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MONOTONE GENERALIZED α-NONEXPANSIVE MAPPINGS ON $CAT_p(0)$ SPACES

МОНОТОННІ УЗАГАЛЬНЕНІ α-НЕРОЗКЛАДНІ ВІДОБРАЖЕННЯ НА ПРОСТОРАХ $CAT_p(0)$

We examine the existence of fixed points of generalized α-nonexpansive mappings on $CAT_p(0)$ spaces. We establish various convergence results for a newly defined algorithm associated with α-nonexpansive mappings. We present some illustrative examples to show the efficiency of the proposed algorithm and to support the above-mentioned results. We also define monotone generalized α-nonexpansive mappings and prove some existence and convergence results for these mappings.

Досліджено існування нерухомих точок узагальнених α-нерозкладних відображень на просторах $CAT_p(0)$. Встановлено різні результати щодо збіжності нового алгоритму, що пов'язаний з α-нерозкладними відображеннями. Наведено кілька ілюстративних прикладів, які демонструють ефективність цього алгоритму та підтверджують вищезгадані результати. Також визначено монотонні узагальнені α-нерозкладні відображення та доведено деякі результати щодо існування та збіжності для цих відображень.

1. Preliminaries and introduction. Let $\varnothing \neq \mathbb{C}$ be a subset of a Banach space $\mathbb{E}$. A single-valued mapping $\mathbb{F}\colon \mathbb{C} \to \mathbb{C}$ is:

(i) nonexpansive if, for all $\nu_1, \nu_2 \in \mathbb{C}$,

$$\|\mathbb{F}\nu_1 - \mathbb{F}\nu_2\| \leq \|\nu_1 - \nu_2\|,$$

(ii) nonspreading [1] if, for all $\nu_1, \nu_2 \in \mathbb{C}$,

$$2\|\mathbb{F}\nu_1 - \mathbb{F}\nu_2\|^2 \leq \|\mathbb{F}\nu_1 - \nu_2\|^2 + \|\mathbb{F}\nu_2 - \nu_1\|^2,$$

(iii) hybrid [2] if, for all $\nu_1, \nu_2 \in \mathbb{C}$,

$$3\|\mathbb{F}\nu_1 - \mathbb{F}\nu_2\|^2 \leq \|\nu_1 - \nu_2\|^2 + \|\mathbb{F}\nu_1 - \nu_2\|^2 + \|\mathbb{F}\nu_2 - \nu_1\|^2,$$

(iv) λ-hybrid [3] if there exists a fixed real number λ such that

$$(1+\lambda)\|\mathbb{F}\nu_1 - \mathbb{F}\nu_2\|^2 - \lambda\|\nu_1 - \mathbb{F}\nu_2\|^2 \leq (1-\lambda)\|\nu_1 - \nu_2\|^2 + \lambda\|\mathbb{F}\nu_1 - \nu_2\|^2$$

for all $\nu_1, \nu_2 \in \mathbb{C}$,

(v) α-nonexpansive (α-NE) [4] if there exists a constant $\alpha < 1$ such that

$$\|\mathbb{F}\nu_1 - \mathbb{F}\nu_2\|^2 \leq (1-2\alpha)\|\nu_1 - \nu_2\|^2 + \alpha\|\mathbb{F}\nu_1 - \nu_2\|^2 + \alpha\|\nu_1 - \mathbb{F}\nu_2\|^2$$

for all $\nu_1, \nu_2 \in \mathbb{C}$.

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It is worth mentioning that all the classes of mappings in (i) – (iii) are independent of each other and are all properly contained in the class of λ-hybrid mappings. It was shown in [4] that the classes of α-NE mappings and λ-hybrid mappings are equivalent in Hilbert spaces for $\lambda < 2$.

Let $(\mathbb{M}, \varrho)$ be a bounded metric space, $\nu_1, \nu_2 \in \mathbb{M}$ and $\varnothing \neq \mathbb{C} \subseteq M$ a set. A mapping $\eta\colon [0, r] \subseteq \mathbb{R} \to \mathbb{M}$ with $\eta(0) = \nu_1,\ \eta(r) = \nu_2$ and $\varrho(\eta(s_1), \eta(s_2)) = |s_1 - s_2|$ for all $s_1, s_2 \in [0, r]$ is called a geodesic path joining ν_1 and ν_2. Observe that η is an isometry and $\varrho(\eta(0), \eta(r)) = r$. The image $\eta([0, r])$ is called geodesic segment (GS) from ν_1 to ν_2. If it is unique, then it is denoted by $[\nu_1, \nu_2]$. The point $\eta(r)$ is denoted by $\eta(r) = (1-r)\nu_1 \oplus r\nu_2$ for $r \in (0, 1)$ and $\eta(r) \in [\nu_1, \nu_2]$ if and only if $\varrho(\eta(r), \nu_2) = (1 - r)\varrho(\nu_1, \nu_2)$ and $\varrho(\eta(r), \nu_1) = r\varrho(\nu_1, \nu_2)$ for any $r \in [0, 1]$. If there exists a geodesic path for any arbitrary $\nu_1, \nu_2 \in \mathbb{M}$, then $(\mathbb{M}, \varrho)$ is called a geodesic space and uniquely geodesic space if that geodesic path is unique. A subset $\mathbb{C} \subseteq M$ is called convex if it contains all geodesic segments joining any pair of its points.

In the geodesic metric space (GMS) $(\mathbb{M}, \varrho)$, a geodesic triangle (GT) $\Delta(\varkappa_1, \varkappa_2, \varkappa_3)$ consists of three points $\varkappa_1,\ \varkappa_2,\ \varkappa_3$ as vertices and three geodesic segments of any pair of these points, that is, $q \in \Delta(\varkappa_1, \varkappa_2, \varkappa_3)$ means that $q \in [\varkappa_1, \varkappa_2] \cup [\varkappa_1, \varkappa_3] \cup [\varkappa_2, \varkappa_3]$. If $\varrho(\varkappa_1, \varkappa_2) = \varrho_2(\overline{\varkappa}_1, \overline{\varkappa}_2)$, $\varrho(\varkappa_1, \varkappa_3) = \varrho_2(\overline{\varkappa}_1, \overline{\varkappa}_3)$ and $\varrho(\varkappa_2, \varkappa_3) = \varrho_2(\overline{\varkappa}_2, \overline{\varkappa}_3)$, then a triangle $\overline{\Delta}(\overline{\varkappa}_1, \overline{\varkappa}_2, \overline{\varkappa}_3)$ in $\mathbb{R}^2$ is called a comparison triangle (CT) for the triangle $\Delta(\varkappa_1, \varkappa_2, \varkappa_3)$. If $\varrho(\varkappa_1, \varpi) = \varrho_2(\overline{\varkappa}_1, \overline{\varpi})$, then $\overline{\varpi} \in [\overline{\varkappa}_1, \overline{\varkappa}_2]$ is called a comparison point for $\varpi \in [\varkappa_1, \varkappa_2]$. A GT $\Delta(\varkappa_1, \varkappa_2, \varkappa_3)$ in $\mathbb{M}$ satisfies $CAT(0)$ inequality if $\varrho(\varpi_1, \varpi_2) \leq \varrho_2(\overline{\varpi}_1, \overline{\varpi}_2)$ for all $\varpi_1, \varpi_2 \in \Delta(\varkappa_1, \varkappa_2, \varkappa_3)$ in which $\overline{\varpi}_1, \overline{\varpi}_2 \in \overline{\Delta}(\overline{\varkappa}_1, \overline{\varkappa}_2, \overline{\varkappa}_3)$ are the comparison points of ϖ_1 and ϖ_2, respectively. A geodesic space is called a $CAT(0)$ space if and only if the CN inequality

$$\varrho^2(\nu_1, (1 - \lambda)\nu_2 \oplus \lambda\nu_3) \leq (1 - \lambda)\varrho^2(\nu_1, \nu_2) + \lambda\varrho^2(\nu_1, \nu_3) - \lambda(1 - \lambda)\varrho^2(\nu_2, \nu_3)$$

holds for $\nu_1, \nu_2, \nu_3 \in \mathbb{M},\ \ \lambda \in [0, 1]$. For some other relations between curvature and metric, we refer the reader, for instance, to [5].

Obviously, every normed space is a geodesic space and every Hilbert space is a $CAT(0)$ space. Khamsi and Shukri [6] defined an extension of $CAT(0)$ spaces by replacing comparison triangles from Euclidean space to a Banach space, especially to l_p spaces.

Definition 1.1 [6]. *Let $(\mathbb{E}, \|\cdot\|)$ be a normed space and $(\mathbb{M}, \varrho)$ be a GMS.*

(1) *Given GT $\Delta(\varkappa_1, \varkappa_2, \varkappa_3)$ in $\mathbb{M}$, a triangle $\overline{\Delta}(\overline{\varkappa}_1, \overline{\varkappa}_2, \overline{\varkappa}_3)$ in $\mathbb{E}$ is said to be a CT whenever $\|\overline{\varkappa}_m - \overline{\varkappa}_n\| = \varrho(\varkappa_m, \varkappa_n)$ holds for $m, n \in \{1, 2, 3\}$. If $\|\overline{\varkappa}_m - \varkappa\| = \varrho(\varkappa_m, \varkappa)$ for any $m, n \in \{1, 2, 3\}$, then $\overline{\varkappa} \in [\overline{\varkappa}_m, \overline{\varkappa}_n]$ is called a comparison point for $\varkappa \in [\varkappa_m, \varkappa_n]$.*

(2) *$\mathbb{M}$ is said to be $CAT_{\mathbb{E}}(0)$ space if, for any GT $\Delta(\varkappa_1, \varkappa_2, \varkappa_3)$ in $\mathbb{M}$, there exists CT $\overline{\Delta}(\overline{\varkappa}_1, \overline{\varkappa}_2, \overline{\varkappa}_3)$ in $\mathbb{E}$ such that $\varrho(w_1, w_2) \leq \|\overline{w}_1 - \overline{w}_2\|$ for all $w_1,\ w_2 \in \Delta(\varkappa_1, \varkappa_2, \varkappa_3)$ and $\overline{w}_1$, $\overline{w}_2 \in \overline{\Delta}(\overline{\varkappa}_1, \overline{\varkappa}_2, \overline{\varkappa}_3)$. If $\mathbb{E} = l_p$, then $\mathbb{M}$ is said to be a $CAT_p(0)$ space.*

Although, a Hilbert space is the only example of normed spaces being a $CAT(0)$ space, yet every normed space $(\mathbb{E}, \|\cdot\|)$ is a $CAT_{\mathbb{E}}(0)$ space. In [7], Bachar and Khamsi showed that generalized (CN) inequality holds for $CAT_p(0)$ spaces with $p \geq 2$. More precisely, they proved the following.

