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EMBEDDINGS INTO COUNTABLY COMPACT HAUSDORFF SPACES

ВКЛАДЕННЯ В ЗЛІЧЕННО КОМПАКТНІ ГАУСДОРФОВІ ПРОСТОРИ

We consider the problem of characterization of topological spaces embedded into countably compact Hausdorff topological spaces. We study the separation axioms for subspaces of Hausdorff countably compact topological spaces and construct an example of a regular separable scattered topological space that cannot be embedded into a Urysohn countably compact topological space.

Розглянуто проблему характеризації топологічних просторів, що вкладаються в зліченно компактні гаусдорфові топологічні простори. Вивчаються аксіоми відокремлення підпросторів зліченно компактних гаусдорфових топологічних просторів та побудовано приклад регулярного сепарабельного розрідженого топологічного простору, який не вкладається у зліченно компактний топологічний простір Урисона.

It is well-known that a topological space X is homeomorphic to a subspace of a compact Hausdorff space if and only if the space X is Tychonoff.

In this paper we discuss the following problem.

Problem 1. *Which topological spaces are homeomorphic to subspaces of countably compact Hausdorff spaces?*

A topological space X is:

compact if each open cover of X has a finite subcover;

ω-bounded if each countable set in X has compact closure in X;

countably compact if each sequence in X has an accumulation point in X;

totally countably compact if each infinite set in X contains an infinite subset with compact closure in X;

ultracompact if each sequence in X has a p-limit for every ultrafilter p on ω;

Lindelöf if each open cover or X has a countable subcover.

These properties relate as follows:

$$\begin{array}{ccccc} \text{compact} & \Longrightarrow & \omega\text{-bounded} & \Longrightarrow & \text{totally countably compact} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{Lindelöf} & & \text{ultracompact} & \Longrightarrow & \text{countably compact.} \end{array}$$

Countably compact topological spaces were investigated in [2, 8 – 12]. The problem of constructing embeddings into ω-bounded or ultracompact spaces was considered in [2] and [1] (see also [7] for basic information on ultracompact spaces). Since the class of countably compact spaces in not

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closed with respect to Tychonoff product, there is no possibility to apply the technique of reflections (applied in [1]) for constructing embeddings into countably compact spaces.

Nonetheless, in this paper we establish some properties of subspaces of countably compact Hausdorff spaces and hence find some necessary conditions of embeddability of topological spaces into Hausdorff countably compact spaces. Also, we construct an example of regular separable first-countable scattered topological space which cannot be embedded into a Urysohn countably compact topological space.

Let $\mathcal{F}$ be a family of closed subsets of a topological space X. The topological space X is called

$\mathcal{F}$-regular if for any set $F \in \mathcal{F}$ and point $x \in X \setminus F$ there exist disjoint open sets $U, V \subset X$ such that $F \subset U$ and $x \in V$.

We recall [6, § 3.6] that the *Wallman extension* $W(X)$ of a topological space X consists of closed ultrafilters, i.e., families $\mathcal{U}$ of closed subsets of X satisfying the following conditions:

$\varnothing \notin \mathcal{U}$;

$A \cap B \in \mathcal{U}$ for any $A, B \in \mathcal{U}$;

a closed set $F \subset X$ belongs to $\mathcal{U}$ if $F \cap U \neq \varnothing$ for every $U \in \mathcal{U}$.

The Wallman extension $W(X)$ of X carries the topology generated by the base consisting of the sets

$$\langle U \rangle = \big\{\mathcal{F} \in W(X) : \exists F \in \mathcal{F},\ \ F \subset U\big\},$$

where U runs over open subsets of X.

By (the proof of) Theorem 3.6.21 in [6], the Wallman extension $W(X)$ is compact.

If X is a T_1-space, then we can consider the map $j_X : X \to W(X)$ assigning to each $x \in X$ the principal ultrafilter consisting of all closed sets $F \subset X$ containing the point x. It is easy to see that the image $j_X(X)$ is dense in $W(X)$. By [6, Theorem 3.6.21], the map $j_X : X \to W(X)$ is a topological embedding, so we can identify the T_1-space X with its image $j_X(X)$ in $W(X)$.

