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# GENERALIZED WEAKLY DEMICOMPACT AND $S$-DEMICOMPACT LINEAR RELATIONS AND THEIR SPECTRAL PROPERTIES

# УЗАГАЛЬНЕНІ СЛАБКО ДЕМІКОМПАКТНІ ТА $S$-ДЕМІКОМПАКТНІ ЛІНІЙНІ ЗВ'ЯЗКИ ТА ЇХНІ СПЕКТРАЛЬНІ ВЛАСТИВОСТІ

We extend the concept of generalized weakly demicompact and relatively weakly demicompact operators on linear relations and present some outstanding results. Moreover, we address the theory of Fredholm and upper semi-Fredholm relations attempting to establish a connection with them.

Розширено поняття узагальнених слабко демікомпактних і відносно слабко демікомпактних операторів на лінійних відношеннях та наведено деякі видатні результати. Крім того, розглянуто теорію фредгольмових та верхніх напівфредгольмових співвідношень з метою встановити з ними зв'язок.

**1. Introduction.** The concept of demicompactness for linear relations was introduced into the functional analysis by A. Ammar, H. Daoud and A. Jeribi [2]. It stands for a generalization of demicompactness for linear operators which are presented by W. V. Petryshyn [20] to discuss fixed points.

In 2018, the notation of relatively demicompactness of linear operators introduced by B. Krichen [18] was extended by A. Ammar, S. Fakhfakh and A. Jeribi [4] on linear relations.

In 2019, I. Ferjani, A. Jeribi and B. Krichen identified in [14] the concept of generalized weakly demicompact operators with respect to weakly closed densely defined linear operators. Recently, in [15] the same authors investigated the notation of generalized weakly $S$-demicompact operators with respect to a weakly closed linear operator $S$.

The main objective of this work lies in extending the concept of generalized weakly demicompact and relatively demicompact operators on linear relations, examining some properties and elaborating a connection with Fredholm and upper semi-Fredholm relations.

This paper is organized as follows. In Section 2, we recall some basic definitions, notations and results that would be invested throughout the paper. In Section 3, we identify a generalized weakly demicompact relations, namely the generalized Riesz relation and we exhibit some pertinent properties and certain prominent results. In Section 4, we tackle the definiton of generalized weakly relatively demicompact relations, we provide certain outstanding results and eventually we enact a connection with Fredholm and upper semi-Fredholm relations.

**2. Preliminary and auxiliary results.** The notion of linear relations generalizes the concept of a linear operator to that of a multivalued linear operator. Linear relations emerged in functional analysis in J. von Neumann [19] triggered by the need to consider adjoints of nondensely defined operators invested in applications to the theory of generalized equations [8] as well as the need to consider the inverses of certain operators which are invested, for instance, in the investigation of certain Cauchy problems related to parabolic type equations in Banach spaces [13]. Certain results

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that are confirmed in the case of linear operators need to be validated within the framework of linear relations, sometimes under supplementary conditions. Let $X, Y, Z$ be vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We call a multivalued operator or a relation between $X$ and $Y$, the mapping $T$ defined on $\mathcal{D}(T) \subseteq X$ with a value in $2^Y \backslash \varnothing = \mathcal{P}(Y) \backslash \varnothing$. $\mathcal{D}(T) = \{x \in X : T(x) \neq \varnothing\}$ is called the domain of $T$. If $T$ maps all the point of $\mathcal{D}(T)$ to singletons, then $T$ is called a single-valued or simply an operator. A relation $T$ is said to be a linear relation, if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ $\forall x, y \in \mathcal{D}(T)$ and $\alpha, \beta \neq 0$. We denote by $LR(X, Y)$, the class of all linear relation from $X$ to $Y$. A linear relation $T \in LR(X, Y)$ is entirely defined by its graph, $G(T)$, which is expressed by

$$G(T) = \big\{(x, y) \in X \times Y : x \in \mathcal{D}(T), y \in Tx\big\}.$$

The linear relation $T^{-1}$ is the inverse of $T$ defined by

$$G(T^{-1}) = \big\{(y, x) \in Y \times X : (x, y) \in G(T)\big\}.$$

Let $M \subset X$, $T \in LR(X, Y)$. We call the range of $M$ by $T$ the set denoted $T(M)$ and defined by

$$T(M) = \bigcup_{m \in M \cap D(T)} T(m).$$

In particular, for $M = X$, $T(X) = R(T)$ is called the range of $T$. $T \in LR(X, Y)$ is said to be surjective if $R(T) = Y$. Let $T \in LR(X, Y)$, $\varnothing \neq H \subset Y$, we call a reciprocal range of $H$ by $T$ the set $T^{-1}(H)$ defined by

$$T^{-1}(H) = \bigcup \big\{T^{-1}(y) : y \in \mathcal{D}(T^{-1}) \cap H\big\} = \big\{x \in \mathcal{D}(T) : T(x) \cap H \neq \varnothing\big\}.$$

In particular, for $y \in R(T)$, we get $T^{-1}(y) = \{x \in \mathcal{D}(T), y \in Tx\}$,

$$D(T^{-1}) = R(T), \qquad R(T^{-1}) = \mathcal{D}(T).$$

We call the kernel of $T$ the subset of $X$ indicated by

$$N(T) = \{x \in X : 0 \in Tx\} = T^{-1}(0).$$

If $N(T) = 0$, that is, $T^{-1}$ is uni-value, we say that $T$ is an injective relation. The identity relation defined on the subset $E$ of $X$ is denoted by $I_E$ or simply $I$. It is represented in terms of

$$G(I_E) = \big\{(e, e) : e \in E\big\}.$$

Let $S, T \in LR(X, Y), \lambda \in \mathbb{K}^*$. The relation $S + T$ is defined by

$$(S + T)x = Sx + Tx \quad \forall x \in \mathcal{D}(S + T),$$

$$\mathcal{D}(S + T) = \mathcal{D}(S) \cap \mathcal{D}(T),$$

$$G(S + T) = \big\{(x, y), x \in \mathcal{D}(S) \cap \mathcal{D}(T) : y = y_1 + y_2 : (x, y_1) \in G(S), (x, y_2) \in G(T)\big\}.$$

We define the relation $\lambda T$ by

$$(\lambda T)x = \lambda Tx \quad \forall x \in \mathcal{D}(\lambda T),$$

$$\mathcal{D}(\lambda T) = \mathcal{D}(T),$$

$$G(\lambda T) = \big\{(x, \lambda y) \colon (x, y) \in G(T)\big\}.$$

For $T \in LR(X,Y)$ and $S \in LR(Y,Z)$ where $R(T) \cap \mathcal{D}(S) \neq \varnothing$, the linear relation $ST$ is the product of $S$ and $T$ defined by

$$ST(x) = S(Tx), \quad x \in X,$$

$$\mathcal{D}(ST) = \big\{x \in X : S(Tx) \neq \varnothing\big\} = \big\{x \in X : Tx \cap \mathcal{D}(S) \neq \varnothing\big\} = T^{-1}(\mathcal{D}(S)),$$

$$G(ST) = \big\{(x,z) \in X \times Z \colon \exists y \in Y \colon (x,y) \in G(T), (y,z) \in G(S)\big\}.$$

Let $M$ be a subset of $X$ such that $M \cap \mathcal{D}(T) \neq \varnothing$. The restriction of $T$ to $M$ denoted $T|_M$ is the relation in $LR(X,Y)$ defined by

$$T|_M x = Tx, \quad x \in M \cap \mathcal{D}(T),$$

$$\mathcal{D}(T|_M) = \mathcal{D}(T) \cap M,$$

$$G(T|_M) = G(T) \cap (M \times Y) = \big\{(x,y) \in G(T) \colon x \in M\big\}.$$

We can easily infer that $T|_M = T|_{M \cap \mathcal{D}(T)}$.

