\left\|t^{1-\rho}\omega_{xx}(x,t)\right\|_{C(\overline{\Omega})} \le C\|\varphi_{xx}\|_{C^a[0,\pi]} + C_{a,q,B}\Big[\|t^{1-\rho}\psi\|_{C[0,T]}$$
$$+ \left\|t^{1-\rho}f(x,t)\right\|_{C_x^a(\overline{\Omega})} + \left\|t^{1-\rho}\omega(x,t)\right\|_{C(\overline{\Omega})}\Big]. \tag{4.2}$$

As a result, we obtained an estimate for $\omega_{xx}(x,t)$ through $\omega(x,t)$. To estimate $\left\|t^{1-\rho}\omega(x,t)\right\|_{C(\overline{\Omega})}$, we will proceed as follows. Applying estimates (2.4) and (2.7), we get

$$\left\|t^{1-\rho}F(x,t)\right\|_{C(\overline{\Omega})} \le \|\varphi\|_{C^a[0,\pi]} + \frac{T^\rho}{\rho}\left[C_{q,B}\|q''\|_{C^a[0,\pi]}\|t^{1-\rho}\psi\|_{C[0,T]} + \|t^{1-\rho}f\|_{C_x^a(\overline{\Omega})}\right].$$

Again by estimate (2.4) we have

$$\left\| t^{1-\rho}\int_0^t (t-s)^{\rho-1}E_{\rho,\rho}(i(t-s)^\rho A)\frac{q''(x)}{B[q(x)]}B[\omega(\cdot,s)]ds \right\|_{C[0,\pi]}$$
$$\le C_{q,B}\|q''\|_{C^a[0,\pi]}\int_0^t (t-s)^{\rho-1}\|\omega(x,s)\|_{C[0,\pi]}.$$

Therefore, from equation (3.7) we obtain an estimate

$$\left\|t^{1-\rho}\omega(x,t)\right\|_{C[0,\pi]} \le \|\varphi\|_{C^a[0,\pi]} + C_{q,\rho,B}\left[\|t^{1-\rho}\psi\|_{C[0,T]} + \|t^{1-\rho}f\|_{C_x^a(\overline{\Omega})}\right]$$
$$+ C_{q,B}\int_0^t (t-s)^{\rho-1}s^{\rho-1}\left\|s^{1-\rho}\omega(x,s)\right\|_{C[0,\pi]}ds$$

for all $t \in [0,T]$. Finally, the Gronwall inequality (2.11) implies

$$\|t^{1-\rho}\omega(x,t)\|_{C(\overline{\Omega})} \le C_{q,\rho,B}\left[\|\varphi\|_{C^a[0,\pi]} + \|t^{1-\rho}\psi\|_{C[0,T]} + \|t^{1-\rho}f\|_{C_x^a(\overline{\Omega})}\right].$$

Substituting this estimate in (4.2) and applying $\|\varphi\|_{C^a[0,\pi]} \le C\|\varphi_{xx}\|_{C^a[0,\pi]}$, we get

$$\|t^{1-\rho}\omega_{xx}\|_{C(\overline{\Omega})} \le C_{\rho,q,B}\left[\|\varphi_{xx}\|_{C^a[0,\pi]} + \|t^{1-\rho}\psi\|_{C[0,T]} + \|t^{1-\rho}f\|_{C_x^a(\overline{\Omega})}\right].$$

To obtain estimate (4.1), it remains to note that

$$\partial_t^\rho\omega(x,t) = \omega_{xx}(x,t) - i\mu(t)q''(x) + f(x,t)$$

and use the estimate

$$\|t^{1-\rho}\mu\|_{C[0,T]} \le C_{q,B}\Big[\|t^{1-\rho}\psi\|_{C[0,T]} + \|t^{1-\rho}\omega\|_{C(\overline{\Omega})}\Big],$$

which follows from definition (3.5) and the conditions of Theorem 1.1.