In the Wallman extension $W(X)$, consider the subspace

$$W_\omega X = \bigcup\big\{\overline{j_X(C)} : C \subset X,\ \ |C| \leq \omega\big\},$$

which is the union of closures of countable subsets of $j_X(X)$ in $W(X)$. The space $W_\omega X$ will be called the *Wallman ω-bounded extension* of X. By Proposition 3.2 from [2], the space $W_\omega X$ is ω-bounded. In [2] (resp., [1]) the Wallman extension was used for constructing embeddings of topological spaces into Hausdorff ω-compact (resp., ultracompact) spaces. In this paper we shall use the Wallman extension in Examples 1 and 3 below.

A topological space X is called

locally countable if each $x \in X$ possesses a countable open neighborhood;

first-countable at a point $x \in X$ if it has a countable neighborhood base at x;

of countable pseudocharacter at a point $x \in X$ if $\{x\} = \bigcap \mathcal{U}$ for a countable family $\mathcal{U}$ of open sets in X;

Fréchet – Urysohn at a point $x \in X$ if for each subset A of X with $x \in \overline{A}$ there exists a sequence $\{a_n\}_{n\in\omega} \subset A$ that converges to x;

regular at a point $x \in X$ if any neighborhood of x contains a closed neighborhood of x;

completely regular at a point $x \in X$ if for any neighborhood $U \subset X$ of x there exists a continuous function $f: X \to [0,1]$ such that $f(x) = 1$ and $f(X \setminus U) \subset \{0\}$.
A topological space X is *first-countable* (resp., *Fréchet–Urysohn*, *regular*, *completely regular*, *of countable pseudocharacter*) if X has that property at each point $x \in X$.

Theorem 1. *Let X be a subspace of a countably compact Hausdorff space Y. If X is first-countable at a point $x \in X$, then X is regular at the point x.*

Proof. Fix a countable neighborhood base $\{U_n\}_{n\in\mathbb{N}}$ at x and assume that X is not regular at x. Consequently, there exists a neighborhood U_0 of x such that $\overline{V} \not\subset U_0$ for any neighborhood V of x. Replacing each basic neighborhood U_n by $\bigcap_{k\le n} U_k$, we can assume that $U_n \subset U_{n-1}$ for every $n \in \mathbb{N}$. The choice of the neighborhood U_0 ensures that, for every $n \in \mathbb{N}$, the set $\overline{U}_n \setminus U_0$ contains some point x_n. Since the space Y is countably compact and Hausdorff, the sequence $(x_n)_{n\in\omega}$ has an accumulation point $y \in Y$. Since $U_0 \cap \{x_n\}_{n\in\omega} = \varnothing$, the point y does not coincide with x. Since Y is Hausdorff, there exists a neighborhood $V \subset Y$ of x such that $y \notin \overline{V}$. Find $n \in \omega$ such that $U_n \subset V$ and observe that $O_y := Y \setminus \overline{V}$ is a neighborhood of y such that $O_y \cap \{x_i : i \in \omega\} \subset \{x_i\}_{i<n}$, which means that y is not an accumulating point of the sequence $(x_i)_{i\in\omega}$.

***Remark* 1.** Example 6.1 from [2] shows that in Theorem 1 the regularity of X at the point x cannot be improved to the complete regularity at x.

***Corollary* 1.** *Let X be a subspace of a countably compact Hausdorff space Y. If X is first-countable, then X is regular.*

The following example shows that Theorem 1 cannot be generalized over Fréchet–Urysohn spaces with countable pseudocharacter.

***Example* 1.** There exists a Hausdorff space X such that:

(1) X is locally countable and hence has countable pseudocharacter;

(2) X is separable and Fréchet–Urysohn;

(3) X is not regular;

(4) X is a subspace of a totally countably compact Hausdorff space.

Proof. Choose any point $\infty \notin \omega \times \omega$ and consider the space $Y = \{\infty\} \cup (\omega \times \omega)$ endowed with the topology consisting of the sets $U \subset Y$ such that if $\infty \in U$, then for every $n \in \omega$ the complement $(\{n\} \times \omega) \setminus U$ is finite. The definition of this topology ensures that Y is Fréchet–Urysohn at the unique nonisolated point ∞ of Y.