For a given closed linear subspace $E$ of $X$, let $Q_E^X$ (or simply $Q_E$) denote the natural quotient map with domain X and null space E. We shall denote $Q_{\overline{T(0)}}^Y$ by $Q_T$ or simply $Q$ when $T$ is understood. We define

$$\|Tx\| := \|Q_T Tx\|, \quad x \in \mathcal{D}(T),$$

$$\|T\| := \|Q_T T\|.$$

Let $A$ and $B$ be nonempty subsets of a normed space. The distance between $A$ and $B$ is defined by the formula

$$d(A,B) = \inf\big\{\|y - z\| \colon y \in A, z \in B\big\}.$$

If $A = \{a\}$, then $d(a,B) = \inf\big\{\|a - z\| \colon z \in B\big\}$. We define the minimum modulus of $T$ by

$$\gamma(T) = \inf\left\{\frac{\|Tx\|}{d(x, N(T))} \colon x \in \mathcal{D}(T), x \notin N(T)\right\}.$$

Conventionally, if $\mathcal{D}(T) \subset \overline{N(T)}$, then we get $\gamma(T) = +\infty$. If $\|T\| < \infty$, $T$ is called continuous and if $\gamma(T) > 0$, $T$ is said to be open. If $\mathcal{D}(T) = X$, $\|T\| < \infty$, then we said that $T$ is bounded. We denote the class of bounded linear relations from $X$ to $Y$ by $BR(X,Y)$. The linear relation $\overline{T}$ is the closure of a linear relation $T$ defined by

$$G(\overline{T}) = \overline{G(T)}.$$

We said that $T$ is closed if its graph $G(T)$ is closed in $X \times Y$, or, equivalently, if $T = \overline{T}$. We denote by $CR(X,Y)$ the class of closed linear relations from $X$ to $Y$. $T \in LR(X,Y)$ is said to be compact if $\overline{Q_T B_X}$ is compact, where $B_X := \big\{x \in X : \|x\| < 1\big\}$. $T \in LR(X,Y)$ is called demicompact if, for every bounded sequence $x_n$ in $\mathcal{D}(T)$, we have $Q_T(I-T)x_n \longrightarrow y$. Then $Q_T x_n$ has a convergent subsequence. If $T, S \in LR(X,Y)$ are densely defined such that $S(0) \subseteq T(0)$ and $\mathcal{D}(T) \subseteq \mathcal{D}(S)$, then $T$ is called $S$-demicompact (or relative demicompact with respect to $S$) if $Q_{S-T}(S-T)x_n = Q_T(S-T)x_n \longrightarrow y \in Y/\overline{T(0)}$ for every bounded sequence $x_n \in \mathcal{D}(T)$, $Q_T S x_n$ has a convergent subsequence. We denote by $X'$, the norm dual of a normed linear space $X$, i.e., the space of all continuous functional $x'$ expressed on $X$, with norm

$$\|x'\| = \inf\{\lambda\colon |x'x| \leq \lambda\|x\| \ \forall x \in X\}.$$

The linear relation $T$ is invertible if $T^{-1}$ is a bounded operator. We call the resolvent set of $T$ the set $\rho(T)$ defined by

$$\rho(T) = \{\lambda \in \mathbb{C}\colon \lambda - T \ \text{ is invertible}\}.$$

Now, the $S$-resolvent set of $T$ is represented in terms of

$$\rho_S(T) = \{\lambda \in \mathbb{C}\colon \lambda S - T \ \text{ is invertible}\}.$$

The complement of $\rho(T)$ is called spectrum of $T$ and is denoted $\sigma(T) = \mathbb{C}\backslash\rho(T)$. The $S$-spectrum of $T$ is defined by $\sigma_S(T) = \mathbb{C}\backslash\rho_S(T)$. Let $T \in CR(X,Y)$. The graph norm $\|.\|_T$ of $x \in \mathcal{D}(T)$ is indicated by $\|x\|_T = \|x\| + \|Tx\|$. We have $X_T = (\mathcal{D}(T), \|.\|_T)$ is a Banach space.

Let $\widetilde{X}$ denote the completion of the normed space $X$ and $\widetilde{T}$ denote the linear relation in $LR(\widetilde{X},\widetilde{Y})$ whose graph is the completion of $G(T)$. Therefore we call $\widetilde{T}$ the completion (or complete closure) of $T$. Let $T \in LR(X,Y)$. We define the nullity of $T$ by $\alpha(T) := \dim N(T)$, the deficiency of $T$ by $\beta(T) := \dim Y/R(T)$, and the index of $T$ by the quantity $i(T) := \alpha(T) - \beta(T)$ provided that $\alpha(T)$ and $\beta(T)$ are not both infinite.

$T \in LR(X,Y)$ is upper semi-Fredholm if and only if there exists a closed, finite, codimensional subspace $M$ of $X$ such that the restriction $T \mid M$ is injective and open. $T \in LR(X,Y)$ is said to be a lower semi-Fredholm linear relation if its conjugate $T'$ is an upper semi-Fredholm linear relation. We denote the set of upper semi-Fredholm linear relations by $F_+(X,Y)$, which we abbreviate as $F_+$, and the set of lower semi-Fredholm linear relations by $F_-(X,Y)$ (or $F_-$). In the case, when $X$ and $Y$ are Banach spaces, we extend the classes of closed single-valued Fredholm type operators given earlier to include closed multivalued operators. Note that the definitions of the classes $F_+(X,Y)$ and $F_-(X,Y)$ are consistent, respectively, with

$$\phi_+(X,Y) := \big\{T \in CR(X,Y)\colon \alpha(T) < \infty \ \text{ and } \ R(T) \ \text{ is closed in } \ Y\big\},$$

$$\phi_-(X,Y) := \big\{T \in CR(X,Y)\colon \beta(T) < \infty \ \text{ and } \ R(T) \ \text{ is closed in } \ Y\big\},$$

$$\phi(X,Y) = \phi_+(X,Y) \cap \phi_-(X,Y).$$

***Remark* 2.1** [9]. For $T \in LR(X,Y)$,
(i) $T \in F_+ \Leftrightarrow Q_T \in F_+$,
(ii) $T \in F_- \Leftrightarrow Q_T \in F_-$.

