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ALMOST EVERYWHERE CONVERGENCE OF T MEANS WITH RESPECT TO THE VILENKIN SYSTEM OF INTEGRABLE FUNCTIONS

ЗБІЖНІСТЬ T СЕРЕДНІХ МАЙЖЕ СКРІЗЬ ЩОДО СИСТЕМИ ІНТЕГРОВНИХ ФУНКЦІЙ ВІЛЕНКІНА

We prove and discuss some new weak-type (1,1) inequalities for the maximal operators of T means with respect to the Vilenkin system generated by monotone coefficients. We also apply these results to prove that these T means are almost everywhere convergent. As applications, we present both some well-known and new results.

Доведено та обговорено деякі нові нерівності слабкого типу (1,1) для максимальних операторів T середніх щодо системи Віленкіна, породженої монотонними коефіцієнтами. Отримані результати застосовано для доведення того факту, що ці T середні збіжні майже скрізь. Як застосування наведено деякі відомі та нові результати.

1. Introduction. The definitions and notations used in this introduction can be found in our next section.

It is well-known (for details see, e.g., [1, 12, 34]) that the Walsh – Paley system is not a Schauder basis in $L_1(G_m)$. Approximation properties of Vilenkin – Fourier series with respect to one- and two-dimensional cases can be found in [2, 27, 36, 37, 43, 44].

Almost everywhere convergence of Walsh – Fourier series of function $f \in L_p(G_m)$ for $1 < p < \infty$ was proved by Sjölin [35] (see also [4, 6]), while for bounded Vilenkin systems by Gosselin [11]. Schipp [31 – 33] (see also [23, 50]) investigated the so-called tree martingales and gave a proof of Carleson's theorem for Vilenkin – Fourier series. In each proof, they show that the maximal operator of the partial sums is bounded on $L_p(G_m)$, i.e., there exists an absolute constant c_p such that

$$\|S^*f\|_p \le c_p\|f\|_p \quad \text{as} \quad f \in L_p(G_m), \quad 1 < p < \infty.$$

Moreover, if we consider subsequences of partial sums, then the following result is true.

Theorem S1. *Let $f \in L_1(G_m)$. Then*

$$y\mu\left\{\sup_{n\in\mathbb{N}}|S_{M_n}f| > y\right\} \le c\|f\|_1, \quad y > 0.$$

Hence (for details see [5, 34]), if $f \in L_1(G_m)$, then $S_{M_n}f \to f$ a.e. on G_m.

In the one-dimensional case the weak-type (1,1) inequality for the maximal operator of Fejér means $\sigma^*f := \sup_{n\in\mathbb{N}}|\sigma_n f|$ can be found in [30] for Walsh series and in [22] for bounded Vilenkin series (see also [48, 49]). It follows that if $f \in L_1(G_m)$ then

$$\sigma_n f(x) \to f(x) \quad \text{a.e. on} \quad G_m.$$

In [42] it was proved that the maximal operator R^* of Riesz means is bounded from the Lebesgue space L_1 to the space weak-L_1. Hence, we get that, for $f \in L_1(G_m)$,

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$$R_n f(x) \to f(x) \quad \text{a.e. on} \quad G_m.$$

Approximation properties of Fejér and Reisz means with respect to Vilenkin systems can be found in [3, 10, 28, 38 – 41] (see also [13, 25, 26, 29]).

Myricz and Siddiqi [14] investigated the approximation properties of some special Nörlund means of Walsh – Fourier series of L_p function in norm. In the two-dimensional case approximation properties of Nörlund means were considered by Nagy [15 – 17] (see also [18 – 21]). In [24] it was proved that the following is true.

Theorem T1. *The maximal operators t^* of Nörlund means defined by*

$$t^* f := \sup_{n\in\mathbb{N}} |t_n f|,$$

either with nondecreasing $\{q_k : k \in \mathbb{N}\}$ sequences or nonincreasing $\{q_k : k \in \mathbb{N}\}$ sequences, satisfying the condition

$$\frac{1}{Q_n} = O\left(\frac{1}{n}\right) \quad as \quad n \to \infty,$$

are bounded from the Lebesgue space L_1 to the space weak-L_1.

