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S-COLOCALIZATION AND ADAMS COCOMPLETION

S-КОЛОКАЛІЗАЦІЯ ТА КОПОПОВНЕННЯ АДАМСА

A relationship between the S-colocalization of an object and the Adams cocompletion of the same object in a complete small $\mathscr{U}$-category ($\mathscr{U}$ is a fixed Grothendieck universe) is established, together with a specific set of morphisms S.

Встановлено зв'язок між S-колокалізацією об'єкта та копоповненням Адамса того самого об'єкта в деякій повній малій $\mathscr{U}$-категорії ($\mathscr{U}$ — фіксований всесвіт Гротендіка), а також специфічну множину морфізмів S.

1. Introduction. Mathematical entities and their relationships can be described in a fundamental and abstract way by introducing the concept of categories. The idea of localization in categories was first introduced by A. K. Bousfield [6]. In addition, he has also explained about the determination of an S-localization functor $E\colon \mathscr{C} \to \mathscr{C}$ by a class of morphisms S in a category $\mathscr{C}$. Deleanu, Frei and Hilton have introduced the idea Adams completion [1 – 3] in a broader way; they have also suggested its dual notion, known to be the Adams cocompletion [7].

The central idea of this note is to deduce an isomorphism between the S-colocalization [10], the dual of S-localization, of an object and Adams cocompletion of the same object in a complete small $\mathscr{U}$-category ($\mathscr{U}$ being a fixed Grothendieck universe [12]) together with a specific set of morphisms S.

Definition 1.1 [7, 8]. *Let $\mathscr{C}$ be a $\mathscr{U}$-category and S be a set of morphisms of $\mathscr{C}$. Let $\mathscr{C}[S^{-1}]$ denote the category of fractions of $\mathscr{C}$ with respect to S and $F\colon \mathscr{C} \to \mathscr{C}[S^{-1}]$ be the canonical functor. Let $\mathscr{S}$ denote the category of sets and functions. Then for a given object Y of $\mathscr{C}$, $\mathscr{C}[S^{-1}](Y,-)\colon \mathscr{C} \to \mathscr{S}$ defines a covariant functor. If this functor is representable by an object Y_S of $\mathscr{C}$, that is, $\mathscr{C}[S^{-1}](Y,-) \cong \mathscr{C}(Y_S,-)$, then Y_S is called the* (*generalized*) *Adams cocompletion of Y with respect to the set of morphisms S or simply the S-cocompletion of Y. We shall often refer to Y_S as the cocompletion of Y.*

Definition 1.2 [7]. *Given a set S of morphisms of $\mathscr{C}$, we define $\bar{S}$, the saturation of S, as the set of all morphisms u in $\mathscr{C}$ such that $F_S(u)$ is an isomorphism in $\mathscr{C}[S^{-1}]$. S is said to be saturated if $S = \bar{S}$.*

2. S-colocalization of an object in a category. Let $\mathscr{C}$ be any $\mathscr{U}$-category and S be a set of morphisms of $\mathscr{C}$.

Definition 2.1 [10]. *An object X in $\mathscr{C}$ is said to be S-colocal if for every $s\colon A \to B$ in S, the map $\operatorname{Hom}(X,s)\colon \operatorname{Hom}(X,A) \to \operatorname{Hom}(X,B)$ is a bijection.*

Definition 2.2 [10]. *An S-colocalization of an object A in $\mathscr{C}$ is a morphism $f\colon X \to A$ with X S-colocal and $f \in S$.*

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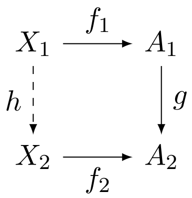
We usually refer X to be the S-colocalization of A and assume the map $X \to A$ understood. The following properties regarding S-colocalization of an object can be easily shown:

1. Any two S-colocalizations of an object A in $\mathscr{C}$ are isomorphic.

2. If A in $\mathscr{C}$ is S-colocal and $f: X \to A$ is an S-colocalization, then f is an isomorphism.

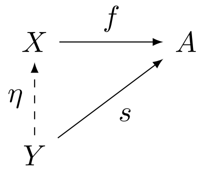
3. Let $f_1: X_1 \to A_1$ and $f_2: X_2 \to A_2$ be S-colocalizations of A_1 and A_2, respectively. Let $g: A_1 \to A_2$ be a morphism. Then there exists a unique morphism $h: X_1 \to X_2$ such that the following diagram commutes:



Next we will show that the S-colocalization $f: X \to A$ of an object A in $\mathscr{C}$ has the following interesting properties:

Proposition 2.1. *The morphism f is terminal among the morphisms $s: Y \to A$ having Y to be S-colocal, that is, f is universal with respect to morphisms $s: Y \to A$ having Y to be S-colocal.*

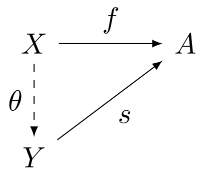
Proof. Since f is S-colocalization of A, then X is S-colocal and $f \in S$. Also, given that Y is S-colocal. So $\mathrm{Hom}(Y, f): \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Y, A)$ is a bijective function. Now $s \in \mathrm{Hom}(Y, A)$.



So there exists a unique morphism $\eta \in \mathrm{Hom}(Y, X)$ such that $\mathrm{Hom}(Y, f)(\eta) = s$, that is, $f\eta = s$.

Proposition 2.2. *The morphism f is initial among the morphisms $s: Y \to A$ with $s \in S$, that is, f is couniversal with respect to morphisms $s: Y \to A$ with $s \in S$.*

Proof. Since f is S-colocalization of A, then X is S-colocal and $f \in S$. Also, it is given that $s \in S$. So $\mathrm{Hom}(X, s): \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, A)$ is a bijective function. As $f \in \mathrm{Hom}(X, A)$ and $\mathrm{Hom}(X, s)$ is a bijection, so there exists a unique morphism $\theta \in \mathrm{Hom}(X, Y)$ such that $\mathrm{Hom}(X, s)(\theta) = f$, that is, $s\theta = f$,



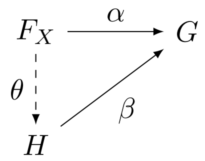
and this proves that f is couniversal with respect to morphisms $s: Y \to A$ with $s \in S$.

But it is not in general true that a morphism $s\colon A \to B$ in S having couniversal property is the S-colocalization map of the object B in $\mathscr{C}$. This fact is clearly seen in the following example.

***Example* 2.1.** Let $\bar{\mathscr{G}}$ be the category whose objects and morphisms are as follows:

objects: groups generated by sets where between every pair of sets there is a bijection,

morphisms: all epimorphisms $f\colon (G, X) \to (H, Y)$ sending each element of X to its image in Y under the bijection (here (G, X) denotes the group generated by the set X).
Let S be class of all morphisms in $\bar{\mathscr{G}}$. Suppose G and F_X are the group and free group generated by X, respectively. Then $\alpha\colon F_X \to G$ is a surjective homomorphism which is the unique homomorphism sending each element in X to its image in X under a bijection [9]. Let $H = (H, Y) \in \bar{\mathscr{G}}$ and $\beta \in S$ be given. Clearly, there is a bijection between X and Y. So there exists a surjective homomorphism $\theta\colon F_X \to H$ which is the unique homomorphism sending each element in X to its image in Y under a bijection.



Now $\beta\theta$ is a surjective homomorphism from F_X to G sending each element in X to its image in X under a bijection. Uniqueness of α gives that $\beta\theta = \alpha$, that is, the above diagram is commutative.