**Lemma 2.1** [9]. *Let $X$, $Y$ be two linear spaces and $T \in LR(X,Y)$. Then*:
(i) *for $x \in \mathcal{D}(T)$, we get $y \in Tx \Longleftrightarrow Tx = y + T(0)$;*
*in particular,* $0 \in Tx \Longleftrightarrow Tx = T(0)$;
(ii) *for $x_1, x_2 \in \mathcal{D}(T)$, we have the equivalence*

$$Tx_1 \cap Tx_2 \neq \varnothing \Longleftrightarrow Tx_1 = Tx_2.$$

**Lemma 2.2** [9]. *Let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$, where $X$, $Y$ and $Z$ are linear spaces. Then*

$$(ST)^{-1} = T^{-1}S^{-1}.$$

**Lemma 2.3** [9]. *Let $X, Y$ be two linear spaces and $T \in LR(X,Y)$. Then*:

(i) $T(0) = TT^{-1}(0)$ *and* $T^{-1}(0) = T^{-1}T(0)$,

(ii) $T^{-1}Tx = x + T^{-1}(0)\ \forall x \in \mathcal{D}(T)$,

(iii) $TT^{-1}y = y + T(0)\ \forall y \in R(T)$.

**Proposition 2.1** [9]. *If $T$ is open and $N(T)$ is closed*, *then $N(T) = N(Q_T)$ and $\gamma(T) = \gamma(Q_T)$.*

**Proposition 2.2** [9]. *let $T \in LR(X,Y)$. Then*:

(i) $Q_T T$ *is single-valued*,

(ii) $\|Tx\| = d(y, T(0))$ *for any* $x \in \mathcal{D}(T), y \in Tx$,

(iii) $\|Tx\| = d(Tx, T(0)) = d(Tx, 0)\ (x \in \mathcal{D}(T))$,

(iv) $\|T\| = \sup_{x \in B_X \cap \mathcal{D}(T)} \|Tx\|$,

(v) $\gamma(T) = \|T^{-1}\|^{-1}$.

Let $T \in LR(X,Y)$. The chain of kernels of the iterates $T^n$, with $n \in \mathbb{N}$, is an increasing chain defined by

$$N(T^0) = 0 \subseteq N(T) \subseteq N(T^2) \subseteq \ldots.$$

Additionally, let the chain of ranges of $T^n$, $n \in \mathbb{N}$, be decreasing chain defined by

$$R(T^0) = X \supseteq R(T) \supseteq R(T^2) \supseteq \ldots.$$

**Definition 2.1** [12]. *Let $T \in LR(X,Y)$, where $X$ and $Y$ are Banach spaces. We define the ascent and the descent of $T$, respectively*, *by*

$$\operatorname{asc}(T) = \inf\left\{n \geq 0 \colon N(T^n) = N(T^{n+1})\right\}$$

*and*

$$\operatorname{des}(T) = \inf\left\{n \geq 0 \colon R(T^n) = R(T^{n+1})\right\}.$$

*The singular chain manifold of $T$ denoted by $R_c(T)$ is defined by $R_c(T) = R_0(T) \cap R_\infty(T)$, where $R_0(T) = \bigcup_{i=1}^{\infty} N(T_i)$ (or $N_\infty(T)$) and $R_\infty(T) = \bigcup_{i=1}^{\infty} T_i(0)$. The linear space $R_c(T)$ is nontrivial if and only if there exist a number $s \in \mathbb{N}$ and elements $x_i \in X$, $1 \leq i \leq s$, not all equal to zero, such that $(0, x_1), (x_1, x_2), \ldots, (x_{s-1}, x_s), (x_s, 0) \in G(T)$.*

The concept of the measure of weak noncompactness has multiple applications in not only topology and functional analysis but also the theories of differential and integral equations (see [5, 11, 16]). In order to recall the concept introduced by De Blasi, let us indicate by $X$ a Banach space, and $\overline{B_X}$ is the closure of $B_X$. We expressed by $\mathcal{M}_X$ the family of all nonempty and bounded subsets of $X$ and its subfamily involving all relatively weakly compact sets by $\mathcal{W}_X$. Additionally, let $\operatorname{conv}(A)$ be the convex hull of a set $A \subset X$. As an example of the regular measure in a Banach space $X$, we mention the measure of weak noncompactness (see [6]) and represented by De Blasi in terms of the following definition.

**Definition 2.2** [10]. *Let $X$ be a Banach space.*
*For all $A \in \mathcal{M}_X$,*

$$w(A) = \inf\big\{t > 0 : \text{ there exists } C \in \mathcal{W}_X \text{ such that } A \subset C + tB_X\big\}.$$

According to the Proposition 2.1 in [15], we conclude the following proposition.

**Proposition 2.3.** *Let $T \in LR(X, Y)$, where $X$ and $Y$ are two complex Banach spaces. Then, for every $A \in \mathcal{M}_X$, we have*:

(i) *For $C \geq 0$ such that, for every $x \in X$, $\|Tx\| \leq C\|x\|$. Then*

$$w(Q_T T(A)) \leq Cw(A).$$

(ii) *For $C \geq 0$ such that, for every $x \in X$, $\|x\| \leq C\|Tx\|$. Then*

$$w(A) \leq Cw(Q_T T(A)).$$

**Definition 2.3** [17]. *Suppose that $V$ is a vector space and $L$ is a nonempty subset of $V$. If there exists a $v \in V$ such that $L + v = \{v + l : l \in L\}$ is a vector subspace of $V$, then $L$ is a linear manifold of $V$. From this perspective, we say that the dimension of $L$ is the dimension of $L + v$ and therefore we write $\dim L = \dim(L + v)$. In the important case $\dim L = \dim V - 1$, $L$ is called a hyperplane.*

**Definition 2.4.** *Assume that $X$, $Y$ are Banach spaces and $T \in CR(X, Y)$. Let there exists number $m = 0, 1, 2, \ldots$ or $\infty$ with the property such that, for any $\varepsilon > 0$, there exists an $m$-dimensional closed linear manifold $N_\varepsilon \subset \mathcal{D}(T)$ such that*

$$\|T\| \leq \varepsilon\|x\| \quad \text{for every} \quad x \in N_\varepsilon.$$

*It is worth noting that this is not true if m is substituted by a larger number. In such a case, we set $\alpha'(T) = m$. By definition, $\alpha'(T)$ takes the values $0, 1, 2, \ldots$ or $\infty$ as well as $\alpha(T)$. $\alpha'(T)$ is therefore called the approximate nullity of $T$.*