Lemma 1.1 [7]. *If $(\mathbb{M}, \varrho)$ is a $CAT_p(0)$ space with $p \geq 2$, then*

$$\varrho(\nu_1, (1 - \lambda)\nu_2 \oplus \lambda\nu_3)^p + C_p\lambda(1 - \lambda)\varrho(\nu_2, \nu_3)^p \leq \lambda\varrho(\nu_1, \nu_2)^p + (1 - \lambda)\varrho(\nu_1, \nu_3)^p$$

holds for any $\nu_1, \nu_2, \nu_3 \in \mathbb{M},\ \lambda \in [0, 1]$, where $C_p = \dfrac{1}{2^{p-1}}$.

Lemma 1.2 [8]. *Let $\{K_i\}_{i\in I}$ be a family of nonempty, convex, and closed subsets of a $CAT_p(0)$ space with $p \geq 2$ such that intersection of finite members of this family is nonempty. Then $\bigcap_{i\in I} K_i \neq \varnothing$.*

2. Generalized α-nonexpansive mappings. Pant and Shukla [9] introduced a class of generalized α-nonexpansive ($G\alpha$-NE) mappings as under:

Definition 2.1. *Let $(\mathbb{M},\varrho)$ be a metric space, $\varnothing \neq \mathbb{C} \subseteq M$ be a set, and $\mathbb{F}\colon \mathbb{C} \to \mathbb{M}$ be a mapping. If there exists an $\alpha \in [0,1)$ such that*

$$\frac{1}{2}\varrho(\nu_1,\mathbb{F}\nu_1) \leq \varrho(\nu_1,\nu_2)$$

$$\Longrightarrow \varrho(\mathbb{F}\nu_1,\mathbb{F}\nu_2) \leq \alpha\varrho(\mathbb{F}\nu_1,\nu_2) + \alpha\varrho(\nu_1,\mathbb{F}\nu_2) + (1-2\alpha)\varrho(\nu_1,\nu_2)$$

for any $\nu_1,\nu_2 \in \mathbb{C}$, then $\mathbb{F}$ is called $G\alpha$-NE mapping.

The class of $G\alpha$-NE mappings is not a subclass of the classes of α-NE and nonexpansive mappings as shown in the following example.

***Example* 2.1.** Let $\mathbb{M} = [-1,1]$ be endowed with $\varrho(\nu_1,\nu_2) = |\nu_1 - \nu_2|$. Define a mapping $\mathbb{F}\colon \mathbb{M} \to \mathbb{M}$ by

$$\mathbb{F}\nu = \begin{cases} \dfrac{\nu^2}{\nu^2+4}, & \nu \in [-1,0], \\ \dfrac{1}{3}, & \nu = 1, \\ -\dfrac{\nu^2}{\nu^2+4}, & v \in (0,1). \end{cases}$$

Observe that $p = 0$ is the unique fixed point of $\mathbb{F}$. (*a*), (*b*), (*c*), and (*d*) of Fig. 1 show that $\mathbb{F}$ is a $G\alpha$-NE mapping with $\alpha = \dfrac{1}{5}$. Now, we show that the mapping $\mathbb{F}$ is neither a nonexpansive nor an α-NE. Indeed, if we take $\nu_1 = 1,\ v_2 \geq 0.59$, then

$$\big|\varrho(\mathbb{F}\nu_1,\mathbb{F}\nu_2)\big| \geq 0.41 \geq |\varrho(\nu_1,\nu_2)|.$$

Therefore, it is not a nonexpansive.

It is also seen in case (*h*) of Fig. 1 that if we take $\nu_1 = 1,\ v_2 \geq \dfrac{2}{3}$, then it is not an α-NE.

The reader interested in approximation of common fixed points of multivalued α-NE type mappings is referred to Oyetunbi and Khan [10].

Now, inspired by [11, Lemma 5], we state the metric version of [9, Lemma 5.1] as follows.

Proposition 2.1. *Let $(\mathbb{M},\varrho)$ be a metric space, $\varnothing \neq \mathbb{C} \subseteq M$ be a set, and $\mathbb{F}\colon \mathbb{C} \to \mathbb{M}$ be a $G\alpha$-NE mapping. Then, for all $\nu_1,\nu_2 \in \mathbb{C}$, the following statements hold:*

i) $\varrho(\mathbb{F}\nu_1,\mathbb{F}^2\nu_1) \leq \varrho(\nu_1,\mathbb{F}\nu_1)$,

ii) $\dfrac{1}{2}\varrho(\nu_1,\mathbb{F}\nu_1) \leq \varrho(\nu_1,\nu_2)$ *or* $\dfrac{1}{2}\varrho(\mathbb{F}\nu_1,\mathbb{F}^2\nu_1) \leq \varrho(\mathbb{F}\nu_1,\nu_2)$,

iii) $\varrho(\mathbb{F}^2\nu_1,\mathbb{F}\nu_2) \leq \alpha\varrho(\mathbb{F}^2\nu_1,\nu_2) + \alpha\varrho(\mathbb{F}\nu_1,\mathbb{F}\nu_2) + (1-2\alpha)\varrho(\mathbb{F}\nu_1,\nu_2)$

or

$$\varrho(\mathbb{F}^2\nu_1,\mathbb{F}\nu_2) \leq \alpha\varrho(\mathbb{F}^2\nu_1,\nu_2) + \alpha\varrho(\mathbb{F}\nu_1,\mathbb{F}\nu_2) + (1-2\alpha)\varrho(\mathbb{F}\nu_1,\nu_2),$$

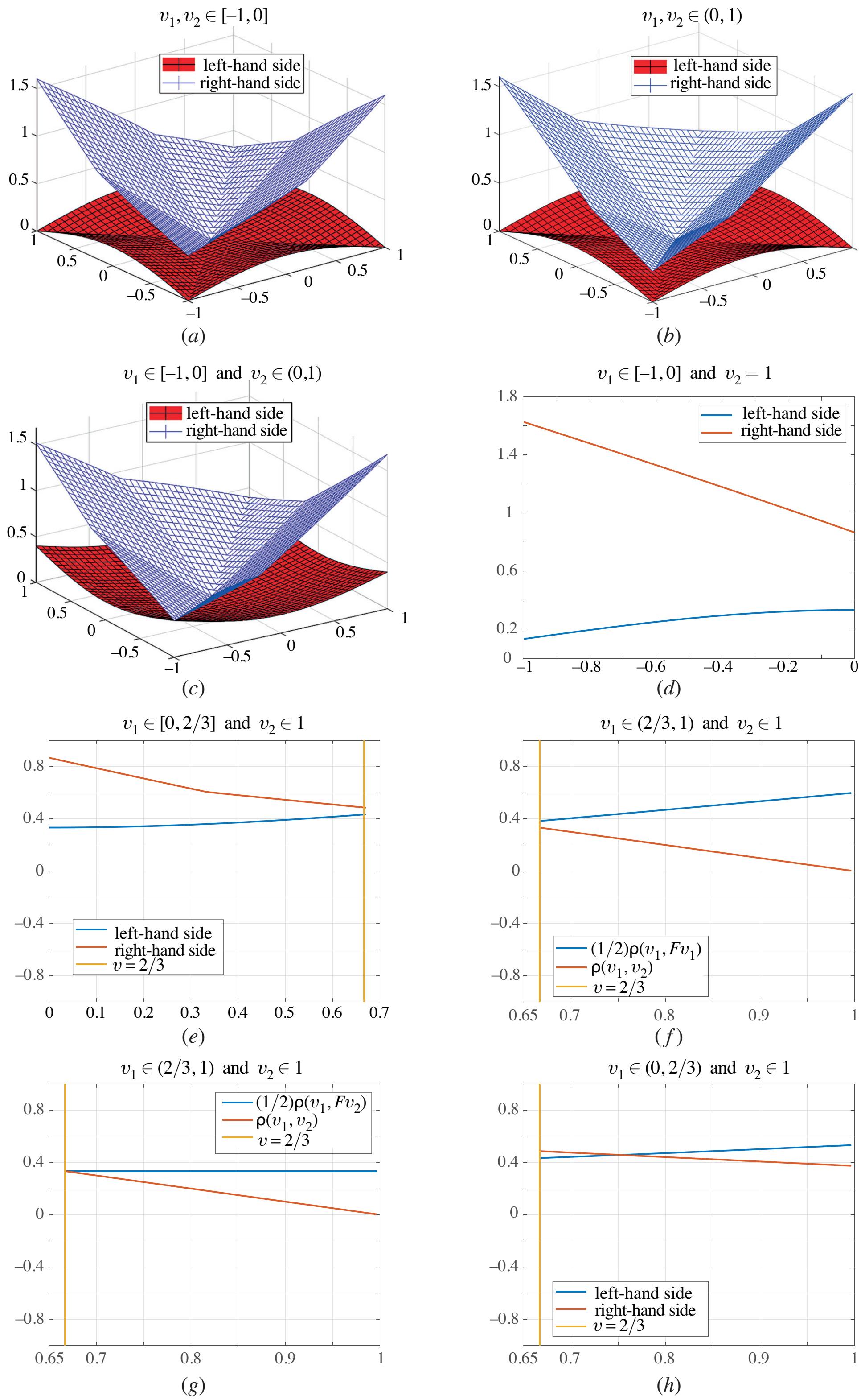

Fig. 1. Graphs for Example 2.1: case 1 (a), case 2 (b), case 3 (c), case 4 (d), 5 (e)–(h). In (a)–(e) and (h) left-hand side is $\varrho(\mathbb{F}v_1, \mathbb{F}v_2)$ and right-hand side is $\alpha\varrho(\mathbb{F}v_1, v_2) + \alpha\varrho(v_1, \mathbb{F}v_2) + (1-2\alpha)\varrho(v_1, v_2)$.