Let $\mathcal{F}$ be the family of closed infinite subsets of Y that do not contain the point ∞. The definition of the topology on Y implies that for every $F \in \mathcal{F}$ and $n \in \omega$ the intersection $(\{n\} \times \omega) \cap F$ is finite. By the Kuratowski–Zorn lemma, the family $\mathcal{F}$ contains a maximal almost disjoint subfamily $\mathcal{A} \subset \mathcal{F}$. The maximality of $\mathcal{A}$ guarantees that each set $F \in \mathcal{F}$ has infinite intersection with some set $A \in \mathcal{A}$.

Consider the space $X = Y \cup \mathcal{A}$ endowed with the topology consisting of the sets $U \subset X$ such that $U \cap Y$ is open in Y and, for any $A \in \mathcal{A} \cap U$, the set $A \setminus U \subset \omega \times \omega$ is finite.

We claim that the space X has properties (1)–(4). The definition of the topology of X implies that X is separable, Hausdorff and locally countable, which implies that X has countable pseudocharacter. Moreover, X is first-countable at all points except for ∞. At the point ∞ the space X is Fréchet–Urysohn (because its open subspace Y is Fréchet–Urysohn at ∞).

The maximality of the maximal almost disjoint family $\mathcal{A}$ guarantees that each neighborhood $U \subset Y \subset X$ of ∞ has an infinite intersection with some set $A \in \mathcal{A}$, which implies that $A \in \overline{U}$ and hence $\overline{U} \not\subset Y$. This means that X is not regular (at ∞).

In the Wallman extension $W(X)$ of the space X consider the subspace $Z := X \cup W_\omega\mathcal{A} = Y \cup W_\omega\mathcal{A}$. We claim that the space Z is Hausdorff and totally countably compact. To prove that Z is Hausdorff, take two distinct ultrafilters $a, b \in Z$. If the ultrafilters a, b are principal, then since X is Hausdorff, they have disjoint neighborhoods in $W(X)$ and hence in Z. Now assume that one of the ultrafilters a or b is principal and the other is not. We lose no generality assuming that a is principal and b is not. If $a \neq \infty$, then we can use the regularity of the space X at a and prove that a and b have disjoint neighborhoods in $W(X) \supset Z$. So, assume that $a = \infty$. It follows from $b \in Z = X \cup W_\omega\mathcal{A}$ that the ultrafilter b contains some countable set $\{A_n\}_{n\in\omega} \subset \mathcal{A}$. Consider the set

$$V = \bigcup_{n\in\omega}\left(\{A_n\} \cup A_n \setminus \bigcup_{k\le n}\{k\}\times\omega\right)$$

and observe that V has finite intersection with every set $\{k\}\times\omega$, which implies that $Y \setminus V$ is a neighborhood of ∞. Then $\langle Y \setminus V\rangle$ and $\langle V\rangle$ are disjoint open neighborhoods of $a = \infty$ and b in $W(X)$.

Finally, assume that both ultrafilters a, b are not principal. Since $a, b \in W_\omega\mathcal{A}$ are distinct, there are disjoint countable sets $\{A_n\}_{n\in\omega}, \{B_n\}_{n\in\omega} \subset \mathcal{A}$ such that $\{A_n\}_{n\in\omega} \in a$ and $\{B_n\}_{n\in\omega} \in b$. Observe that the sets

$$V = \bigcup_{n\in\omega}\left(\{A_n\} \cup A_n \setminus \bigcup_{k\le n} B_k\right) \qquad \text{and} \qquad W = \bigcup_{n\in\omega}\left(\{B_n\} \cup B_n \setminus \bigcup_{k\le n} A_k\right)$$

are disjoint and open in X. Then $\langle V\rangle$ and $\langle W\rangle$ are disjoint open neighborhoods of the ultrafilters a, b in $W(X)$, respectively.