*We define the approximate deficiency of $T$ by*

$$\beta'(T) = \alpha'(T^*).$$

***Remark* 2.2.** $\alpha'(T) \geq \alpha(T)$, $\beta'(T) \geq \beta(T)$.

**Theorem 2.1.** *Let $T \in CR(X, Y)$ where $X$ and $Y$ are Banach spaces such that $T \in \Phi_+(X, Y) \cup \Phi_-(X, Y)$. Let $S \in BR(X, Y)$ such that $\overline{\mathcal{D}(T)} \subset \mathcal{D}(S)$, $S(0) \subset T(0)$ and $\|S\| < \gamma(T)$. Then the linear relation $T + S$ is closed and has a closed range ($\gamma(T + S) > 0$). Furthermore, we have*

$$i(T + S) = i(T).$$

***Proof.*** By Lemma 1 in [1], $T + S$ is a closed linear relation defined by $\mathcal{D}(T + S) = \mathcal{D}(T)$. To demonstrate that $T + S$ has a closed range and $i(T + S) = i(T)$ holds, it is sufficient to show that

$$\alpha(T + S) \leq \alpha(T), \qquad \beta(T + S) \leq \beta(T). \tag{2.1}$$

Based on Remark 2.2, to prove (2.1), it is sufficient to show that

$$\alpha'(T + S) \leq \alpha(T), \qquad \beta'(T + S) \leq \beta(T).$$

Suppose that there exists a closed linear manifold $M_\varepsilon$ such that $\|(T+S)u\| \leq \varepsilon\|u\|$ for every $u \in M_\varepsilon$. Then

$$\big(\|S\|+\varepsilon\big)\|u\| \geq \|Su\| + \big\|(T+S)u\big\| \geq \|Tu\| \geq \gamma(T)\|\overline{u}\|,$$

where $\overline{u} \in \overline{X} = X/N(T)$. If $\varepsilon$ is so small that $0 < \varepsilon < \gamma(T) - \|S\|$, it follows that $d(u, N(T)) = \|\overline{u}\| < \|u\|$ if $0 \neq u \in M_\varepsilon$. Hence, based on Lemma 242 in [17], we have $\dim M_\varepsilon \leq \dim N(T) = \alpha(T)$, which implies that $\alpha'(T+S) \leq \alpha(T)$. For the other inequality, we get

$$\beta'(T+S) = \alpha'((T+S)^*) = \alpha'(T^*+S^*) \leq \alpha(T^*) \leq \alpha'(T^*) = \beta(T).$$

Theorem 2.1 is proved.

**Lemma 2.4.** *Let $T \in CR(X,Y)$ be densely defined on $X$. Then $T \in \phi_+(X,Y)$ with $i(T) \leq 0$ if and only if $T = S + F$ on $\mathcal{D}(T)$, where $S \in \phi_+(X,Y)$ with $\alpha(S) = 0$ and $F$ is a finite rank operator on $X$.*

***Proof.*** On the one hand, suppose that $T = S + F$ with $S \in \phi_+(X,Y)$ and $F$ is a finite rank operator on $X$. Then, from Theorem 2.1, we have $T \in \phi_+(X,Y)$ with $i(T) = i(S+F) = i(S) \leq 0$. On the other hand, assume that $T \in \phi_+(X,Y)$ and $i(T) \leq 0$. Let $\alpha(T) = n$. If $n = 0$, we take $F$ to be the zero operator and therefore the result holds. Now, suppose that $n > 0$, $i(T) \leq 0$. As a result, $\beta(T) > n$. Hence, there exist $n$ elements $y_1, y_2, \ldots, y_n$ in $X$ which are linearly independent modules $R(T)$. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for $N(T)$. Let $x_i'$ in the dual space $X'$ of $X$, $i = 1, 2, \ldots, n$, be chosen so that

$$x_i'(x_j) = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j, \end{cases} \qquad i, j = 1, 2, \ldots, n.$$

Define $F$ by

$$Fx = \sum_{i=l}^{n} x_i'(x) y_i \quad \text{for each} \quad x \in X.$$

Consequently, $F$ is a finite rank operator. Define $S$ on $\mathcal{D}(T)$ by $Sx = Tx - Fx$ for $x \in \mathcal{D}(T)$, $S \in \phi_+(X)$ by Theorem 2.1.We assume also that $\alpha(S) = 0$. Suppose that $x \in N(S)$. Then

$$Tx = Fx + S(0) = \sum_{i=l}^{n} x_i'(x) y_i + S(0).$$

The $y_i$ are linearly independent modules $R(T)$. Therefore, $x_i'(x) = 0$ for $i = 1, 2, \ldots, n$. Hence, $Tx = S(0) = T(0)$. Thus, $x \in N(T)$. As a matter of fact,

$$x \in \sum_{i=l}^{n} c_i x_i + T(0),$$

where $c_i$ are scalars. Now, $x_i'(x) = 0$ for all $i$ and $x_i'(x_j) = \delta_{ij}$. Therefore, $c_i = 0$ for all $i$. Hence, $x \in T(0)$. Thus, $\alpha(T) < \infty$.

Lemma 2.4 is proved.

**Definition 2.5.** *$T \in LR(X,Y)$ is called closed, if $T(0)$ is closed and $x_n \to x$ in $X$ for every sequence $(x_n)_n$ in $\mathcal{D}(T)$ and $Q_T T x_n \to y$ in $Y/\overline{T(0)}$ implies that $x \in \mathcal{D}(T)$ and $Q_T T x = y$.*

**Definition 2.6.** *Let* $T \in LR(X,Y)$, *where* $X$ *and* $Y$ *are Banach spaces.* $T$ *is called weakly demicompact, if* $\big(Q_{I-T}(I-T)x_n\big)_n$ *converges weakly in* $Y/\overline{T(0)}$ *for every bounded sequence* $(x_n)_n$ *in* $\mathcal{D}(T)$. *Then* $(Q_T x_n)_n$ *has a weakly convergent subsequence.*

**Definition 2.7.** *Let* $T \in LR(X,Y)$, *where* $X$ *and* $Y$ *are Banach spaces.* $T$ *is called Riesz relation if*:

(i) $T - \lambda \in \phi(X,Y)$ $\forall \lambda \in \mathbb{C} \setminus \{0\}$,

(ii) *the ascent and the descent of* $T - \lambda$ *are finite for all* $\lambda \in \mathbb{C} \setminus \{0\}$,

(iii) *the multiplicity of all eigenvalues* $\lambda \in \sigma(T) \setminus \{0\}$ *is finite and they don't have accumulation points only possibly in zero.*

**3. Generalized weak demicompact linear relations.**

**Definition 3.1.** *Let* $T \in LR(X,Y)$, *where* $X$ *and* $Y$ *are Banach spaces.* $T$ *is said to be generalized Riesz linear relation if there exists a finite subset* $E$ *of* $\mathbb{C}$ *involving* $0$, *such that*:

(i) $T - \lambda \in \phi(X,Y)$ $\forall \lambda \in \mathbb{C} \setminus E$,

(ii) *the ascent and the descent of* $T - \lambda$ *are finite for all* $\lambda \in \mathbb{C} \setminus E$,

(iii) *the multiplicity of all eigenvalues* $\lambda \in \sigma(T) \setminus E$ *is finite and they don't have accumulation points only possibly in one point of* $E$.

**Theorem 3.1.** *Let* $T \in LR(X,Y)$, *where* $X$ *and* $Y$ *are Banach spaces such that* $R_c(T) = \{0\}$. $T$ *is a generalized Riesz relation if and only if there exists a finite subset* $E$ *of* $\mathbb{C}$ *involving* $0$ *such that* $\lambda - T \in \Phi_+(X,Y)$ *for all* $\lambda \in \mathbb{C} \setminus E$.