iv) *if* $\mathrm{Fix}(\mathbb{F}) \neq \varnothing$, *then* $\mathbb{F}$ *is a quasi-nonexpansive mapping.*

We are now in a position to present an existence result for $G\alpha$-NE mappings under minimal number of conditions.

Theorem 2.1. *Let* $(\mathbb{M}, \varrho)$ *be a complete* $CAT_p(0)$ *metric space with* $p \geq 2$, $\varnothing \neq \mathbb{C} \subseteq M$ *be a convex, closed, and bounded set, and* $\mathbb{F} : \mathbb{C} \to \mathbb{C}$ *be a* $G\alpha$*-NE mapping. Then* $\mathrm{Fix}(\mathbb{F})$ *is a nonempty, convex, and closed set.*

Proof. Let $\nu_0 \in \mathbb{C}$ and $\nu_k = \mathbb{F}^k \nu_0 = \mathbb{F}\nu_{k-1}$ for all $k \in \mathbb{N}$. Set

$$\Theta(\nu) = \limsup_{k\to\infty} \varrho(\nu_k, \nu) \quad \text{for all} \quad \nu \in \mathbb{C}.$$

It is shown in [6] that Θ has a unique minimum point $w \in \mathbb{C}$ such that

$$\Theta^p(w) + C_p \varrho(w, \nu)^p \leq \Theta^p(\nu) \quad \text{for all} \quad \nu \in \mathbb{C}.$$

Now, by Proposition 2.1(ii), we have $\frac{1}{2}\varrho(\nu_k, \nu_{k+1}) \leq \varrho(\nu_k, w)$ or $\frac{1}{2}\varrho(\nu_{k+1}, \nu_{k+2}) \leq \varrho(\nu_{k+1}, w)$. If $\frac{1}{2}\varrho(\nu_k, \nu_{k+1}) \leq \varrho(\nu_k, w)$ for all $k \in \mathbb{N}$, then we have

$$\begin{aligned} \varrho(\nu_{k+1}, \mathbb{F}w) &= \varrho(\mathbb{F}\nu_k, \mathbb{F}w) \\ &\leq \alpha\varrho(\mathbb{F}\nu_k, w) + \alpha\varrho(\nu_k, \mathbb{F}w) + (1-2\alpha)\varrho(\nu_k, w) \\ &= \alpha\varrho(\nu_{k+1}, w) + \alpha\varrho(\nu_k, \mathbb{F}w) + (1-2\alpha)\varrho(\nu_k, w). \end{aligned} \tag{1}$$

By taking the limit superior on both sides of inequality (1), we obtain

$$\begin{aligned} \limsup_{k\to\infty} \varrho(\nu_{k+1}, \mathbb{F}w) \leq \alpha \limsup_{k\to\infty} \varrho(\nu_{k+1}, w) \\ + \alpha \limsup_{k\to\infty} \varrho(\nu_k, \mathbb{F}w) + (1-2\alpha) \limsup_{k\to\infty} \varrho(\nu_k, w), \end{aligned}$$

which implies

$$\Theta(\mathbf{F}w) = \limsup_{k\to\infty} \varrho(\nu_{k+1}, \mathbf{F}w) \leq \limsup_{k\to\infty} \varrho(\nu_k, w) = \Theta(w).$$

Hence, $w = \mathbb{F}w$. The same concerns the case $\frac{1}{2}\varrho(\nu_{k+1}, \nu_{k+2}) \leq \varrho(\nu_{k+1}, w)$.

Now we prove that $\mathrm{Fix}(\mathbb{F})$ is a convex and closed set. First, we shall show that $\mathrm{Fix}(\mathbb{F})$ is a closed set.

Assume that $(\nu_k) \subseteq \mathrm{Fix}(\mathbb{F})$ with $\nu_k \to p \in \mathbb{M}$. As $\frac{1}{2}\varrho(\nu_k, \mathbb{F}\nu_k) \leq \varrho(\nu_k, p)$, so we obtain $\varrho(\nu_k, \mathbb{F}p) = \varrho(\mathbb{F}\nu_k, \mathbb{F}p) \leq \alpha\varrho(\nu_k, p) + \alpha\varrho(\nu_k, \mathbb{F}p) + (1-2\alpha)\varrho(\nu_k, p)$, which yields $\varrho(\nu_k, \mathbb{F}p) \leq \varrho(\nu_k, p)$. Therefore, $p = \mathbb{F}p \in \mathrm{Fix}(\mathbb{F})$.

Next, we show that $\mathrm{Fix}(\mathbb{F})$ is a convex set. Assume that $p_1, p_2 \in \mathrm{Fix}(\mathbb{F})$ and $p_1 \neq p_2$. Set $p = (1-\rho)p_1 \oplus \rho p_2$ for $\rho \in [0,1]$. From $\frac{1}{2}\varrho(p_1, \mathbb{F}p_1) \leq \varrho(p_1, p)$ and $\frac{1}{2}\varrho(p_2, \mathbb{F}p_2) \leq \varrho(p_2, p)$, we have

$$\varrho(p_1, \mathbb{F}p) \leq \varrho(p_1, p) \qquad \text{and} \qquad \varrho(p_2, \mathbb{F}p) \leq \varrho(p_2, p).$$

Therefore,

$$\begin{aligned}\varrho(p_1, p_2) &= \varrho(p_1, p) + \varrho(p, p_2) \\ &\leq \varrho(p_1, \mathbb{F}p) + \varrho(\mathbb{F}p, p_2) \leq \varrho(p_1, p) + \varrho(p_2, p) = \varrho(p_1, p_2),\end{aligned}$$

which, by the uniqueness of geodesics, implies the existence of $\rho_1 \in [0,1]$ such that $\mathbb{F}p = (1 - \rho_1)p_1 \oplus \rho_1 p_2$. Now, the following inequalities:

$$\rho_1 \varrho(p_1, p_2) = \varrho(p_1, \mathbb{F}p) \leq \varrho(p_1, p) = \rho\varrho(p_1, p_2)$$

and

$$(1 - \rho_1)\varrho(p_1, p_2) = \varrho(p_2, \mathbb{F}p) \leq \varrho(p_2, p) = (1 - \rho)\varrho(p_1, p_2)$$

imply that $(1 - \rho_1) \leq (1 - \rho)$ and $\rho_1 \leq \rho$, so $\rho_1 = \rho$. Hence, $p = \mathbb{F}p$.

Theorem 2.1 is proved.

Next, we extend Theorem 2.1 to an arbitrary family of $G\alpha$-NE and commuting mappings.

Theorem 2.2. *Let* $(\mathbb{M}, \varrho)$ *be a complete* $CAT_p(0)$ *metric space with* $p \geq 2$, $\varnothing \neq \mathbb{C}$ *be a bounded*, *convex*, *and closed subset of* $\mathbb{M}$, *and* $\mathbb{F}\colon \mathbb{C} \to \mathbb{C}$ *be a* $G\alpha$*-NE mapping*. *If*

$$\begin{aligned}\mathcal{F} = \big\{\mathbb{F}_i\colon \mathbb{C} \to \mathbb{C};\ \ \mathbb{F}_i \ \textit{ is a } \ G\alpha_i\textit{-NE } \ \textit{mapping} \\ \textit{with } \ \alpha_i \in [0,1) \ \textit{ and } \ i \in I \ \textit{ for some index set } \ I\big\}\end{aligned}$$

is a family of commutative mappings, *then* $\operatorname{Fix}(\mathcal{F}) = \bigcap_{i\in I} \operatorname{Fix}(\mathbb{F}_i)$ *is a nonempty*, *closed*, *and convex set*.

Proof. By Theorem 2.1, we have $\operatorname{Fix}(\mathbb{F}_i) \neq \varnothing$ for all $i \in I$. Furthermore, $\mathbb{F}_i\big(\operatorname{Fix}(\mathbb{F}_j)\big) \subseteq \operatorname{Fix}(\mathbb{F}_j)$ for all $i, j \in I$, we can conclude that $\operatorname{Fix}(\mathbb{F}_i) \cap \operatorname{Fix}(\mathbb{F}_j)$ is a nonempty, convex, and closed set. Consequently, $\bigcap_{i\in J} \operatorname{Fix}(\mathbb{F}_i)$ is also a nonempty, convex, and closed set for finite subset J of I. Applying Lemma 1.2, we can deduce that $\operatorname{Fix}(\mathcal{F})$ is a nonempty, convex, and closed set.