T means are generalizations of the Fejér and the Reisz logarithmic means. According to this fact it is of prior interest to study the behavior of operators related to T means of Vilenkin – Fourier series. Moreover, if we define maximal operator of T means by

$$T^* f := \sup_{n\in\mathbb{N}} |T_n f|,$$

in [46] it was proved that if $\{q_k : k \in \mathbb{N}\}$ is nonincreasing or nondecreasing and satisfying the condition

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right) \quad \text{as} \quad n \to \infty, \tag{1}$$

then

$$y\mu\{T^* f > y\} \le c\|f\|_1, \qquad f \in L^1(G_m), \quad y > 0.$$

The boundedness of the maximal operator of T means does not hold from $L_1(G_m)$ to the space $L_1(G_m)$. However, $\|T_n f - f\|_p \to 0$ as $n \to \infty$ for $f \in L^p(G_m)$, $1 \le p < \infty$.

In this paper, we prove and discuss some new weak-type (1,1) inequalities of maximal operators of of T means with respect to the Vilenkin system generated by monotone coefficients. We also apply these results to prove almost everywhere convergence of such T means. As applications, both some well-known and new results are pointed out.

This paper is organized as follows. In order not to disturb our discussions later on some definitions and notations are presented in Section 2. For the proofs of the main results we need some auxiliary lemmas, some of them are new and of independent interest. These results are presented in Section 3. The main results and some of their consequences can be found in Section 4. The detailed proofs of the main results are also given in Section 4.

2. Definitions and notation. Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_{0,}\ m_1, \ldots)$ be a sequence of the positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of Z_{m_j}'s.

The direct product μ of the measures $\mu_k\big(\{j\}\big) := 1/m_k,\ j \in Z_{m_k}$, is the Haar measure on G_m with $\mu(G_m) = 1$.

In this paper we discuss bounded Vilenkin groups, i.e., the case when $\sup_{n\in\mathbb{N}} m_n < \infty$.

The elements of G_m are represented by sequences

$$x := \big(x_0, x_1, \ldots, x_j, \ldots\big), \quad x_j \in Z_{m_j}.$$

Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G_m$, the nth coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$. It is easy to give a basis for the neighborhoods of G_m:

$$I_0(x) := G_m, \qquad I_n(x) := \big\{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\big\},$$

where $x \in G_m,\ n \in \mathbb{N}$.

If we define $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m / I_n$, then

$$\overline{I_N} = \left(\bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l}\right) \bigcup \left(\bigcup_{k=1}^{N-1} I_N^{k,N}\right), \tag{2}$$

where

$$I_N^{k,l} := \begin{cases} I_N(0, \ldots, 0, x_k \neq 0, 0, \ldots, 0, x_l \neq 0, x_{l+1}, \ldots, x_{N-1}, \ldots) & \text{for} \quad k < l < N, \\ I_N(0, \ldots, 0, x_k \neq 0, 0, \ldots, x_{N-1} = 0, x_N, \ldots) & \text{for} \quad l = N. \end{cases}$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \qquad M_{k+1} := m_k M_k, \quad k \in \mathbb{N},$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j},\ j \in \mathbb{N}_+$, and only a finite number of n_j's differ from zero.

We introduce on G_m an orthonormal system which is called the Vilenkin system. At first, we define the complex-valued function $r_k(x) \colon G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k / m_k), \qquad i^2 = -1, \quad x \in G_m, \quad k \in \mathbb{N}.$$

Next, we define the Vilenkin system $\psi := (\psi_n \colon n \in \mathbb{N})$ on G_m by

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad n \in \mathbb{N}.$$

Specifically, we call this system the Walsh – Paley one when $m \equiv 2$.

The norms (or quasinorms) of the spaces $L_p(G_m)$ and weak-$L_p(G_m)$, $0 < p < \infty$, are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p d\mu, \qquad \|f\|_{\text{weak}-L_p}^p := \sup_{\lambda>0} \lambda^p \mu(f > \lambda) < +\infty.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [47]).

Now, we introduce analogues of the usual definitions in Fourier analysis. If $f \in L_1(G_m)$ we can define Fourier coefficients, partial sums and Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f\overline{\psi}_n d\mu, \qquad S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k, \qquad D_n := \sum_{k=0}^{n-1} \psi_k, \quad n \in \mathbb{N}_+.$$

It is well-known that (see [1]) if $n \in \mathbb{N}$, then

$$D_{M_n}(x) = \begin{cases} M_n, & x \in I_n, \\ 0, & x \notin I_n. \end{cases}$$

Moreover, if $n = \sum_{i=0}^{\infty} n_i M_i$ and $1 \leq s_n \leq m_n - 1$, then we have the following identity:

$$D_n = \psi_n \left(\sum_{j=0}^{\infty} D_{M_j} \sum_{k=m_j-n_j}^{m_j-1} r_j^k \right).$$

It immediately follows that

$$|D_n(x)| \leq cM_s, \qquad x \in I_s \backslash I_{s+1}, \quad s = 0, \ldots, N-1, \tag{3}$$

where c is an absolute constant.