***Proof.*** Assume that there exists a finite subset $E$ of $\mathbb{C}$ involving 0 such that $\lambda - T \in \Phi_+(X,Y)$ for all $\lambda \in \mathbb{C} \setminus E$. Then $\alpha(\lambda - T) < \infty$. Therefore, referring to Proposition in [12], we have that $\operatorname{asc}(\lambda - T) < \infty$ for all $\lambda \in \mathbb{C} \setminus E$. We denote by $p = \operatorname{asc}(\lambda - T) < \infty$ for any $\lambda \in \sigma(T) \setminus E$. Hence, we have

$$\operatorname{mult}(T,\lambda) = \dim N_\infty(\lambda - T) = \dim N((\lambda - T)^p).$$

Relying on Sandovici [21]

$$\dim N((\lambda - T)^p) \leq p \dim N(\lambda - T).$$

Thus, $\operatorname{mult}(T,\lambda) < \infty$.

To confirm the opposite, let us first verify that $\alpha(\lambda - T) < \infty$. Since $T$ is a generalized Riesz relation, then $p = \operatorname{asc}(\lambda - T) < \infty$. Consequently,

$$\operatorname{mult}(T,\lambda) = \dim N_\infty(\lambda - T) = \dim N((\lambda - T)^p) < \infty.$$

Hence, departing from the inclusion

$$N(\lambda - T) \subset N((\lambda - T)^p),$$

we get $\dim N(\lambda - T) = \alpha(\lambda - T) < \infty$. Now, we assert that $R(\lambda - T)$ is closed. Since $T$ is a generalized Riesz relation, then $\operatorname{desc}(\lambda - T) < \infty$. As a matter of fact, according to Theorem in [12], $\beta(\lambda - T) < \infty$. Consequently, $R(\lambda - T)$ is closed.

Theorem 3.1 is proved.

**Definition 3.2.** *Let* $T \in LR(X,Y)$, *where* $X$ *and* $Y$ *are Banach spaces.* $T$ *is called generalized weakly demicompact relation if there exists a finite subset* $E$ *of* $\mathbb{C}$ *involving* $0$ *such that*:

(i) $\dfrac{1}{\lambda}T$ *is a weakly demicompact relation fol all* $\lambda \in \mathbb{C} \setminus E$,

(ii) *the ascent of* $T - \lambda$ *is finite fol all* $\lambda \in \mathbb{C} \setminus E$,

(iii) *the multiplicity of all eigenvalues* $\lambda \in \sigma(T) \setminus E$ *is finite and they don't have accumulation points only possibly in one point of* $E$.

$E$ *is the generalized set of* $T$.

***Remark* 3.1.** 1. It is noteworthy that if $G$ is a finite subset of $\mathbb{C}$ involving $E$, where $E$ is a generalized set of $T$, then $G$ is a generalized set of $T$.

2. If $T \in LR(X,Y)$ is a generalized Riesz linear relation and, for all $\lambda \in \mathbb{C} \setminus E$ $\dfrac{1}{\lambda}T$ is a weakly demicompact relation, then $T$ is generalized weakly demicompact.

Now, we shall demonstrate that there exists a connection between upper semi-Fredholm linear relations and the class of generalized weakly demicompact linear relations acting on a Banach space. First, let us recall the following definition.

**Definition 3.3.** *Let* $T \in LR(X,Y)$ *is called weakly closed. If* $T(0)$ *is closed and* $x_n \rightharpoonup x$ *in* $X$ *for every sequence* $(x_n)_n$ *in* $\mathcal{D}(T)$, *and* $Q_T T x_n \rightharpoonup y$ *in* $Y/\overline{T(0)}$, *then* $x \in \mathcal{D}(T)$ *and* $Q_T T x = y$.

$CR_W(X,Y)$ denotes the set of all weakly closed densely defined linear relations from $X$ into $Y$.

***Remark* 3.2.** Obviously, if $T$ is a weakly closed linear relation, then $T$ is closed.

The following key lemma will be invested for some proofs.

**Lemma 3.1.** *Let* $T \in CR(X,Y)$, *where* $X$ *and* $Y$ *are Banach spaces such that* $\overline{T(0)}$ *is compact. If* $T$ *is demicompact*, *then* $T$ *is weakly demicompact.*

***Proof.*** Let $y_n := Q_T(I-T)x_n \rightharpoonup y$, where $(x_n)_n$ be a bounded sequence. Suppose that $T$ is a demicompact relation. Grounded on Theorem 3.3 in [2], we have that $I - T \in \Phi_+(X,Y)$, and based upon Remark 2.1, we deduce that $Q_T(I-T) \in \Phi_+(X,Y)$, which implies that $Q_{\widehat{T}}(I-\widehat{T}) \in \Phi_+(X_T, X)$. The use of $\alpha(Q_T(I-T)) < \infty$ entails that $X = N(Q_T(I-T)) \oplus X_0$, where $X_0$ is a closed subspace of $X$. As $T \in CR(X,Y)$ and $X_0 \subset X$ is closed, $X_0 \cap \mathcal{D}(T)$ is a closed subset of $X_T$. Hence, $(X_0 \cap \mathcal{D}(T), \|.\|_T)$ is a Banach space. Thus,

$$X_T = N\big(Q_{\widehat{T}}(I-\widehat{T})\big) \oplus [D(T) \cap X_0].$$

So, $(Q_{\widehat{T}}(I-\widehat{T})) \mid_{X_0 \cap \mathcal{D}(T)} : (X_0 \cap D(T), \|.\|_T) \longrightarrow \big(R(Q_T(I-T)), \|.\|\big)$ is invertible with a bounded inverse on $R(Q_T(I-T)) = R(Q_{\widehat{T}}(I-\widehat{T}))$. As a result, from the boundedness of $(Q_{\widehat{T}}(I-\widehat{T}) \mid_{X_0 \cap \mathcal{D}(T)})^{-1}$ on $R(Q_{\widehat{T}}(I-\widehat{T}))$, we get $x_n \rightharpoonup (Q_{\widehat{T}}(I-\widehat{T}))^{-1}(y)$. From this perspective, by using Proposition 3.5 in [7], we have that $\big(Q_{\widehat{T}}(I-\widehat{T})\big)x_n \to y$. Thus, $Q_T(I-T)x_n \to y$. From the demicompactness of $T$, there exists a convergent subsequence $(Q_T x_{\varphi(n)})_n$ of $(Q_T x_n)_n$ such that $Q_T x_{\varphi(n)} \to Q_T x$. Therefore, $Q_T x_{\varphi(n)} \rightharpoonup Q_T x$.

Lemma 3.1 is proved.

Now, we present the following result.

**Theorem 3.2.** *Let* $T \in CR_W(X,Y)$, *where* $X$ *and* $Y$ *are Banach spaces such that* $\overline{T(0)}$ *is compact and* $R_c(T) = \{0\}$. $T$ *is a generalized weakly demicompact relation if and only if there exists a finite subset* $E$ *of* $\mathbb{C}$ *involving* $0$ *such that* $\lambda - T \in \Phi_+(X,Y)$ $\forall \lambda \in \mathbb{C} \setminus E$.