***Proof of Theorem* 1.2.** Apply (3.4) to get

$$\|t^{1-\rho}p(t)\|_{C[0,T]} \le C_{q,B}\Big[\|t^{1-\rho}\partial_t^\rho\omega\|_{C(\overline{\Omega})} + \|t^{1-\rho}\partial_t^\rho\psi\|_{C[0,T]}\Big].$$

Equations (3.3) and (3.2) imply

$$\partial_t^\rho u(x,t) = \partial_t^\rho\omega(x,t) + p(t)q(x).$$

Hence, from estimates of $\partial_t^\rho\omega(x,t)$ and $p(t)$, we obtain an estimate for $\partial_t^\rho u(x,t)$. On the other hand, by virtue of equation (1.1), we have

$$-u_{xx}(x,t) = -i\partial_t^\rho u(x,t) + p(t)q(x) + f(x,t).$$

Now, to establish estimate (4.1), it suffices to use the statement of Theorem 4.1.

The authors are grateful to A. O. Ashyralyev for posing the problem and they convey thanks to Sh. A. Alimov for discussions of these results. The authors acknowledge financial support from the Ministry of Innovative Development of the Republic of Uzbekistan (Grant No. F-FA-2021-424).

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. A. V. Pskhu, *Fractional partial differential equations* (in Russian), Nauka, Moscow (2005).
2. S. Umarov, *Introduction to fractional and pseudo-differential equations with singular symbols*, Springer (2015).
3. R. Ashurov, O. Muhiddinova, *Initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary elliptic differential operator*, Lobachevskii J. Math., **42**, № 3, 517 – 525 (2021).
4. A. Ashyralyev, M. Urun, *Time-dependent source identification problem for the Schrödinger equation with nonlocal boundary conditions*, AIP Conf. Proc., **2183**, Article 070016, Amer. Inst. Phys. (2019).
5. A. Ashyralyev, M. Urun, *On the Crank – Nicolson difference scheme for the time-dependent source identification problem*, Bull. Karaganda Univ., Math., **102**, № 2, 35 – 44 (2021).
6. A. Ashyralyev, M. Urun, *Time-dependent source identification Schrödinger type problem*, Int. J. Appl. Math., **34**, № 2, 297 – 310 (2021).
7. Y. Liu, Z. Li, M. Yamamoto, *Inverse problems of determining sources of the fractional partial differential equations*, Handbook of Fractional Calculus with Applications, vol. 2, De Gruyter (2019), p. 411 – 430.
8. S. I. Kabanikhin, *Inverse and ill-posed problems*. *Theory and applications*, De Gruyter (2011).
9. A. I. Prilepko, D. G. Orlovsky, I. A. Vasin, *Methods for solving inverse problems in mathematical physics,* Marcel Dekkers, New York (2000).
10. K. Sakamoto, M. Yamamoto, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. and Appl., **382**, 426 – 447 (2011).
11. P. Niu, T. Helin, Z. Zhang, *An inverse random source problem in a stochastic fractional diffusion equation*, Inverse Problems, **36**, № 4, Article 045002 (2020).
12. M. Slodichka, *Uniqueness for an inverse source problem of determining a space-dependent source in a non-autonomous time-fractional diffusion equation*, Fract. Calc. and Appl. Anal., **23**, № 6, 1703 – 1711 (2020); DOI: 10.1515/fca-2020-0084.
13. M. Slodichka, K. Sishskova, V. Bockstal, *Uniqueness for an inverse source problem of determining a space dependent source in a time-fractional diffusion equation*, Appl. Math. Lett., **91**, 15 – 21 (2019).