Next for any $s\colon A \to B$ in S, $\mathrm{Hom}(F_X, s) = s_*\colon \mathrm{Hom}(F_X, A) \to \mathrm{Hom}(F_X, B)$ is defined by $s_*(p) = s \circ p$ for all $p \in \mathrm{Hom}(F_X, A)$. Suppose $p_1, p_2 \in \mathrm{Hom}(F_X, A)$ and $s_*(p_1) = s_*(p_2)$. Then $s \circ p_1 = s \circ p_2$, that is, $s(p_1(c)) = s(p_2(c))$ for all $c \in F_X$. From this we can not conclude $p_1(c) = p_2(c)$ for all $c \in F_X$, that is, s_* is not injective. So s_* is not a bijection and it proves that F_X is not S-colocal. So even if $\alpha \in S$ and has couniversal property, but it is not the S-colocalization map of G in $\bar{\mathscr{G}}$.

The following result shows that the converse of Proposition 2.2 is always true when there is an additional condition on S.

Proposition 2.3. *Suppose that*:

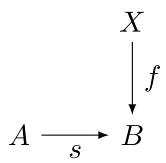
(a) $w\colon X \to M$ *is in* S,

(b) w *is couniversal with respect to all the morphisms to* M *belonging to* S,

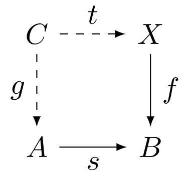
(c) S *admits a calculus of right fractions.*

Then w *is the* S*-colocalization of* M.

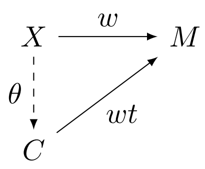
Proof. We have to show w is the S-colocalization of M, that is, to show $w \in S$ and X is S-colocal. Since $w \in S$ is given, it is enough to show X is S-colocal, that is, to show each $s\colon A \to B$ in S induces a bijection $\mathrm{Hom}(X, s)\colon \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$. Consider the following diagram:



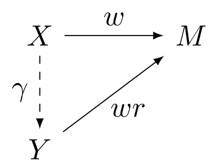
in $\mathscr{C}$ with $s \in S$. As S admits a calculus of right fractions, this diagram can be embedded to a weak pull-back diagram



in $\mathscr{C}$ with $t \in S$. Next it is clear that $wt \in S$. From (b) we conclude that there exists a unique morphism $\theta\colon X \to C$ making the following diagram commutative:



that is, $wt\theta = w$. Next $\operatorname{Hom}(X, s)(g\theta) = sg\theta = ft\theta$. If we can show $t\theta = 1_X$, then $\operatorname{Hom}(X, s)$ will be surjective. Since w is couniversal with respect to all the morphisms to M in S, there exists a unique morphism $\eta\colon X \to X$ such that $w\eta = w$. Also, $w1_X = w$. As η is unique, $\eta = t\theta = 1_X$ and hence $\operatorname{Hom}(X, s)(g\theta) = f$, showing the surjectivity of $\operatorname{Hom}(X, s)$. Next consider $p, q \in \operatorname{Hom}(X, A)$ such that $\operatorname{Hom}(X, s)(p) = \operatorname{Hom}(X, s)(q)$, that is, $sp = sq$. As S admits a calculus of right fractions, there exists a morphism $r\colon Y \to X$ in S such that $pr = qr$. By using condition (b), we get a unique morrphism $\gamma\colon X \to Y$ such that $wr\gamma = w$. In other words, the following diagram commutes:



Again from the couniversal property of w we will have a unique morphism $\beta\colon X \to X$ such that $w\beta = w$. But $w1_X = w$. From the uniqueness of β, it can be concluded that $\beta = r\gamma = 1_X$. So $p = p1_X = pr\gamma = qr\gamma = q1_X = q$ shows the injectivity of $\operatorname{Hom}(X, s)$. Thus $\operatorname{Hom}(X, s)$ is a bijection, which shows that X is S-colocal. Hence, $w\colon X \to M$ is S-colocalization of M.