***Proof.*** To prove the first hypothesis assume that there exists a finite subset $E$ of $\mathbb{C}$ involving $0$ such that $\lambda - T \in \Phi_+(X,Y)$ $\forall \lambda \in \mathbb{C} \setminus E$. Then $T$ is a generalized Riesz relation. Therefore, based upon Remark 3.1 to demonstrate that $T$ is a generalized weakly demicompact linear relation, it is sufficient to show that $\dfrac{1}{\lambda}T$ is a weakly demicompact linear relation for all $\lambda \in \mathbb{C} \setminus E$. From Theorem 3.4 in [2], since $\lambda - T \in \Phi_+(X,Y)$ for all $\lambda \in \mathbb{C} \setminus E$, we have $\dfrac{1}{\lambda}T$ which is a demicompact relation for all $\lambda \in \mathbb{C} \setminus E$. Then, from Lemma 3.1, we infer the result.

To confirm the opposite, let us first verify that $\alpha(\lambda - T) < \infty$. As $T$ is a generalized weakly demicompact relation, $\operatorname{asc}(\lambda - T) < \infty\ \forall\lambda \in \mathbb{C} \setminus E$. We denote by $p = \operatorname{asc}(\lambda - T) < \infty$. Moreover, for any eigenvalue $\lambda \in \sigma(T) \setminus E$, we get that $\operatorname{mult}(T, \lambda) < \infty$, where $\operatorname{mult}(T, \lambda)$ denotes the multiplicity of all $\lambda \in \sigma(T) \setminus E$. Hence,

$$\operatorname{mult}(T, \lambda) = \dim N_\infty(\lambda - T) = \dim N((\lambda - T)^p) < \infty.$$

As a consequence, departing from $N(\lambda - T) \subset N\big((\lambda - T)^p\big)$, we obtain that $\dim N(\lambda - T) = \alpha(\lambda - T) < \infty$. Now, we verify that $R(\lambda - T)$ is closed. Let us consider $y_n = Q_T(\lambda - T)x_n = Q_T\left(\lambda\left(I - \frac{1}{\lambda}T\right)\right)x_n = \lambda Q_T\left(I - \frac{1}{\lambda}T\right)x_n$ for any sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $y_n \to y \in X$. As $\frac{1}{\lambda}T$ is weakly demicompact and

$$Q_T x_n - \frac{1}{\lambda} Q_T T x_n \rightharpoonup \frac{1}{\lambda} y \in X,$$

there exists a subsequence $(Q_T x_{\varphi(n)})_n$ of $(Q_T x_n)_n$ such that $Q_T x_{\varphi(n)} \rightharpoonup Q_T x \in X$. The use of

$$Q_T x_{\varphi(n)} \rightharpoonup Q_T x \qquad \text{and} \qquad Q_T(\lambda - T)x_{\varphi(n)} \rightharpoonup y,$$

with the closedness of $Q_T(\lambda - T)$, we deduce that

$$x \in \mathcal{D}(T) \qquad \text{and} \qquad y = Q_T(\lambda - T)x.$$

As a result, $y \in R\big(Q_T(\lambda - T)\big)$. Therefore, $R\big(Q_T(\lambda - T)\big)$ is closed. Hence, $R(\lambda - T)$ is closed.

Theorem 3.2 is proved.

**Theorem 3.3.** *Let $T \in CR_W(X, Y)$, where $X$ and $Y$ are Banach spaces such that $R_c = \{0\}$. If $\mu T$ is a generalized weakly demicompact relation for each $\mu \in [0, 1]$ with a generalized subset $E$, then $\lambda - T \in \Phi(X, Y)$ and $i(\lambda - T) = 0\ \forall\lambda \in \mathbb{C} \setminus E$.*

***Proof.*** For $\mu \in [0, 1]$, as $\mu T$ is a generalized weakly demicompact relation $\lambda - \mu T \in \Phi_+(X, Y)$ for all $\lambda \in \mathbb{C} \setminus E$. In addition, as $\mu$ is an integer including infinite values and $i(\lambda - \mu T)$ is continuous in $\mu$, the index $i(\lambda - \mu T)$ must be constant for every $\mu \in [0, 1]$. Confirming that $i(\lambda - \mu T) = i(\lambda - T) = i(\lambda) = 0$. Thus, $\lambda - T \in \Phi(X, Y)$ and $i(\lambda - T) = 0$ for all $\lambda \in \mathbb{C} \setminus E$.

**4. Generalized weak $S$-demicompact linear relations.**

**Definition 4.1.** *Let $X$ and $Y$ be Banach spaces, $T : \mathcal{D}(T) \subset X \longrightarrow Y$ and $S : \mathcal{D}(S) \subset X \longrightarrow Y$ be densely defined linear relations with $S(0) \subseteq T(0)$ and $\mathcal{D}(S) \subset \mathcal{D}(T)$. $T$ is called weakly $S$-demicompact (or weak relative demicompact with respect to $S$), if $(Q_{S-T}(S - T)x_n)_n$ converges weakly in $Y/\overline{T(0)}$ every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$. Then $(Q_T S x_n)_n$ has a weakly convergent subsequence.*

**Definition 4.2.** *Let $T, S \in CR(X, Y)$, where $X$ and $Y$ are Banach spaces such that $\mathcal{D}(T) \subset \mathcal{D}(S)$. $T$ is called generalized weakly $S$-demicompact relation if there exists a finite subset $E$ of $\mathbb{C}$ involving $0$ such that:*

(i) *$\frac{1}{\lambda}T$ is a weakly $S$-demicompact relation for all $\lambda \in \mathbb{C} \setminus E$,*

(ii) *the ascent of $\lambda S - T$ is finite for all $\lambda \in \mathbb{C} \setminus E$,*

(iii) *the multiplicity of all eigenvalues $\lambda \in \sigma_S(T) \setminus E$ is finite and they don't have accumulation points only possibly in one point of $E$.*

*$E$ is the generalized set of $T$.*

***Remark* 4.1.** For $S = I$, we go back to the usual definition of generalized weak demicompactness of a linear relation.

**Lemma 4.1.** *Let $T \in CR(X,Y)$ $X$ and $Y$ are Banach spaces such that $\overline{T(0)}$ is compact and $S \in BR(X,Y)$ such that $0 \in \rho(S)$, $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) = T(0)$, $S - T$ is closed, $Q_T - Q_S$ is compact, $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$ and $S(\mathcal{D}(T)) \subset \mathcal{D}(T)$. If $T$ is an $S$-demicompact relation, then $T$ is a weakly $S$-demicompact relation.*