Theorem 2.1 holds for l_p spaces as shown in the following example.

***Example* 2.2.** Consider $\mathbb{M} = l_p$ with $p = 3$. Let $\mathbb{C} = \Big\{(\nu_k) \in l_p\colon \sum|\nu_k|^p < 10\Big\} \subset \mathbb{X}$ be endowed with the metric $\varrho(\nu_k, w_k) = \Big(\sum|\nu_k - w_k|^p\Big)^{1/p}$. Let $\mathbb{F}\colon \mathbb{C} \to \mathbb{C}$ be defined as

$$\mathbb{F}\nu_k = \begin{cases} \left(\dfrac{\arctan \nu_k}{2}\right), & \sum |\nu_k|^3 > 1, \\[2ex] \left(\dfrac{\sin \nu_k}{2}\right), & \sum |\nu_k|^3 < 1. \end{cases}$$

The mapping $\mathbb{F}$ is well-defined and is a $G\alpha$-NE mapping with $\alpha = \dfrac{1}{4}$. Indeed,

Case 1. If $\sum|\nu_k|^3 > 1$ and $\sum|w_k|^3 > 1$ or $\sum|\nu_k|^3 < 1$ and $\sum|w_k|^3 < 1$, then the result is obvious.

Case 2. Let $\sum|\nu_k|^3 > 1$ and $\sum|w_k|^3 < 1$. Then we have

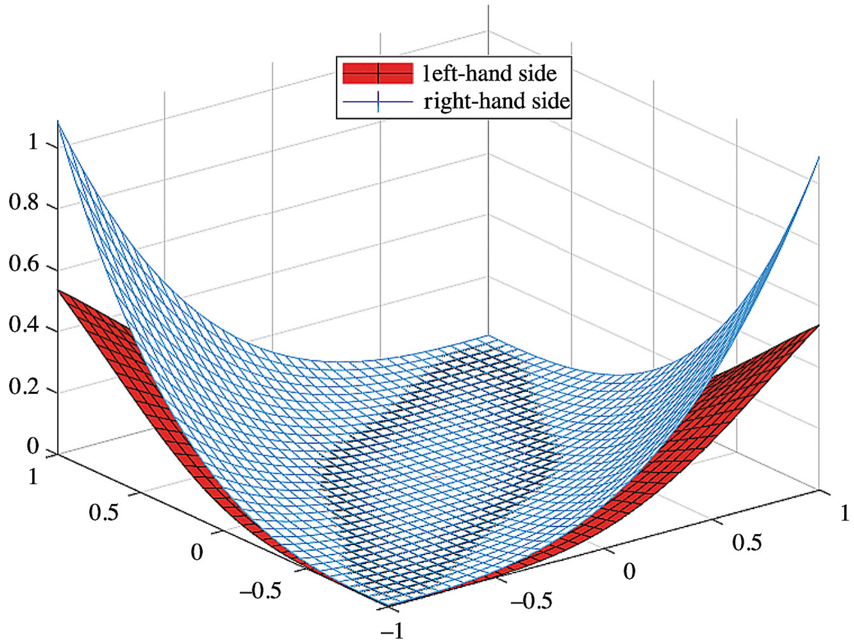

Fig. 2. Graphs for Example 2.2: case 2, left-hand side is $\varrho(\mathbb{F}v_1, \mathbb{F}v_2)$ and right-hand side is $\alpha\varrho(\mathbb{F}v_1, v_2) + \alpha\varrho(v_1, \mathbb{F}v_2) + (1-2\alpha)\varrho(v_1, v_2)$.

$$\left|\frac{\arctan\nu}{2} - \frac{\sin w}{2}\right|^3 \leq \left(\frac{1}{4}\right)^3\left|\nu - \frac{\sin w}{2}\right|^3 + \left(\frac{1}{4}\right)^3\left|\frac{\arctan\nu}{2} - w\right|^3 + \left(\frac{1}{2}\right)^3|\nu - w|^3 \quad \text{for all} \quad \nu, w \in \mathbb{R}$$

(see Fig. 2), which yields

$$\sum_k\left|\frac{\arctan\nu_k}{2} - \frac{\sin w_k}{2}\right|^3 \leq \sum_k\left(\frac{1}{4}\right)^3\left|\nu_k - \frac{\sin w_k}{2}\right|^3 + \sum_k\left(\frac{1}{4}\right)^3\left|\frac{\arctan\nu_k}{2} - w_k\right|^3 + \sum_k\left(\frac{1}{2}\right)^3|\nu_k - w_k|^3.$$

Therefore, in each case, we get the following desired inequality:

$$\varrho(\mathbb{F}(\nu_k), \mathbb{F}(w_k)) \leq \frac{1}{4}\varrho(\mathbb{F}(\nu_k), (w_k)) + \frac{1}{4}\varrho((\nu_k), \mathbb{F}(w_k)) + \frac{1}{2}\varrho((\nu_k), (w_k)).$$

On the other hand, we obtain $\operatorname{Fix}(\mathbb{F}) = \{q = (0, 0, 0, \ldots)\} \neq \varnothing$, and it is well-known that a singleton set is a convex and closed set. Therefore, all the requirements of Theorem 2.1 are met and its conclusion is verified.

3. An iterative algorithm for generalized α-nonexpansive mappings. The solution of most of the real-world problems encountered in different fields of research are usually modelled in equations or systems of equations of linear or nonlinear mappings. Sometimes, such an equation/a system of equations, even if it is linear, can be too large to be solved by a direct algorithm in a plausible amount of time. In this case, iteration algorithms become the only mathematical tools to get approximate solutions to such equations and are often used, for nonlinear equations or large scale of sparse linear equations, due to the computer memory limitation and the efficiency requirement. Since there are no universal iteration algorithms that can be applied to all types of linear or nonlinear equations, a lot of iteration algorithms with different structures and capabilities have been designed by many researchers to serve their purposes. For example, for nonlinear equations of contraction mappings, the Picard iteration algorithm is the simplest to implement and works very well in case of the determined initial approximation for the solution of a tackled problem is close to the exact solution. However, it fails badly to approximate the solutions of the nonlinear equations of nonexpansive mappings, and therefore the Mann iteration algorithm was invented to approximate the solutions of the equations formed by the aforementioned mappings. Similarly, the Mann iteration algorithm fails to approximate the solutions of the equations generated by the pseudo-contraction mappings and to overcome this problem, the Ishikawa iteration algorithm was invented. Continuing this trend, many iteration algorithms have been derived for solving different problems by making modifications or enhancements to the classical Picard, Mann, and Ishikawa iterative algorithms, and their qualitative properties like data dependence, convergence, stability, and rate of convergence have been extensively studied (see, e.g., [12, 13] and the references therein).

We present an iterative algorithm of normal-S type for a $G\alpha$-NE mapping as follows:

$$\begin{aligned} &\sigma_0 \in C, \\ &\sigma_{k+1} = \mathbb{F}\big[(1-\zeta_{k,1}-\zeta_{k,2})\sigma_k \oplus \zeta_{k,1}\mathbb{F}(\sigma_k) \oplus \zeta_{k,2}\mathbb{F}^2(\sigma_k)\big] \quad \text{for all} \quad k \in \mathbb{N}, \end{aligned} \tag{2}$$

where $(\zeta_{k,1})$ and $(\zeta_{k,2})$ are real sequences in $[0,1]$.

***Remark* 3.1.** 1. Algorithm (2) is quite general and it reduces to (i) Picard iterative algorithm [14] if $\zeta_{k,1}=\zeta_{k,2}=0$ for all $k\in\mathbb{N}$, (ii) normal-S iterative algorithm [15] if $\zeta_{k,2}=0$ for all $k\in\mathbb{N}$, (iii) iterative algotithm (1.7) of [16] if $\zeta_{k,1}=0$ for all $k\in\mathbb{N}$.

2. The beauty of algorithm (2) lies in the fact that it is completely independent of Mann [17] and Ishikawa [18] iterative algorithms.

The concept of Δ-convergence originally given by Lim [19] goes as follows:

Definition 3.1. *Let $(\mathbb{M},\varrho)$ be a metric space and $\varnothing\neq\mathbb{C}\subseteq M$ be a set. A sequence (ν_k) is said to be Δ-convergent to $\nu\in\mathbb{C}$ if, for any subsequence $\big(\nu_{k(n)}\big)$ of (ν_k), the following holds:*

$$\limsup_{n\to\infty}\varrho\big(\nu_{k(n)},\nu\big)\le\limsup_{n\to\infty}\varrho\big(\nu_{k(n)},w\big)$$

for any $w\in\mathbb{C}$.

Below, we show that the sequence (σ_k) in (2) is an approximate fixed point of a $G\alpha$-NE mapping $\mathbb{F}$ and Δ-converges to a fixed point of $\mathbb{F}$.

Theorem 3.1. *Let $(\mathbb{M},\varrho)$ be a complete $CAT_p(0)$ metric space with $p\ge 2$, $\varnothing\neq\mathbb{C}\subseteq M$ be a convex, bounded, and closed set, and $\mathbb{F}\colon\mathbb{C}\to\mathbb{C}$ be a $G\alpha$-NE mapping. If (σ_k) is a sequence generated by* (2) *with real sequences $(\zeta_{k,1})$ and $(\zeta_{k,2})$ in $[\tau_1,\tau_2]$ in which $\tau_1,\tau_2\in\left(0,\dfrac{1}{2}\right)$, then it is an approximate fixed point sequence of $\mathbb{F}$, i.e., $\lim_{k\to\infty}\varrho(\sigma_k,\mathbb{F}(\sigma_k))=0$ and (σ_k) Δ-converges to an element of* $\operatorname{Fix}(\mathbb{F})$, *say p.*

Proof. By Theorem 2.1, we have that $\operatorname{Fix}(\mathbb{F})\neq\varnothing$. Let $p\in\operatorname{Fix}(\mathbb{F})$. First, we show that $(\varrho(\sigma_k,p))$ is a decreasing sequence and $\lim_{k\to\infty}\varrho(\sigma_k,\mathbb{F}(\sigma_k))=0$. Then we prove that the minimal point, say w, of $\Theta(\sigma)=\limsup_{k\to\infty}\varrho(\sigma_k,\sigma)$ is a fixed point of $\mathbb{F}$.