14. Y. Zhang, X. Xu, *Inverse scource problem for a fractional differential equations*, Inverse Problems, **27**, № 3, 31 – 42 (2011).
15. M. Ismailov, I. M. Cicek, *Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions*, Appl. Math. Model., **40**, 4891 – 4899 (2016).
16. M. Kirane, A. S. Malik, *Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time*, Appl. Math. and Comput., **218**, 163 – 170 (2011).
17. M. Kirane, B. Samet, B. T. Torebek, *Determination of an unknown source term and the temperature distribution for the subdiffusion equation at the initial and final data*, Electron. J. Different. Equat., **217**, 1 – 13 (2017).
18. H. T. Nguyen, D. L. Le, V. T. Nguyen, *Regularized solution of an inverse source problem for a time fractional diffusion equation*, Appl. Math. Model., **40**, 8244 – 8264 (2016).
19. B. T. Torebek, R. Tapdigoglu, *Some inverse problems for the nonlocal heat equation with Caputo fractional derivative*, Math. Methods Appl. Sci., **40**, 6468 – 6479 (2017).
20. R. Ashurov, Yu. Fayziev, *Determination of fractional order and source term in a fractional subdiffusion equation*; https: //www.researchgate.net/publication/354997348.
21. Z. Li, Y. Liu, M. Yamamoto, *Initial-boundary value problem for multi-term time-fractional diffusion equation with positive constant coefficients*, Appl. Math. and Comput., **257**, 381 – 397 (2015).
22. W. Rundell, Z. Zhang, *Recovering an unknown source in a fractional diffusion problem*, J. Comput. Phys., **368**, 299 – 314 (2018).
23. N. A. Asl, D. Rostamy, *Identifying an unknown time-dependent boundary source in time-fractional diffusion equation with a non-local boundary condition*, J. Comput. and Appl. Math., **335**, 36 – 50 (2019).
24. L. Sun, Y. Zhang, T. Wei, *Recovering the time-dependent potential function in a multi-term time-fractional diffusion equation*, Appl. Numer. Math., **135**, 228 – 245 (2019).
25. S. A. Malik, S. Aziz, *An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions*, Comput. and Math. Appl., **3**, 7 – 19 (2017).
26. M. Ruzhansky, N. Tokmagambetov, B. T. Torebek, *Inverse source problems for positive operators. I, Hypoelliptic diffusion and subdiffusion equations*, J. Inverse Ill-Possed Probl., **27**, 891 – 911 (2019).
27. R. Ashurov, O. Muhiddinova, *Inverse problem of determining the heat source density for the subdiffusion equation*, Different. Equat., **56**, № 12, 1550 – 1563 (2020).
28. K. M. Furati, O. S. Iyiola, M. Kirane, *An inverse problem for a generalized fractional diffusion*, Appl. Math. and Comput., **249**, 24 – 31 (2014).
29. M. Kirane, A. M. Salman, A. Mohammed Al-Gwaiz, *An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions*, Math. Methods Appl. Sci. (2012); DOI: 10.1002/mma.2661.
30. A. Muhammad, A. M. Salman, *An inverse problem for a family of time fractional diffusion equations*, Inverse Probl. Sci. and Eng., **25**, № 9, 1299 – 1322 (2016); DOI: 10.1080/17415977.2016.1255738.
31. Zh. Shuang, R. Saima, R. Asia, K. Khadija, M. A. Abdullah, *Initial boundary value problems for a multi-term time fractional diffusion equation with generalized fractional derivatives in time*, AIMS Math., **6**, № 11, 12114 – 12132 (2021); DOI, 10.3934/math.2021703.
32. R. Ashurov, Y. Fayziev, *On the nonlocal problems in time for time-fractional subdiffusion equations*, Fractal and Fractional, **6**, 41 (2022); https: //doi.org/10.3390/fractalfract6010041.
33. R. Ashurov, Yu. Fayziev, *Uniqueness and existence for inverse problem of determining an order of time-fractional derivative of subdiffusion equation*, Lobachevskii J. Math., **42**, № 3, 508 – 516 (2021).
34. R. Ashurov, Yu. Fayziev, *Inverse problem for determining the order of the fractional derivative in the wave equation*, Math. Notes, **110**, № 6, 842 – 852 (2021).
35. M. M. Dzherbashian, *Integral transforms and representation of functions in the complex domain* (in Russian), Nauka, Moscow (1966).
36. R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogozin, *Mittag-Leffler functions, related topics and applications*, Springer (2014).
37. R. Ashurov, A. Cabada, B. Turmetov, *Operator method for construction of solutions of linear fractional differential equations with constant coefficients*, Fract. Calc. Appl. Anal., **1**, 229 – 252 (2016).
38. R. R. Ashurov, Yu. E. Fayziev, *On construction of solutions of linear fractional differentional equations with constant coefficients and the fractional derivatives*, Uzb. Math. J., **3**, 3 – 21 (2017).
39. A. Zygmund, *Trigonometric series*, vol. 1, Cambridge (1959).
40. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Math. Stud., vol. 204 (2006).

Received 10.02.22