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The nth T means T_n and Nörlund mean t_n for a Fourier series of f are respectively defined by

$$T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f \qquad \text{and} \qquad t_n f = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \tag{4}$$

where $Q_n := \sum_{k=0}^{n-1} q_k$.

It is obvious that

$$T_n f(x) = \int_{G_m} f(t) F_n(x-t) d\mu(t) \qquad \text{and} \qquad t_n f(x) = \int_{G_m} f(t) F_n^{-1}(x-t) d\mu(t),$$

where

$$F_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_k D_k \qquad \text{and} \qquad F_n^{-1} := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_{n-k} D_k \tag{5}$$

are called the T kernels and Nörlund kernels, respectively.

If $q_k \equiv 1$ in (4) and (5), we respectively define the Fejér means σ_n and Fejér kernels K_n as follows:

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f, \qquad K_n := \frac{1}{n}\sum_{k=1}^{n} D_k.$$

It is well-known that (for details see [1, 7]) if $n > t,\ t, n \in \mathbb{N}$, then

$$K_{M_n}(x) = \begin{cases} \dfrac{M_t}{1 - r_t(x)}, & x \in I_t \backslash I_{t+1}, \quad x - x_t e_t \in I_n, \\ \dfrac{M_n - 1}{2}, & x \in I_n, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, for all $n \in \mathbb{N}$,

$$n|K_n| \le c \sum_{l=0}^{|n|} M_l |K_{M_l}| \qquad \text{and} \qquad \|K_n\|_1 \le c < \infty.$$

The well-known example of Nörlund summability is the so-called (C, α) means (Cesàro means) for $0 < \alpha < 1$, which are defined by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha}\sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f, \quad \text{where} \quad A_0^\alpha := 0, \qquad A_n^\alpha := \frac{(\alpha+1)\ldots(\alpha+n)}{n!}.$$

We also consider the "inverse" (C, α) means, which is an example of T means:

$$U_n^\alpha f := \frac{1}{A_n^\alpha}\sum_{k=0}^{n-1} A_k^{\alpha-1} S_k f, \quad 0 < \alpha < 1.$$

Let V_n^α denote the T mean, where $\left\{q_0 = 0,\ q_k = k^{\alpha-1} : k \in \mathbb{N}_+\right\}$, that is,

$$V_n^\alpha f := \frac{1}{Q_n}\sum_{k=1}^{n-1} k^{\alpha-1} S_k f, \quad 0 < \alpha < 1.$$

The nth Riesz logarithmic mean R_n and the Nörlund logarithmic mean L_n are defined by

$$R_n f := \frac{1}{l_n}\sum_{k=1}^{n} \frac{S_k f}{k} \qquad \text{and} \qquad L_n f := \frac{1}{l_n}\sum_{k=1}^{n-1} \frac{S_k f}{n-k},$$

respectively, where $l_n := \sum_{k=1}^{n-1} 1/k$.

Up to now we have considered T means in the case when the sequence $\{q_k : k \in \mathbb{N}\}$ is bounded, but now we consider T summabilities with unbounded sequence $\{q_k : k \in \mathbb{N}\}$.

If we define the sequence $\{q_k : k \in \mathbb{N}\}$ by $\left\{q_0 = 0,\ q_k = \log k : k \in \mathbb{N}_+\right\}$, then we get the class B_n of T means with nondecreasing coefficients:

$$B_n f := \frac{1}{Q_n}\sum_{k=1}^{n} \log(k+1) S_k f, \quad \text{where} \quad Q_n = \sum_{k=1}^{n} \log(k+1).$$