It follows from (2) and quasi-nonexpansiveness of $\mathbb{F}$ that

$$\begin{aligned}\varrho(\sigma_{k+1},p) &= \varrho\big(\mathbb{F}\big[(1-\zeta_{k,1}-\zeta_{k,2})\sigma_k \oplus \zeta_{k,1}\mathbb{F}(\sigma_k) \oplus \zeta_{k,2}\mathbb{F}^2(\sigma_k)\big],p\big)\\ &\le \varrho\big((1-\zeta_{k,1}-\zeta_{k,2})\sigma_k \oplus \zeta_{k,1}T(\sigma_k) \oplus \zeta_{k,2}\mathbb{F}^2(\sigma_k),p\big)\\ &\le (1-\zeta_{k,2})\varrho\left(\left(\frac{1-\zeta_{k,1}-\zeta_{k,2}}{1-\zeta_{k,2}}\right)\sigma_k \oplus \left(\frac{\zeta_{k,1}}{1-\zeta_{k,2}}\right)\mathbb{F}(\sigma_k),p\right) + \zeta_{k,2}\varrho\big(\mathbb{F}^2(\sigma_k),p\big)\\ &\le (1-\zeta_{k,2})\left[\left(\frac{1-\zeta_{k,1}-\zeta_{k,2}}{1-\zeta_{k,2}}\right)\varrho(\sigma_k,p) + \left(\frac{\zeta_{k,1}}{1-\zeta_{k,2}}\right)\varrho(\mathbb{F}(\sigma_k),p)\right] + \zeta_{k,2}\varrho(\mathbb{F}(\sigma_k),p)\\ &\le (1-\zeta_{k,2})\varrho(\sigma_k,p) + \zeta_{k,2}\varrho(\sigma_k,p) = \varrho(\sigma_k,p).\end{aligned}$$

Now, it is obvious that $(\varrho(\sigma_k,p))$ is a decreasing sequence. Thus, we get $\lim_{k\to\infty}\varrho(\sigma_k,p) = \lim_{k\to\infty}\varrho(\varphi_k,p) = M \ge 0$, in which

$$\varphi_k = (1-\zeta_{k,2})\psi_k \oplus \zeta_{k,2}\mathbb{F}^2(\sigma_k), \tag{3}$$

and

$$\psi_k = \left(\frac{1-\zeta_{k,1}-\zeta_{k,2}}{1-\zeta_{k,2}}\right)\sigma_k \oplus \left(\frac{\zeta_{k,1}}{1-\zeta_{k,2}}\right)\mathbb{F}(\sigma_k) \tag{4}$$

for all $k \ge 0$. If $M = 0$, then the proof is obvious. Assume that $M > 0$. By (4), we have

$$\varrho(\psi_k,p) \le \left(\frac{1-\zeta_{k,1}-\zeta_{k,2}}{1-\zeta_{k,2}}\right)\varrho(\sigma_k,p) + \left(\frac{\zeta_{k,1}}{1-\zeta_{k,2}}\right)\varrho(\mathbb{F}(\sigma_k),p) \le \varrho(\sigma_k,p). \tag{5}$$

As $(\zeta_{k,2})$ and $(\zeta_{k,1})$ are from $[\tau_1,\tau_2]$ in which $\tau_1,\tau_2 \in \left(0,\dfrac{1}{2}\right)$, so we have the following inequalities:

$$1-\tau_2 \le 1-\zeta_{k,2} \le 1-\tau_1, \tag{6}$$

$$1-\tau_2 \le 1-\zeta_{k,1} \le 1-\tau_1, \tag{7}$$

$$1-2\tau_2 \le 1-\zeta_{k,1}-\zeta_{k,2} \le 1-2\tau_1. \tag{8}$$

It follows from Lemma 1.1, quasi-nonexpansivity of $\mathbb{F}$, (3), and (6) that

$$\begin{aligned}\varrho(p,\varphi_k)^p + C_p(1-\tau_2)\tau_1\varrho\big(\psi_k,\mathbb{F}^2(\sigma_k)\big)^p &\le \varrho(p,\varphi_k)^p + C_p(1-\zeta_{k,2})\zeta_{k,2}\,\varrho\big(\psi_k,\mathbb{F}^2(\sigma_k)\big)^p\\ &\le (1-\zeta_{k,2})\varrho(p,\psi_k)^p + \zeta_{k,2}\varrho\big(p,\mathbb{F}^2(\sigma_k)\big)^p\\ &\le (1-\zeta_{k,2})\varrho(p,\psi_k)^p + \zeta_{k,2}\varrho(p,\sigma_k)^p\\ &\le (1-\zeta_{k,2})\varrho(p,\sigma_k)^p + \zeta_{k,2}\varrho(p,\sigma_k)^p = \varrho(p,\sigma_k)^p,\end{aligned} \tag{9}$$

which implies that

$$\lim_{k\to\infty} C_p(1-\tau_2)\tau_1\varrho\big(\psi_k,\mathbb{F}^2(\sigma_k)\big)^p = 0 \le \lim_{k\to\infty}\varrho(p,\sigma_k)^p - \lim_{k\to\infty}\varrho(p,\varphi_k)^p = 0,$$

since $(\zeta_{k,1})$ and $(\zeta_{k,2})$ are from $[\tau_1, \tau_2]$ in which $\tau_1, \tau_2 \in \left(0, \dfrac{1}{2}\right)$. From (9) and (6), we have

$$\varrho(p, \varphi_k)^p + C_p(1-\tau_2)\tau_1 \varrho\big(\psi_k, \mathbb{F}^2(\sigma_k)\big)^p \le (1-\tau_1)\varrho(p, \psi_k)^p + \tau_1 \varrho(p, \sigma_k)^p. \tag{10}$$

Utilizing Lemma 1.1, quasi-nonexpansivity of $\mathbb{F}$, (4), (7) and (8), we obtain

$$\begin{aligned}
&\varrho(p, \psi_k)^p + C_p\left(\frac{1-2\tau_2}{1-\tau_1}\right)\left(\frac{\tau_1}{1-\tau_1}\right)\varrho(\sigma_k, \mathbb{F}(\sigma_k))^p \\
&\quad \le \varrho(p, \psi_k)^p + C_p\left(\frac{1-\zeta_{k,1}-\zeta_{k,2}}{1-\zeta_{k,2}}\right)\left(\frac{\zeta_{k,1}}{1-\zeta_{k,2}}\right)\varrho(\sigma_k, \mathbb{F}(\sigma_k))^p \\
&\quad \le \left(\frac{1-\zeta_{k,1}-\zeta_{k,2}}{1-\zeta_{k,2}}\right)\varrho(p, \sigma_k)^p + \left(\frac{\zeta_{k,1}}{1-\zeta_{k,2}}\right)\varrho(p, \mathbb{F}(\sigma_k))^p \le \varrho(p, \sigma_k)^p
\end{aligned}$$

or, equivalently,

$$\varrho(p, \psi_k)^p \le \varrho(p, \sigma_k)^p - C_p\left(\frac{1-2\tau_2}{1-\tau_1}\right)\left(\frac{\tau_1}{1-\tau_1}\right)\varrho(\sigma_k, \mathbb{F}(\sigma_k))^p. \tag{11}$$

Substituting (11) into (10), we get

$$\begin{aligned}
&\varrho(p, \varphi_k)^p + C_p(1-\tau_2)\tau_1 \varrho\big(\psi_k, \mathbb{F}^2(\sigma_k)\big)^p \\
&\quad \le (1-\tau_1)\left[\varrho(p, \sigma_k)^p - C_p\left(\frac{1-2\tau_2}{1-\tau_1}\right)\left(\frac{\tau_1}{1-\tau_1}\right)\varrho(\sigma_k, \mathbb{F}(\sigma_k))^p\right] + \tau_1 \varrho(p, \sigma_k)^p \\
&\quad \le \varrho(p, \sigma_k)^p - C_p(1-2\tau_2)\left(\frac{\tau_1}{1-\tau_1}\right)\varrho(\sigma_k, \mathbb{F}(\sigma_k))^p,
\end{aligned}$$

which gives

$$\begin{aligned}
&C_p(1-2\tau_2)\left(\frac{\tau_1}{1-\tau_1}\right)\varrho(\sigma_k, \mathbb{F}(\sigma_k))^p \\
&\qquad \le \varrho(p, \sigma_k)^p - \varrho(p, \varphi_k)^p - C_p(1-\tau_2)\tau_1 \varrho\big(\psi_k, \mathbb{F}^2(\sigma_k)\big)^p.
\end{aligned} \tag{12}$$

Since $\lim_{k\to\infty} C_p(1-\tau_2)\tau_1 \varrho\big(\psi_k, \mathbb{F}^2(\sigma_k)\big)^p = 0$ and $\lim_{k\to\infty} \varrho(p, \sigma_k)^p - \varrho(p, \varphi_k)^p = 0$, by passing to the limit in (12), we obtain that $\lim_{k\to\infty} \varrho(\sigma_k, \mathbb{F}(\sigma_k)) = 0$.