***Proof.*** Assume that $T$ is an $S$-demicompact relation. Let $y_n := Q_T(S-T)x_n \rightharpoonup y$ where $(x_n)_n$ is a bounded sequence. As $T$ is an $S$-demicompact relation, grounded on Theorem 3.2 in [3], $S-T \in \Phi_+(X,Y)$, and based upon Remark 2.1 we deduce that $Q_T(S-T) \in \Phi_+(X,Y)$, which implies that $Q_{\widehat{T}}(S-\widehat{T}) \in \Phi_+(X_T, X)$. The use of $\alpha(Q_T(S-T)) < \infty$ entails that $X = N(Q_T(S-T)) \oplus X_0$, where $X_0$ is a closed subspace of $X$. As $T \in CR(X,Y)$ and $X_0 \subset X$ is closed, $X_0 \cap \mathcal{D}(T)$ is a closed subset of $X_T$. Hence, $\big(X_0 \cap \mathcal{D}(T), \|.\|_T\big)$ is a Banach space. Thus,

$$X_T = N\big(Q_{\widehat{T}}(S-\widehat{T})\big) \oplus [D(T) \cap X_0].$$

So, $\big(Q_{\widehat{T}}(S-\widehat{T})\big)\,|_{X_0 \cap \mathcal{D}(T)} : (X_0 \cap D(T), \|.\|_T) \longrightarrow \big(R(Q_T(S-T)), \|.\|\big)$ is invertible with a bounded inverse on $R\big(Q_T(S-T)\big) = R\big(Q_{\widehat{T}}(S-\widehat{T})\big)$. Consequently, departing from the boundedness of $\big(Q_{\widehat{T}}(S-\widehat{T})\,|_{X_0 \cap \mathcal{D}(T)}\big)^{-1}$ on $R\big(Q_{\widehat{T}}(S-\widehat{T})\big)$, we have that $x_n \rightharpoonup (Q_{\widehat{T}}(S-\widehat{T}))^{-1}(y)$. From this perspective, by using Proposition 3.5 in [7], we get that $(Q_{\widehat{T}}(S-\widehat{T}))x_n \to y$, so $Q_T(S-T)x_n \to y$. From the $S$-demicompactness of $T$, there exists a convergent subsequence $(Q_T x_{\varphi(n)})_n$ of $(Q_T S x_n)_n$ such that $Q_T S x_{\varphi(n)} \to Q_T S x$. Hence, $Q_T S x_{\varphi(n)} \rightharpoonup Q_T S x$ and the proof holds.

Lemma 4.1 is proved.

Now, we shall demonstrate that there exists a connection between upper semi-Fredholm linear relations and the class of generalized weakly S-demicompact relations acting on a Banach space.

**Theorem 4.1.** *Let $T \in CR(X,Y)$, where $X$ and $Y$ are Banach spaces such that $\overline{T(0)}$ is compact and $S \in BR(X,Y)$ such that $0 \in \rho(S)$, $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subseteq T(0)$, $S - T$ is closed, $Q_T - Q_S$ is compact, $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$, $S(\mathcal{D}(T)) \subset \mathcal{D}(T)$, and $R_c(S^{-1}T) = \{0\}$. Therefore, $T$ is a generalized weakly $S$-demicompact if and only if there is $E \subset \mathbb{C}$ a finite subset involving $0$ such that $\lambda S - T \in \Phi_+(X,Y)$ $\forall \lambda \in \mathbb{C} \setminus E$.*

***Proof.*** To prove the first hypothesis assume that, there exists a finite subset $E \subset \mathbb{C}$ involving $0$ such that $\lambda S - T \in \Phi_+(X,Y)$ $\forall \lambda \in \mathbb{C} \setminus E$. Since $S$ is an invertible relation, we get

$$\lambda S - T = S(\lambda I - S^{-1}T).$$

As $S$ is bounded, applying Theorem V.2.16 of [9], $\lambda I - S^{-1}T \in \Phi_+(X,Y)$ $\forall \lambda \in \mathbb{C} \setminus E$. Thus, by Theorem 3.1, $S^{-1}T$ is a generalized Riesz relation. According to Definition 3.1, the multiplicity of all eigenvalues $\lambda \in \sigma(S^{-1}T) \setminus E$ is finite and they don't have accumulation points only possibly in one point of $E$. Now, since $S$ is bounded and invertible, we have

$$\sigma(S^{-1}T) = \sigma_S(T).$$

Consequently, we infer that the multiplicity of all eigenvalues $\lambda \in \sigma_S(T) \backslash E$, denoted by $\mathrm{mult}(T, \lambda S)$, is finite and they don't have accumulation points only possibly in one point of $E$. Now, we verify that $\mathrm{asc}(\lambda S - T) < \infty$. Note that $n_0 = \mathrm{asc}(\lambda S - T)$. Therefore, we have

$$\mathrm{mult}(T, \lambda S) = \dim N_\infty(\lambda S - T) = \dim N((\lambda S - T)^{n_0}) < n_0 \dim N((\lambda S - T) < \infty.$$

As $\dim N((\lambda S - T) < \infty,\ n_0 < \infty$ and it follows that $\mathrm{asc}(\lambda S - T) < \infty\ \forall\lambda \in \mathbb{C} \setminus E$. Now, let us move on to the last point. Indeed, using the fact that $\lambda S - T \in \phi_+(X, Y)$ and investing Theorem 3.2 in [3], we have $\frac{1}{\lambda}T$ which is $S$-demicompact. To confirm the opposite, let us first verify that $\alpha(\lambda S - T) < \infty$. As $T$ is a generalized weakly $S$-demicompact relation, $\mathrm{asc}(\lambda S - T) < \infty$ $\forall\lambda \in \mathbb{C} \setminus E$. Note that $p = \mathrm{asc}(\lambda S - T) < \infty$. Furthermore, for any eigenvalue $\lambda \in \sigma_S(T) \setminus E$, we have $\mathrm{mult}(T, \lambda S) < \infty$, then

$$\mathrm{mult}(T, \lambda S) = \dim N^{\infty}(\lambda S - T) = \dim N((\lambda S - T)^p) < \infty.$$

Therefore, referring to

$$N(\lambda S - T) \subset N((\lambda S - T)^p),$$

we have $\dim N(\lambda S - T) = \alpha(\lambda S - T) < \infty$. Now, we assert that $R(\lambda S - T)$ is closed. Let $y_n = Q_T(\lambda S - T)x_n$ for any sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $y_n \to y \in X$. As $\frac{1}{\lambda}T$ is a weakly $S$-demicompact and

$$Q_T S x_n - \frac{1}{\lambda} Q_T T x_n \rightharpoonup \frac{1}{\lambda} y \in X,$$

there exists a subsequence $Q_T S(x_{\varphi(n)})_n$ of $Q_T S(x_n)_n$ such that $Q_T S x_{\varphi(n)} \rightharpoonup Q_T S x \in X$. The use of $Q_T S x_{\varphi(n)} \rightharpoonup Q_T S x$ and $Q_T(\lambda S - T)x_{\varphi(n)} \rightharpoonup y$ with the closedness of $Q_T(\lambda S - T)$, we deduce that $x \in \mathcal{D}(T)$ and $y = Q_T(\lambda S - T)x$. As a result, $y \in R\big(Q_T(\lambda S - T)\big)$. Therefore, $R\big(Q_T(\lambda S - T)\big)$ is closed. Hence, $R(\lambda S - T)$ is closed.

Theorem 4.1 is proved.

The following result corroborates the connection between generalized weakly $S$-demicompact relations and Fredholm relations having null indices.