3. Auxiliary lemmas. Next lemma is very important to study problems of almost everywhere convergence.

Lemma 1. *Suppose that the σ-sublinear operator V is bounded from L_{p_1} to L_{p_1} for some $1 < p_1 \leq \infty$ and*

$$\int_{\overline{I}} |Vf| d\mu \leq C\|f\|_1$$

for $f \in L_1$ and Vilenkin interval I which satisfy

$$\operatorname{supp} f \subset I, \qquad \int_{G_m} f d\mu = 0. \tag{6}$$

Then the operator V is of weak-type (1,1), *i.e.,*

$$\sup_{y>0} y\mu\big(\{Vf > y\}\big) \leq \|f\|_1.$$

Lemma 2. *Let $T, T_n \colon L^p(G_m) \to L^p(G_m)$ are sublinear operators for some $1 \leq p < \infty$ with T bounded and $T_n f \to Tf$ a.e. on G_m as $n \to \infty$ for each $f \in X_0$, where X_0 is dense in $L^p(G_m)$. Set*

$$T^* f := \sup_{n \in \mathbb{N}} |T_n f|, \quad f \in X.$$

If there iexists a constant $C > 0$, independent of f and n, such that

$$y^p \mu\Big(\big\{|Tf| > y\big\}\Big) \leq C\|f\|_X^p$$

and

$$y^p \mu\Big(\big\{T^* f > y\big\}\Big) \leq C\|f\|_X^p,$$

for all $y > 0$ and $f \in L^p(G_m)$, then

$$Tf = \lim_{n \to \infty} T_n f$$

a.e. on G_m for every $f \in L^p(G_m)$.

We need the following auxiliary lemmas.

Lemma 3 (see [7]). *Let $n \in \mathbb{N}$ and $x \in I_N^{k,l}$, where $k < l$. Then*

$$K_{M_n}(x) = 0, \quad \textit{if} \quad n > l, \tag{7}$$

and

$$|K_{M_n}(x)| \leq cM_k. \tag{8}$$

For the proof of our main results we also need the following lemmas.

Lemma 4 (see [45]). *Let $n \in \mathbb{N}$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence either of nonincreasing numbers or nondecreasing numbers satisfying condition* (1). *Then*

$$\|F_n\|_1 < c.$$

Lemma 5 (see [45]). *Let* $\{q_k : k \in \mathbb{N}\}$ *be a sequence of nonincreasing numbers and* $n > M_N$. *Then*

$$\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x)\right| \leq \frac{c}{M_N}\sum_{j=0}^{|n|} M_j|K_{M_j}|,$$

where c *is an absolute constant.*

Lemma 6 (see [45]). *Let* $\{q_k : k \in \mathbb{N}\}$ *be a sequence of nondecreasing numbers satisfying* (1). *Then*

$$\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x)\right| \leq \frac{c}{M_N}\sum_{j=0}^{|n|} M_j|K_{M_j}|,$$

where c *is an absolute constant.*

The next two lemmas are very important for our further investigations to prove almost everywhere convergence of T means generated by nonincreasing sequences $\{q_k : k \in \mathbb{N}\}$.

Lemma 7 (see [45]). *Let* $\{q_k : k \in \mathbb{N}\}$ *be a sequence of nonincreasing numbers. Then*

$$\int_{G_m} \sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x)\right| d\mu(x) \leq c < \infty,$$

where c *is an absolute constant.*

Proof. Let $n > M_N$ and $x \in I_N^{k,l}$, $k = 0, \ldots, N-2$, $l = k+1, \ldots, N-1$. Combining Lemma 5 with (7) and (8), we get that

$$\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x)\right| \leq \frac{1}{M_N}\sum_{i=0}^{l} M_i|K_{M_i}(x)|$$

$$\leq \frac{1}{M_N}\sum_{i=0}^{l} M_i M_k \leq \frac{cM_l M_k}{M_N}$$

and

$$\sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x)\right| \leq \frac{1}{M_N}\sum_{i=0}^{|n|} M_i|K_{M_i}(x)| \leq \frac{cM_l M_k}{M_N}. \tag{9}$$

Let $n > M_N$ and $x \in I_N^{k,N}$. By using (3), we can conclude that

$$\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x)\right| \leq \frac{c}{Q_n}\sum_{j=M_N}^{n-1} q_j M_k \leq \frac{Q_n - Q_{M_N}}{Q_n} M_k \leq cM_k,$$

so that

$$\sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j} D_j(x)\right| \leq cM_k. \tag{10}$$

Hence, combining (2) and estimates (9) and (10), we get that

$$\int\limits_{G_m\setminus I_N} \sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x)\right| d\mu$$

$$= \sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{x_j=0, j\in\{l+1,\ldots,N-1\}}^{m_{j-1}} \int\limits_{I_N^{k,l}} \sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x)\right| d\mu$$

$$+ \sum_{k=0}^{N-1} \int\limits_{I_N^{k,N}} \sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x)\right| d\mu$$

$$\leq c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{m_{l+1}\ldots m_{N-1}}{M_N}\frac{M_l M_k}{M_N} + c\sum_{k=0}^{N-1}\frac{M_k}{M_N}$$

$$\leq \sum_{k=0}^{N-2}\frac{(N-k)M_k}{M_N} + c < C < \infty.$$

The lemma is proved.