Define

$$\Theta(\sigma) = \limsup_{k\to\infty} \varrho(\sigma_k, \sigma) \quad \text{for all} \quad \sigma \in \mathbb{C}.$$

Then Θ has a unique minimum point $w \in \mathbb{C}$ (see [6]). Next, we show that w is a fixed point of $\mathbb{F}$.

Now, there are two possibilities. Assume that w is not the limit of any subsequence of (σ_k). Since $\lim_{k\to\infty} \varrho(\sigma_k, \mathbb{F}(\sigma_k)) = 0$, we can find $N_0 \in \mathbb{N}$ such that

$$\frac{1}{2}\varrho(\sigma_k, \mathbb{F}(\sigma_k)) \le \varrho(\sigma_k, w)$$

for all $k \ge N_0$. Observe that

$$\begin{aligned}\varrho(\sigma_k, \mathbb{F}(w)) &= \varrho(\sigma_k, \mathbb{F}(\sigma_k)) + \varrho(\mathbb{F}(\sigma_k), \mathbb{F}(w)) \\ &\leq \varrho(\sigma_k, \mathbb{F}(\sigma_k)) + \alpha\varrho(\mathbb{F}(\sigma_k), w) + \alpha\varrho(\sigma_k, \mathbb{F}(w)) + (1-2\alpha)\varrho(\sigma_k, w) \\ &\leq \varrho(\sigma_k, \mathbb{F}(\sigma_k)) + \alpha[\varrho(\mathbb{F}(\sigma_k), \sigma_k) + \varrho(\sigma_k, w)] + \alpha\varrho(\sigma_k, \mathbb{F}(w)) + (1-2\alpha)\varrho(\sigma_k, w),\end{aligned}$$

which leads to

$$\Theta\big(\mathbb{F}(w)\big) = \limsup_{k\to\infty} \varrho(\sigma_k, \mathbb{F}(w)) \leq \limsup_{k\to\infty} \varrho(\sigma_k, w) = \Theta(w).$$

Therefore, $w = \mathbb{F}(w) \in \mathrm{Fix}(\mathbb{F})$.

For the other possibility, assume that $\big(\sigma_{k(n)}\big)$ is a subsequence of (σ_k) converging to w. If $w = p$, then the proof is over. Assume that $w \neq p$. Set $u = (1-\varepsilon)w \oplus \varepsilon p$ for fixed $\varepsilon \in (0,1)$. Then $u \in \mathbb{C}$. Since $\mathbb{F}$ is a quasi-nonexpansive mapping,

$$\varrho(p, \mathbb{F}(u)) \leq \varrho(p, u) = (1-\varepsilon)\varrho(p, w).$$

Since $\lim_{n\to\infty} \varrho\big(\sigma_{k(n)}, u\big) = \varrho(w, u) = \varepsilon\varrho(p, w)$, there exists $N_0 \in \mathbb{N}$ such that

$$\frac{1}{2}\varrho\big(\sigma_{k(n)}, \mathbb{F}\big(\sigma_{k(n)}\big)\big) \leq \varrho\big(\sigma_{k(n)}, u\big)$$

holds for $k(n) \geq N_0$. Hence, for $k(n) \geq N_0$, we have

$$\varrho\big(\mathbb{F}\big(\sigma_{k(n)}\big), \mathbb{F}(u)\big) \leq \alpha\varrho\big(\mathbb{F}\big(\sigma_{k(n)}\big), u\big) + \alpha\varrho\big(\sigma_{k(n)}, \mathbb{F}(u)\big) + (1-2\alpha)\varrho\big(\sigma_{k(n)}, u\big) \tag{13}$$

and by the triangle inequality

$$\varrho\big(\sigma_{k(n)}, \mathbb{F}(u)\big) \leq \varrho\big(\sigma_{k(n)}, \mathbb{F}\big(\sigma_{k(n)}\big)\big) + \varrho\big(\mathbb{F}\big(\sigma_{k(n)}\big), \mathbb{F}(u)\big) \tag{14}$$

is satisfied. By using (13) and (14), we get

$$\begin{aligned}\varrho\big(\sigma_{k(n)}, \mathbb{F}(u)\big) &\leq \varrho\big(\sigma_{k(n)}, \mathbb{F}\big(\sigma_{k(n)}\big)\big) + \alpha\varrho\big(\mathbb{F}\big(\sigma_{k(n)}\big), u\big) \\ &\qquad + \alpha\varrho\big(\sigma_{k(n)}, \mathbb{F}(u)\big) + (1-2\alpha)\varrho\big(\sigma_{k(n)}, u\big) \\ &\leq \varrho\big(\sigma_{k(n)}, \mathbb{F}\big(\sigma_{k(n)}\big)\big) + \alpha\big[\varrho\big(\mathbb{F}\big(\sigma_{k(n)}\big), \sigma_{k(n)}\big) + \varrho\big(\sigma_{k(n)}, u\big)\big] \\ &\qquad + \alpha\varrho\big(\sigma_{k(n)}, \mathbb{F}(u)\big) + (1-2\alpha)\varrho\big(\sigma_{k(n)}, u\big),\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \varrho\big(\sigma_{k(n)}, \mathbb{F}(u)\big) \leq \lim_{n\to\infty} \varrho\big(\sigma_{k(n)}, u\big). \tag{15}$$

Therefore, by (15), we have

$$\begin{aligned}\varrho(w, \mathbb{F}(u)) &= \lim_{n\to\infty} \varrho\big(\sigma_{k(n)}, \mathbb{F}(u)\big) \\ &\leq \lim_{n\to\infty} \varrho\big(\sigma_{k(n)}, u\big) = \varrho(w, u) = \varepsilon\varrho(p, w).\end{aligned} \tag{16}$$

By using (16), we get

$$\begin{aligned}\varrho(u,\mathbb{F}(u)) &\leq \varrho\big(u,\sigma_{k(n)}\big)+\varrho\big(\sigma_{k(n)},w\big)+\varrho(w,\mathbb{F}(u))\\ &\leq \varrho\big(u,\sigma_{k(n)}\big)+\varrho\big(\sigma_{k(n)},w\big)+\varepsilon\varrho(p,w). \end{aligned}\tag{17}$$

Since $\lim_{n\to\infty}\varrho\big(\sigma_{k(n)},u\big)=\varrho(w,u)=\varepsilon\varrho(p,w)$, by passing to the limit in (17), we obtain

$$\varrho(u,\mathbb{F}(u))\leq 2\varepsilon\varrho(p,w).$$

Since ε is arbitrarily small, we have $u=\mathbb{F}(u)$. Bearing in mind,

$$\lim_{n\to\infty}\varrho\big(\sigma_{k(n)},u\big)=\varrho(w,u)=\varepsilon\varrho(p,w)$$

and using the triangle inequality and nonexpansivity of $\mathbb{F}$, we obtain

$$\begin{aligned}\varrho(w,\mathbb{F}(w)) &\leq \varrho(w,u)+\varrho(u,\mathbb{F}(w))\\ &\leq \varrho(w,u)+\varrho(u,w)\leq 2\varepsilon\varrho(p,w),\end{aligned}$$

which implies that $w=\mathbb{F}(w)$, as ε is arbitrarily small.

Now we show that (σ_k) Δ-converges to w. Let $\big(\sigma_{k(n)}\big)$ be a subsequence of (σ_k). Set

$$\overline{\Theta}(\sigma)=\limsup_{n\to\infty}\varrho\big(\sigma_{k(n)},\sigma\big)\quad\text{for all}\quad\sigma\in\mathbb{C}.$$

As before, $\overline{\Theta}$ has a unique minimum point, say $\overline{w}$. Since $\lim_{n\to\infty}\varrho\big(\sigma_{k(n)},\mathbb{F}\big(\sigma_{k(n)}\big)\big)=0$, we have $\overline{w}\in\mathrm{Fix}(\mathbb{F})$. Also as $(\varrho(\sigma_k,w))$ is a decreasing sequence, so for all $k(n)\geq k$, we have

$$\begin{aligned}\Theta(\overline{w}) &= \limsup_{k\to\infty}\varrho(\sigma_k,\overline{w})=\limsup_{n\to\infty}\varrho\big(\sigma_{k(n)},\overline{w}\big)\\ &\leq \limsup_{n\to\infty}\varrho\big(\sigma_{k(n)},w\big)\leq\limsup_{k\to\infty}\varrho(\sigma_k,w)=\Theta(w),\end{aligned}$$

which implies that $w=\overline{w}$. Hence, (σ_k) Δ-converges to $w\in\mathrm{Fix}(\mathbb{F})$ by the uniqueness of minimum point of Θ.

Theorem 3.1 is proved.

Here is our strong convergence result for iterative algorithm (2) under the compactness condition on the set $\mathbb{C}\subseteq\mathbb{M}$.

Theorem 3.2. *Let $(\mathbb{M},\varrho)$ be a complete $CAT_p(0)$ metric space with $p\geq 2$, $\varnothing\neq\mathbb{C}\subseteq M$ be a compact and convex set, and $\mathbb{F}\colon\mathbb{C}\to\mathbb{C}$ be a $G\alpha$-NE mapping. If (σ_k) is the iterative sequence generated by* (2) *with real sequences $(\zeta_{k,1})$ and $(\zeta_{k,2})$ are from $[\tau_1,\tau_2]$ in which $\tau_1,\tau_2\in\left(0,\dfrac{1}{2}\right)$, then (σ_k) strongly converges to an element of* $\mathrm{Fix}(\mathbb{F})$, *say p.*

Proof. By Theorem 3.1, we have that $\lim_{k\to\infty}\varrho(\sigma_k,\mathbb{F}(\sigma_k))=0$ and $(\varrho(\sigma_k,p))$ is a decreasing sequence for all $p\in\mathrm{Fix}(\mathbb{F})$. Since $\mathbb{C}$ is a compact set, there exists a subsequence $\big(\sigma_{k(n)}\big)$ of (σ_k) converging to $w\in\mathbb{C}$. Assume that $w\neq p$. Set $u=(1-\varepsilon)w\oplus\varepsilon p$ for fixed $\varepsilon\in(0,1)$ and for $p\in\mathrm{Fix}(\mathbb{F})$. Then we have $w=\mathbb{F}w\in\mathrm{Fix}(\mathbb{F})$ by an argument similar to that in proof of Theorem 3.1. Hence, (σ_k) is convergent to w.