**Theorem 4.2.** *Let $T \in CR(X, Y)$ and $S \in BR(X, Y)$, where $X$ and $Y$ are Banach spaces such that $0 \in \rho(S)$, $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subseteq T(0)$, $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$, $S(\mathcal{D}(T)) \subset \mathcal{D}(T)$, and $R_c(S^{-1}T) = \{0\}$. If $\mu T$ is a generalized weakly $S$-demicompact relation for each $\mu \in [0, 1]$ with a generalized subset $E$, then $\lambda S - T \in \Phi(X, Y)$ and $i(\lambda S - T) = i(\lambda S)$ $\forall\lambda \in \mathbb{C} \setminus E$.*

***Proof.*** For $\mu \in [0, 1]$, as $\mu T$ is a generalized weakly $S$-demicompact relation $\lambda S - \mu T \in \Phi_+(X, Y)$ $\forall\lambda \in \mathbb{C} \setminus E$. In addition, as $\mu$ is an integer including infinite values and $i(\lambda S - \mu T)$ is continuous in $\mu$, the index $i(\lambda S - \mu T)$ must be constant for every $\mu \in [0, 1]$. Confirming that $i(\lambda S - \mu T) = i(\lambda S - T) = i(\lambda S)$, yields that $\lambda S - T \in \Phi(X, Y)$ and $i(\lambda S - T) = i(\lambda S)$ for $\lambda \in \mathbb{C} \setminus E$.

Theorem 4.2 is proved.

Now, we consider the assumption $(\mathcal{L})$: for a given linear relation $T$. $(\mathcal{L})$ : If $(x_n)_n$ is a weakly convergent sequence in $X$, then $(Q_T T x_n)_n$ has a strongly convergent subsequence in $X$.

**Lemma 4.2.** *Let $T \in CR(X, Y)$ and $S \in LR(X, Y)$ where $X$ and $Y$ are Banach spaces such that $0 \in \rho(S)$. $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subseteq T(0)$, $S - T$ is closed, $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$ and $S(\mathcal{D}(T)) \subset \mathcal{D}(T)$. If $T$ is a weakly $S$-demicompact satisfying $(\mathcal{L})$, then $S - T \in \phi_+(X, Y)$.*

***Proof.*** First, we verify that $\alpha(S - T) < \infty$. Assume that

$$H := \big\{x \in \mathcal{D}(T) : Q_T(S - T)x = 0 \ \text{ and } \ \|x\| \leq 1\big\}$$

and $(x_n)_n$ be a sequence of $H$. As $T$ is a weakly $S$-demicompact relation satisfying $(\mathcal{L})$, there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ such that $x_{\varphi(n)} \rightharpoonup x \in X$ and $Q_T T x_{\varphi(n)} \to y$. Thus, from the boundedness of $(Q_T S)^{-1}$ it follows that $x_{\varphi(n)} \to (Q_T S)^{-1} y = x$. Now, as $T \in CR(X, Y)$, $x \in$

$\mathcal{D}(T)$ and $Q_T Tx = y$. Therefore, $x \in N\big(Q_T(S-T)\big)$. Additionally, departing from the assumption that $x_{\varphi(n)} \rightharpoonup x$, we get $\|x\| \leq \liminf \|x_{\varphi(n)}\| \leq 1$. Thus, $x \in H$ and so $\alpha\big(Q_T(S-T)\big) < \infty$. Hence, $\alpha(S-T) < \infty$.

Now, it remains for us to confirm that $R(S-T)$ is closed. Let $y_n = Q_T(S-T)x_n$ for any sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $y_n \to y \in X$. Since $T$ is a weakly $S$- demicompact relation and $Q_T Sx_n - Q_T Tx_n \rightharpoonup y \in X$, there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ such that $x_{\varphi(n)} \rightharpoonup x \in X$. Investing both $x_{\varphi(n)} \rightharpoonup x$ and $Q_T(S-T)x_{\varphi(n)} \rightharpoonup y$, together with the closedness of $(S-T)$, we deduce that $x \in \mathcal{D}(T)$ and $y = Q_T(S-T)x$. As a result, $y \in R\big(Q_T(S-T)\big)$. Consequently, $R(Q_T(S-T))$ is closed. Thus, $R(S-T)$ is closed.

Lemma 4.2 is proved.

**Theorem 4.3.** *Let $T \in CR(X,Y)$ and $S \in BR(X,Y)$, where $X$ and $Y$ are Banach spaces such that $0 \in \rho(S)$, $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subseteq T(0)$, $S-T$ is closed, $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$, $S(\mathcal{D}(T)) \subset \mathcal{D}(T)$, and $R_c(S^{-1}T) = \{0\}$. Assuming that $T$ satisfies $(\mathcal{L})$, $T$ is a generalized weakly $S$-demicompact if and only if there exist a finite subset $E$ of $\mathbb{C}$ involving $0$ and a positive constant $C$ such that, for all bounded sets $A \subset \mathcal{D}(T)$,*

$$w(A) \leq Cw(Q_T(\lambda S - T)(A)) \quad \forall \lambda \in \mathbb{C} \setminus E.$$

***Proof.*** Assume that $T$ be a generalized weakly $S$-demicompact. Hence, relying upon Theorem 4.1, there exists a finite subset $E$ of $\mathcal{C}$ involving $0$ such that $\lambda S - T \in \Phi_+(X,Y)\ \forall\ \lambda \in \mathbb{C} \setminus E$. Now, since $\operatorname{asc}(\lambda S - T)$ is finite, we get $i(\lambda S - T) \leq 0$. Based on Lemma 2.4, $\lambda S - T = K + T_0$ where exists a compact operator $K$ and $T_0$ is a bounded linear relation. As $T_0$ is bounded below such that

$$\|x\| \leq C\|T_0 x\| \quad \forall x \in \mathcal{D}(T) \quad \text{such that} \quad C \quad \text{is a positive constant.}$$

Thus, based upon Proposition 2.3, for any bounded set $A \subset \mathcal{D}(T)$, we get

$$w(A) \leq Cw(Q_T(T_0)(A)) \leq C\big(w(Q_T(\lambda S - T)(A)) + w(Q_T(-K)(A))\big).$$

Therefore, since $K$ is compact, then by Definition 2.2 we have

$$w(A) \leq Cw\big(Q_T(\lambda S - T)(A)\big).$$

To confirm the converse, assume that there exist a finite subset $E$ of $\mathbb{C}$ involving $0$ and a positive constant $C$ such that, for every bounded set $A$ of $X$,

$$w(A) \leq Cw(Q_T(\lambda S - T)(A)) \quad \forall \lambda \in \mathbb{C} \setminus E.$$

Assume that

$$(\lambda S - T)x_n \rightharpoonup x \in X \quad \text{for any bounded sequence} \quad (x_n)_n \quad \text{in} \quad \mathcal{D}(T).$$

Choose $A = \{x_n : n \in \mathbb{N}\}$. It is obvious that $A$ is a bounded set of $\mathcal{D}(T)$ such that $w(Q_T(\lambda S - T)(A)) = 0$. Thus, $w(A) = 0$ which infer that $(x_n)_n$ has a weakly convergent subsequence. As a matter of fact, $\dfrac{1}{\lambda}T$ is a weakly $S$-demicompact for all $\lambda \in \mathbb{C} \setminus E$. Now, as $T$ satisfies $(\mathcal{L})$, and referring to Lemma 4.2, we get that $Q_T(\lambda S - T) \in \Phi_+(X,Y)$ for all $\lambda \in \mathbb{C} \setminus E$. Therefore, $\lambda S - T \in \Phi_+(X,Y)$ for all $\lambda \in \mathbb{C} \setminus E$. Hence, resting on Theorem 4.1, $T$ is a generalized weakly $S$-demicompact relation.

Theorem 4.3 is proved.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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