Lemma 8. *Let* $\{q_k : k\in\mathbb{N}\}$ *be a sequence of nondecreasing numbers satisfying* (1). *Then, for any* $n, N \in \mathbb{N}_+$,

$$\int\limits_{G_m} \sup_{n>N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j(x)\right| d\mu(x) \leq c < \infty,$$

Proof. If we apply Lemma 6 the proof is analogous to Lemma 7, so, we leave out the details.

4. Main result.

Theorem 1. *Let* $T_n f$ *be the* T *means and* F_n *be the corresponding kernels such that*

$$\int\limits_{\overline{I_N}} \sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{k=M_N+1}^{n-1} q_k D_k(x)\right| d\mu(x) < c < \infty.$$

If the maximal operator T^* *of* T *means is bounded from* L^{p_1} *to* L^{p_1} *for some* $1 < p_1 \leq \infty$, *then the operator* T^* *is of weak-type* (1,1), *i.e., for all* $f \in L^1(G_m)$,

$$\sup_{y>0} y\mu\{T^* f > y\} \leq \|f\|_1.$$

Proof. In view of Lemma 1 we obtain that the proof is complete if we show that

$$\int\limits_{\overline{I}} |T^* f(x)| d\mu(x) \leq c\|f\|_1 \qquad (11)$$

for every function f, which satisfies conditions in (6), where I denotes the support of the function f.

Without lost the generality we may assume that f is a function with support I and $\mu(I) = M_N$. We may also assume that $I = I_N$. It is easy to see that $T_n f = 0$ when $n \leq M_N$. Therefore, we can suppose that $n > M_N$. Moreover, $S_n f = 0$ for $n \leq M_N$,

$$\frac{1}{Q_n}\left(\sum_{k=0}^{M_N} q_k S_k f(x)\right) = 0 \qquad \text{and} \qquad \int_{I_N} \frac{1}{Q_n}\left(\sum_{k=0}^{M_n} q_k D_k(x-t)\right) f(t)d\mu(t) = 0.$$

Hence,

$$|T^* f(x)| \leq \sup_{n>M_N}\left|\int_{I_N} \frac{1}{Q_n}\left(\sum_{k=0}^{M_N} q_k D_k(x-t)\right) f(t)d\mu(t)\right|$$

$$+ \sup_{n>M_N}\left|\int_{I_N} \frac{1}{Q_n}\left(\sum_{k=M_N+1}^{n-1} q_k D_k(x-t)\right) f(t)d\mu(t)\right|$$

$$= \sup_{n>M_N}\left|\int_{I_N} \frac{1}{Q_n}\left(\sum_{k=M_N+1}^{n-1} q_k D_k(x-t)\right) f(t)d\mu(t)\right|. \tag{12}$$

Let $t \in I_N$ and $x \in \overline{I_N}$. Then $x - t \in \overline{I_N}$ and (12) implies that

$$\int_{\overline{I_N}} \left|T^* f(x)\right| d\mu(x) \leq \int_{\overline{I_N}} \sup_{n>M_N} \int_{I_N} \left|\frac{1}{Q_n}\left(\sum_{k=M_N+1}^{n-1} q_k D_k(x-t)\right) f(t)\right| d\mu(t)d\mu(x)$$

$$\leq \int_{\overline{I_N}} \int_{I_N} \sup_{n>M_N} \left|\frac{1}{Q_n}\left(\sum_{k=M_N+1}^{n-1} q_k D_k(x-t)\right) f(t)\right| d\mu(t)d\mu(x)$$

$$\leq \int_{I_N} \int_{\overline{I_N}} \sup_{n>M_N} \left|\frac{1}{Q_n}\left(\sum_{k=M_N+1}^{n-1} q_k D_k(x-t)\right) f(t)\right| d\mu(x)d\mu(t)$$

$$\leq \int_{I_N} \int_{\overline{I_N}} \sup_{n>M_N} \left|\frac{1}{Q_n}\left(\sum_{k=M_N+1}^{n-1} q_k D_k(x)\right) f(t)\right| d\mu(x)d\mu(t)$$

$$\leq \int_{I_N} |f(t)|d\mu(t) \int_{\overline{I_N}} \sup_{n>M_N} \left|\frac{1}{Q_n}\left(\sum_{k=M_N+1}^{n-1} q_k D_k(x)\right)\right| d\mu(x)$$

$$= \|f\|_1 \int_{\overline{I_N}} \sup_{n>M_N} \left|\frac{1}{Q_n}\left(\sum_{k=M_N+1}^{n-1} q_k D_k(x)\right)\right| d\mu(x) \leq c\|f\|_1.$$