Note that Theorem 3.1 still holds if we change metric ϱ by ϱ^2 in the definition of $\mathbb{F}$. Therefore, Theorems 3.1 and 3.2 are true if $\mathrm{Fix}(\mathbb{F})\neq\varnothing$ and we change the condition on $\mathbb{F}$ in Definiton 2.1 as follows:

Number of iterations	Picard	Algorithm (2)	Algorithm (1.7) in [16]	Mann	Ishikawa	Normal-S
0	1	1	1	1	1	1
1	0.333333333	-0.056844116	-0.097578875	0.833336667	0.717163589	-0.147930002
2	-0.027027027	6.65087E-05	0.000929014	0.506252683	0.464950916	0.002337427
3	0.000182582	-5.59856E-11	-7.76774E-08	0.293830208	0.284436783	-5.33185E-07
4	-8.33402E-09	2.63404E-23	5.13E-16	0.167848491	0.168193128	2.55862E-14
5	1.7364E-17	-4.15499E-48	-2.15086E-32	0.09499832	0.097212556	-5.5692E-29
6	-7.53769E-35	7.74268E-98	3.65946E-65	0.053320487	0.055242336	2.53196E-58
7	1.42042E-69	-2.0891E-197	-1.0333E-130	0.029682304	0.030973356	-5.0712E-117
8	-5.044E-139	0	8.0751E-262	0.016392447	0.017175739	1.9843E-234
9	6.3604E-278	0	0	0.0089857	0.009436939	0
10	0	0	0	0.004892161	0.005144511	0
11	0	0	0	0.002647203	0.002785759	0
12	0	0	0	0.001424621	0.001499781	0
13	0	0	0	0.000762962	0.000803387	0
14	0	0	0	0.000406849	0.000428456	0
15	0	0	0	0.000216121	0.000227613	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮

$$\frac{1}{2}\varrho(\nu_1,\mathbb{F}\nu_1)\le\varrho(\nu_1,\nu_2)$$
$$\Longrightarrow \varrho^2(\mathbb{F}\nu_1,\mathbb{F}\nu_2)\le\alpha\varrho^2(\mathbb{F}\nu_1,\nu_2)+\alpha\varrho^2(\nu_1,\mathbb{F}\nu_2)$$
$$+(1-2\alpha)\varrho^2(\nu_1,\nu_2)\quad\text{for all}\quad \nu_1,\nu_2\in\mathbb{C}.$$

***Example* 3.1.** Let $\mathbb{M}$, ϱ, and $\mathbb{F}$ be as in Example 2.1. For $\nu_0=1$, $\zeta_{k,1}=\dfrac{1}{2}\left(0.99999-\dfrac{1}{k+1}\right)$, and $\zeta_{k,2}=\dfrac{1}{2}\left(0.99999-\dfrac{1}{k+2}\right)$ for all $k\in\mathbb{N}$, the convergence results for (2), Picard [14], Mann [17], Ishikawa [18], and normal-S [15] iterative algorithms to the fixed point $p=0$ are listed in the table and depicted in Fig. 3. Obviously, iterative algorithm (2) converges to $p=0$ in earlier steps than Picard [14], Mann [17], Ishikawa [18], and normal-S [15] iterative algorithms and it has high accuracy even in early steps of the iterations. We also calculate with MATLAB tic-toc function the time of calculation for the number of iterations to reach $p=0$ with an accuracy up to 10^{-300}. The details of our observation are as follows: for the Picard: $2.8400E-05$ (in ms), for the algorithm (2): $1.9900E-05$ (in ms), for the algorithm (1.7) in [16]: $2.01E-05$ (in ms), for the Mann: $3.0250E-04$ (in ms), for the Ishikawa: $4.8250E-04$ (in ms), for the normal-S: $2.8010E-05$ (in ms).

4. Monotone generalized α-nonexpansive mappings. In this section, we will carry over the results obtained in Section 3 to monotone operators. Assume that $(\mathbb{M},\varrho,\preceq)$ is a partially ordered metric space and all order intervals are closed and convex. Any two elements $\nu_1,\nu_2\in\mathbb{M}$ will be called comparable elements if either $\nu_1\preceq\nu_2$ or $\nu_2\preceq\nu_1$. Let $\mathbb{C}$ be a closed and convex subset of $\mathbb{M}$ and $\mathbb{F}:\mathbb{C}\to\mathbb{C}$ be a map. $\mathbb{F}$ will be called monotone map if $\nu_1\preceq\nu_2$ (or $\nu_2\preceq\nu_1$) implies $\mathbb{F}(\nu_1)\preceq\mathbb{F}(\nu_2)$ (or $\mathbb{F}(\nu_2)\preceq\mathbb{F}(\nu_1)$) for all $\nu_1,\nu_2\in\mathbb{C}$. We modify Definition 2.1 for monotonic mappings as follows.

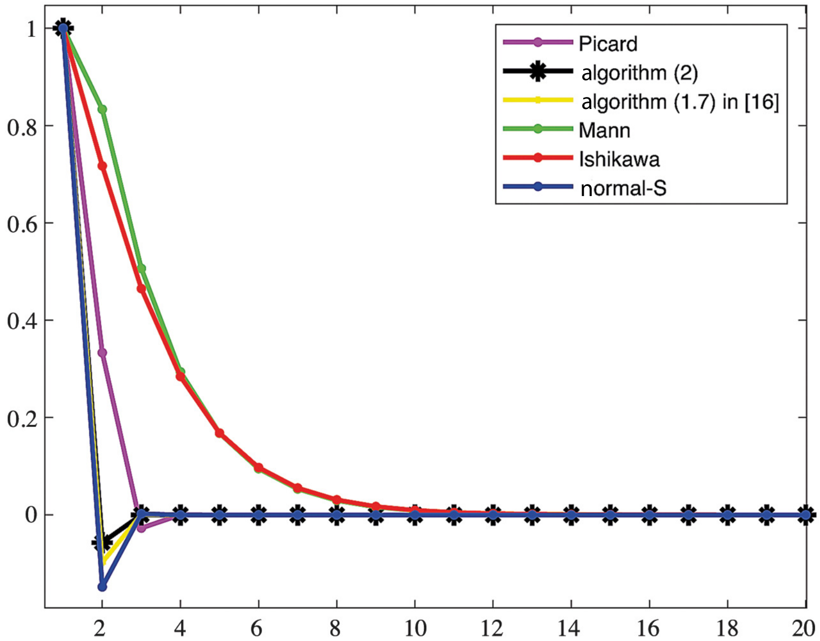

Fig. 3. Comparison of the rate of convergence among various algorithms for Example 3.1.

Definition 4.1. *Let $(\mathbb{M}, \varrho, \preceq)$ be a partially ordered metric space, $\varnothing \neq \mathbb{C} \subseteq M$ be a set, and $\mathbb{F}\colon \mathbb{C} \to \mathbb{M}$ be a map. The mapping $\mathbb{F}$ is said to be monotone generalized α-nonexpansive (MGα-NE) mapping if $\mathbb{F}$ is monotone and there exists an $\alpha \in [0,1)$ such that*

$$\frac{1}{2}\varrho(\nu_1, \mathbb{F}(\nu_1)) \leq \varrho(\nu_1, \nu_2)$$

$$\Longrightarrow \varrho(\mathbb{F}\nu_1, \mathbb{F}\nu_2) \leq \alpha\varrho(\mathbb{F}\nu_1, \nu_2) + \alpha\varrho(\nu_1, \mathbb{F}\nu_2) + (1-2\alpha)\varrho(\nu_1, \nu_2)$$

for any comparable $\nu_1, \nu_2 \in \mathbb{C}$.

Theorem 4.1. *Let $(\mathbb{M}, \varrho, \preceq)$ be a partially ordered $CAT_p(0)$ space with $p \geq 2$, $\varnothing \neq \mathbb{C} \subseteq M$ be a bounded, convex, and closed set, and $\mathbb{F}$ be an MGα-NE mapping. If there exists an element $\nu_0 \in \mathbb{C}$ such that ν_0 and $\mathbb{F}(\nu_0)$ are comparable, then there exists a fixed point of $\mathbb{F}$ comparable to ν_0.*

Proof. Let $\nu_0 \in \mathbb{C}$ and $\nu_k = \mathbb{F}^k\nu_0$ for all $k \in \mathbb{N}$. Assume that $\nu_0 \preceq \mathbb{F}(\nu_0)$. Then we have

$$\nu_0 \preceq \mathbb{F}\nu_0 \preceq \mathbb{F}^2(\nu_0) \preceq \ldots \preceq \mathbb{F}^k(\nu_0) \preceq \ldots.$$

Let

$$\mathbb{C}_\infty = \bigcap_{k\in\mathbb{N}} \{\nu \in \mathbb{C} \colon \nu_k \preceq \nu\}.$$

By Lemma 1.2, $\mathbb{C}_\infty$ is a nonempty, closed and convex subset of $\mathbb{C}$ and $\mathbb{F}\big(\mathbb{C}_\infty\big) \subseteq \mathbb{C}_\infty$. Define

$$\Theta(\nu) = \limsup_{k\to\infty} \varrho(\nu_k, \nu) \quad \text{for all} \quad \nu \in \mathbb{C}_\infty.$$

Now Θ has a unique minimum point $w \in \mathbb{C}_\infty$ such that

$$\Theta^p(w) + C_p \varrho(w, \nu)^p \leq \Theta^p(\nu)$$

for all $\nu \in \mathbb{C}_\infty$. The rest of the proof goes along the same lines as the proof of Theorem 2.1 by replacing $\mathbb{C}$ with $\mathbb{C}_\infty$, and is thus omitted. In addition, $\nu_0 \preceq w$ as $w \in \mathbb{C}_\infty$.