3. Correspondence between S-colocalization and Adams cocompletion. Before establishing the main result, at first we recall some results related to Adams cocompletion. Consider a complete small $\mathscr{U}$-category $\mathscr{A}$, where $\mathscr{U}$ is a fixed Grothendieck universe and a set of morphisms S admitting a calculus of right fractions. Moreover, assume a compatibility condition with product in S, that is, if each $s_i\colon X_i \to Y_i$ for $i \in I$ is an element of S, where the index set I is an element of $\mathscr{U}$, then $\prod_{i\in I} s_i\colon \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ is an element of S. Then from dual of Deleanu's theorem [8] and Theorem 2 of [11], it follows that every object X of $\mathscr{A}$ has an Adams cocompletion X_S with respect to the set of morphisms S.

In many cases, the set of morphisms S is not saturated. Behera and Nanda [4] have given the generalization of Deleanu, Frei and Hilton's characterization of Adams completion in terms of a couniversal property and the dualization is as follows:

Theorem 3.1 ([5, p. 224], Proposition 1.1)**.** *Let S be a set of morphisms of $\mathscr{A}$ admitting a calculus of right fractions. Then an object A_S of $\mathscr{A}$ is the cocompletion of the object A with respect to S if and only if there exists a morphism $e\colon A_S \to A$ in $\bar{S}$ which is couniversal with respect to morphisms of S: given a morphism $s\colon B \to A$ in S there exists a unique morphism $t\colon A_S \to B$ in $\bar{S}$ such that $st = e$. In other words, the following diagram is commutative*:

$$\begin{array}{ccc} A_S & \xrightarrow{\ e\ } & A \\ {\scriptstyle t}\,\downarrow & \nearrow{\scriptstyle s} & \\ B & & \end{array}$$

By using Behera and Nanda's result ([4, p. 533], dual of Theorem 1.3), the following can be easily deduced.

Theorem 3.2. *The morphism $e\colon A_S \to A$ as constructed in Theorem* 3.1 *is in S.*

Next we present the relation between the S-colocalization and Adams cocompletion of an object A in $\mathscr{A}$. Let $f\colon X \to A$ be the S-colocalization morphism of A and A_S be the Adams cocompletion of A.

Theorem 3.3. *In assumptions of Theorem* 3.1 *there is an isomorphism between X and A_S, that is, $X \cong A_S$.*

Proof. Both e and f are in S. By the couniversal property of e (Theorem 3.1), it can be concluded that there exists a unique morphism $\alpha\colon A_S \to X$ such that $f\alpha = e$, that is, the following diagram is commutative:

$$\begin{array}{ccc} A_S & \xrightarrow{\ e\ } & A \\ {\scriptstyle \alpha}\,\downarrow & \nearrow{\scriptstyle f} & \\ X & & \end{array}$$

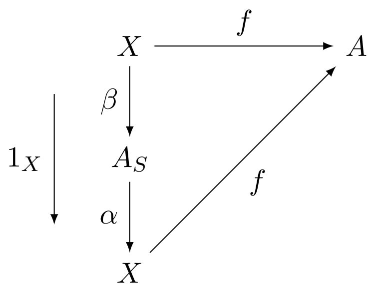
As f is couniversal with respect to all the morphisms in S (Proposition 2.2), there exists a unique morphism $\beta\colon X \to A_S$ making the following triangle commutative:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & A \\ {\scriptstyle \beta}\,\downarrow & \nearrow{\scriptstyle e} & \\ A_S & & \end{array}$$

that is, $e\beta = f$. From the diagram

$$\begin{array}{cccc} & A_S & \xrightarrow{\quad e\quad} & A \\ & {\scriptstyle \alpha}\,\downarrow & & \\ {\scriptstyle 1_{A_S}}\,\downarrow & X & \nearrow{\scriptstyle e} & \\ & {\scriptstyle \beta}\,\downarrow & & \\ & A_S & & \end{array}$$

we have $e\beta\alpha = f\alpha = e$ and from the uniqueness condition of the couniversal property of e we get $\beta\alpha = 1_{A_S}$. Consider the following diagram:



$f\alpha\beta = e\beta = f$. The uniqueness condition of the couniversal property of f shows that $\alpha\beta = 1_X$.

Now we have $\beta\alpha = 1_{A_S}$ and $\alpha\beta = 1_X$. Hence $X \cong A_S$.

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