Thus, (11) holds so the proof is complete.

Theorem 2. *Let $f \in L_1$ and T_n be the regular T means with nonincreasing sequence $\{q_k : k \in \mathbb{N}\}$. Then*

$$T_n f \to f \quad \text{a.e. as} \quad n \to \infty.$$

Proof. According to the regularity of T means with nonincreasing sequence $\{q_k : k \in \mathbb{N}\}$, we obtain that $T_n P \to P$ a.e. as $n \to \infty$, where $P \in \mathcal{P}$ is dense in the space L_1.

On the other hand, combining Lemmas 4 and 7 and Theorem 1, we can conclude that the maximal operator T^* of T means with nonincreasing sequences $\{q_k : k \in \mathbb{N}\}$ is bounded from the space L_1 to the space weak-L_1, that is,

$$\sup_{y>0} y\mu\big\{x \in G_m : |T^* f(x)| > y\big\} \le \|f\|_1.$$

Hence, according to Lemma 2, we obtain almost everywhere convergence of T means with nonincreasing sequence $\big\{q_k : k \in \mathbb{N}\big\}$.

The theorem is proved.

***Corollary* 1.** *Let $f \in L_1$. Then*

$$R_n f \to f \quad \text{a.e. as} \quad n \to \infty,$$

$$V_n^{\alpha} f \to f \quad \text{a.e. as} \quad n \to \infty,$$

$$U_n^{\alpha} f \to f \quad \text{a.e. as} \quad n \to \infty.$$

Theorem 3. *Let $f \in L_1$ and T_n be the regular T means with nondecreasing sequence $\big\{q_k : k \in \mathbb{N}\big\}$ satisfying condition* (1). *Then*

$$T_n f \to f \quad \text{a.e. as} \quad n \to \infty.$$

Proof. If we apply Theorem 1 and Lemma 8 the proof is analogous to Theorem 2, so we leave out the details.

Theorem 4. *Let $f \in L_1$ and T_n be the regular T means with nondecreasing sequence $\big\{q_k : k \in \mathbb{N}\big\}$. Then*

$$T_{M_n} f \to f \quad \text{a.e. as} \quad n \to \infty.$$

Proof. If we use (for details see [8, 9]) $D_{M_n - j}(x) = D_{M_n}(x) - \psi_{M_n - 1}(x)\overline{D}_j(x), j < M_n$, we get that

$$\begin{aligned}
F_{M_n}(x) &= \frac{1}{Q_{M_n}} \sum_{k=1}^{M_n} q_{M_n - k} D_k(x) \\
&= \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n - 1} q_k D_{M_n - k}(x) \\
&= \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n - 1} q_k \big(D_{M_n}(x) - \psi_{M_n - 1}(x)\overline{D}_k(x)\big) \\
&= D_{M_n}(x) - \psi_{M_n - 1}(x)\overline{F^{-1}}_{M_n}(x).
\end{aligned}$$

Hence,

$$\sup_{n\in\mathbb{N}}|T_{M_n}f| \le \sup_{n\in\mathbb{N}}|S_{M_n}f| + \sup_{n\in\mathbb{N}}|t_{M_n}f|.$$

Combining Theorem S1 and Theorem T1, we immediately have that

$$y\mu\left\{\sup_{n\in\mathbb{N}}|T_{M_n}f| > y\right\} \le c\|f\| \quad \text{for any} \quad f\in L_1(G_m), \quad y>0.$$

On the other hand, if we repeat analogous steps of Theorem 2, we immediately get the proof of theorem.

***Corollary* 2.** *Let $f\in L_1$. Then*

$$\sigma_n f \to f \quad \textit{a.e. as} \quad n\to\infty,$$

$$B_n f \to f \quad \textit{a.e. as} \quad n\to\infty.$$

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