Note that the convexity condition on $\mathbb{C}$ can be replaced by a weaker condition: if $\nu_1, \nu_2 \in \mathbb{C}$ are two comparable elements, then $(1-\rho)\nu_1 \oplus \rho\nu_2 \in \mathbb{C}$ for $\rho \in [0,1]$. In this case, Theorem 4.1 still holds. The following result shows that algorithm (2) produces a sequence of the comparable elements.

Lemma 4.1. *Let* $(\mathbb{M}, \varrho, \preceq)$ *be a partially ordered* $CAT_p(0)$ *space with* $p \geq 2$, $\varnothing \neq \mathbb{C} \subseteq M$ *be a convex set, and* $\mathbb{F}$ *be a monotone map. If* (σ_k) *is a sequence generated by* (2) *with real sequences* $(\zeta_{k,1})$ *and* $(\zeta_{k,2})$ *in* $[\tau_1, \tau_2]$ *in which* $\tau_1, \tau_2 \in (0,1)$, *then the following statements are fulfilled:*

(i) *if* $\sigma_0 \preceq \mathbb{F}(\sigma_0)$, *then* $\sigma_k \preceq \mathbb{F}(\sigma_k) \preceq \sigma_{k+1} \preceq \mathbb{F}(\sigma_{k+1})$ *for all* $k \in \mathbb{N}$,

(ii) *if* $\mathbb{F}(\sigma_0) \preceq \sigma_0$, *then* $\mathbb{F}(\sigma_{k+1}) \preceq \sigma_{k+1} \preceq \mathbb{F}(\sigma_k) \preceq \sigma_k$ *for all* $k \in \mathbb{N}$.

Proof. (i) Let $\sigma_{k+1} = \mathbb{F}(\varphi_k)$, where $\varphi_k = (1-\zeta_{k,2})\psi_k \oplus \zeta_{k,2}\mathbb{F}^2(\sigma_k)$ and $\psi_k = \left(\dfrac{1-\zeta_{k,1}-\zeta_{k,2}}{1-\zeta_{k,2}}\right)\sigma_k \oplus \dfrac{\zeta_{k,1}}{1-\zeta_{k,2}}\mathbb{F}(\sigma_k)$ for all $k \geq 0$. As $\sigma_0 \preceq \mathbb{F}(\sigma_0) \preceq \mathbb{F}^2(\sigma_0)$, so we get

$$\sigma_0 \preceq \psi_0 = \left(\frac{1-\zeta_{0,1}-\zeta_{0,2}}{1-\zeta_{0,2}}\right)\sigma_0 \oplus \frac{\zeta_{0,1}}{1-\zeta_{0,2}}\mathbb{F}(\sigma_0) \preceq \mathbb{F}(\sigma_0)$$

and

$$\psi_0 \preceq \varphi_0 = (1-\zeta_{0,2})\psi_0 \oplus \zeta_{0,2}\mathbb{F}^2(\sigma_0) \preceq \mathbb{F}^2(\sigma_0).$$

Since $\mathbb{M}$ is uniquely geodesic, then $\sigma_0 \preceq \varphi_0 \preceq \mathbb{F}(\sigma_0)$ or $\mathbb{F}(\sigma_0) \preceq \varphi_0 \preceq \mathbb{F}^2(\sigma_0)$. If $\sigma_0 \preceq \varphi_0 \preceq \mathbb{F}(\sigma_0)$, then

$$\varphi_0 \preceq \mathbb{F}(\sigma_0) \preceq \mathbb{F}(\varphi_0) = \sigma_1,$$

which gives

$$\sigma_1 \preceq \mathbb{F}(\sigma_1).$$

If $\mathbb{F}(\sigma_0) \preceq \varphi_0 \preceq \mathbb{F}^2(\sigma_0)$, then

$$\varphi_0 \preceq \mathbb{F}^2(\sigma_0) \preceq \mathbb{F}(\varphi_0) = \sigma_1,$$

which gives

$$\sigma_1 \preceq \mathbb{F}(\sigma_1).$$

Therefore, we have $\sigma_0 \preceq \mathbb{F}(\sigma_0) \preceq \sigma_1 \preceq \mathbb{F}\sigma_1$. Hence, it can be proved by induction that

$$\sigma_k \preceq \mathbb{F}(\sigma_k) \preceq \sigma_{k+1} \preceq \mathbb{F}(\sigma_{k+1})$$

for all $k \geq 0$.

(ii) Similar to that of part i), and is thus omitted.

By imposing an additional monotonicity condition on the mapping $\mathbb{F}$, we obtain the following results corresponding to Theorems 3.1 and 3.2, respectively.

Theorem 4.2. *Let* $(\mathbb{M}, \varrho, \preceq)$ *be a partially ordered* $CAT_p(0)$ *space with* $p \geq 2$, $\varnothing \neq \mathbb{C} \subseteq M$ *be a bounded, convex, and closed set, and* $\mathbb{F}$ *be an MG*α*-NE mapping. If* (σ_k) *is the iterative sequence generated by* (2) *with real sequences* $(\zeta_{k,1})$ *and* $(\zeta_{k,2})$ *in* $[\tau_1, \tau_2]$ *in which* $\tau_1, \tau_2 \in \left(0, \frac{1}{2}\right)$ *and initial guess* $\sigma_0 \in \mathbb{C}$ *such that* σ_0 *and* $\mathbb{F}(\sigma_0)$ *are comparable, then* $\lim_{k\to\infty} \varrho(\sigma_k, \mathbb{F}(\sigma_k)) = 0$ *and* (σ_k) *is* Δ*-convergent to an element of* $\mathrm{Fix}(\mathbb{F})$.

Proof. Assume that $\sigma_0 \preceq \mathbb{F}(\sigma_0)$. By Theorem 4.1, there exists a fixed point p of $\mathbb{F}$ such that $\sigma_0 \preceq p$. Let

$$\mathbb{C}_\infty = \bigcap_{k\in\mathbb{N}} \{\sigma \in \mathbb{C} \colon \sigma_k \preceq \sigma\}.$$

Then, by Lemmas 1.2 and 4.1, $\mathbb{C}_\infty$ is a nonempty, closed, and convex subset of $\mathbb{C}$, $\mathbb{F}\big(\mathbb{C}_\infty\big) \subseteq \mathbb{C}_\infty$ and $p \in \mathbb{C}_\infty$. As in the proof of Theorem 3.1, we set

$$\Theta(x) = \lim_{k\to\infty}\sup \varrho(\sigma_k, \sigma) \quad \text{for all} \quad \sigma \in \mathbb{C}_\infty$$

and

$$\overline{\Theta}(\sigma) = \lim_{n\to\infty}\sup \varrho(\sigma_{k(n)}, \sigma) \quad \text{for all} \quad \sigma \in \mathbb{C}_\infty.$$

Now this proof can be completed on the lines of the proof of Theorem 3.1.

Theorem 4.3. *Let* $(\mathbb{M}, \varrho, \preceq)$ *be a partially ordered* $CAT_p(0)$ *space with* $p \geq 2$, $\varnothing \neq \mathbb{C} \subseteq M$ *be a bounded, convex, and compact set, and* $\mathbb{F}$ *be an MG*α*-NE mapping. If* (σ_k) *is the iterative sequence generated by* (2) *with real sequences* $(\zeta_{k,1})$ *and* $(\zeta_{k,2})$ *in* $[\tau_1, \tau_2]$ *in which* $\tau_1, \tau_2 \in \left(0, \frac{1}{2}\right)$ *and initial guess* $\sigma_0 \in \mathbb{C}$ *such that* σ_0 *and* $\mathbb{F}(\sigma_0)$ *are comparable, then* $\lim_{k\to\infty} \varrho(\sigma_k, \mathbb{F}(\sigma_k)) = 0$ *and* (σ_k) *is convergent to an element of* $\mathrm{Fix}(\mathbb{F})$.

Proof. Similar to that of Theorem 3.2.

***Remark* 4.1.** (i) Theorem 2.1 generalizes [9, Theorem 4.1] on a nonlinear domain, namely, $CAT_p(0)$ space and improves it in the sense that the approximate fixed point sequence requirement is not needed.

(ii) Theorem 3.1 is an improvement of [9, Theorems 5.4, 5.6 and 5.8] on a $CAT_p(0)$ space without requiring approximate fixed point sequence and some other conditions such as Opial's property or differentiability of Frèchet norm.

(iii) Theorem 3.2 extends [9, Theorem 5.9] to $CAT_p(0)$ spaces by replacing the strong and somewhat strange condition $\lim_{k\to\infty} \inf \varrho\big(\sigma_k, \mathrm{Fix}(\mathbb{F})\big)$ with compactness of the domain of the mapping $\mathbb{F}$.

(iv) Theorems 4.1, 4.2, and 4.3 are generalizations of Theorems 2.1, 3.1, and 3.2, respectively, for monotonic mappings which generalize/improve the corresponding results of [20, Theorems 4.1, 5.3, 5.4].

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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