To see that Z is totally countably compact, take any infinite set $I \subset Z$. We should find an infinite set $J \subset I$ with compact closure $\overline{J}$ in Z. We lose no generality assume that I is countable and $\infty \notin I$. If $J = I \cap W_\omega\mathcal{A}$ is infinite, then $\overline{J}$ is compact by the ω-boundedness of $W_\omega\mathcal{A}$ (see [2]). If $I \cap W_\omega\mathcal{A}$ is finite, then $I \cap Z \setminus W_\omega\mathcal{A} = I \cap Y = I \cap (\omega\times\omega)$ is infinite. If for some $n \in \omega$ the set $J_n = I \cap (\{n\}\times\omega)$ is infinite, then $\overline{J}_n = J_n \cup \{\infty\}$ is compact by the definition of the topology of the space Y. If for every $n \in \omega$ the set $I \cap (\{n\}\times\omega)$ is finite, then $I \cap (\omega\times\omega) \in \mathcal{F}$ and by the maximality of the family $\mathcal{A}$, for some set $A \in \mathcal{A}$ the intersection $J = A \cap I$ is infinite, and then $\overline{J} = J \cup \{A\}$ is compact.

A topological space X is called *locally countably compact* if for each $x \in X$ there exists an open neighborhood U of x such that $\overline{U}$ is countably compact.

Theorem 2. *A first-countable topological space X can be embedded as an open subspace into a Hausdorff countably compact topological space Y if and only if X is locally countably compact.*

Proof. Assume that a first-countable topological space X is an open subspace of a countably compact topological space Y and X is not locally countably compact. Then there exists $x \in X$ such that, for each open neighborhood U of x, the closure of U in X is not countably compact. Fix any countable base $\{U_n\}_{n\in\omega}$ at the point x such that $U_n \subset U_m$, whenever $n > m$. Then there exists a family $\{A_n\}_{n\in\omega}$ of closed discrete subsets of X such that $A_n \subset U_n$ for each $n \in \omega$. Since Y

is countably compact for each $n \in \omega$, the set A_n has an accumulation point $y_n \in Y$. Since A_n is closed in X, $y_n \in Y \setminus X$, $n \in \omega$. Using one more time countably compactness of Y we can find an accumulation point z of the set $\{y_n\}_{n\in\omega}$. Since X is open in Y, $z \in Y \setminus X$. It is easy to see that $z \in \overline{U}_n$ for all $n \in \omega$ which contradicts the Hausdorffness of Y.

Let X be a locally countably compact topological space. Put $Y = X \cup \{\infty\}$ where $\infty \notin X$. Let τ be the topology on Y which satisfies the following conditions:

X is an open subspace of Y;

if $\infty \in U \in \tau$, then $X \setminus U$ is closed and countably compact.

It is easy to check that the space Y is Hausdorff and countably compact.

The following example shows that Theorem 2 does not hold for topological spaces of character ω_1.

***Example* 2.** By $[0, \omega_1]$ we denote the ordinal $\omega_1 + 1$ endowed with the order topology. By X we denote the subspace $\{\omega_1\} \cup \{\alpha \in \omega_1 \mid \alpha \text{ is isolated in } [0, \omega_1]\}$ of $[0, \omega_1]$. Obviously, X is not locally countably compact (at the point ω_1) and the character of X is equal ω_1. Nevertheless, X can be embedded as an open subspace into a Hausdorff countably compact space Y. Let Y be the set $\omega_1 + 1$ endowed with the topology τ which satisfies the following conditions:

X is open in Y;

if $\alpha \in U \in \tau$, then there exists an ordinal $\beta \leq \alpha$ such that $\{\gamma \mid \beta < \gamma \leq \alpha\} \subset U$.

Observe that $Y \setminus \{\omega_1\}$ is homeomorphic to ω_1 endowed with the order topology. At this point it is easy to see that Y is countably compact.

A topological space X is called *weakly ∞-regular* if for any infinite closed subset $F \subset X$ and point $x \in X \setminus F$ there exist disjoint open sets $V, U \subset X$ such that $x \in V$ and $U \cap F$ is infinite.

Proposition 1. *Each subspace X of a Hausdorff countably compact space Y is weakly ∞-regular.*

Proof. Given an infinite closed subset $F \subset X$ and a point $x \in X \setminus F$, consider the closure $\overline{F}$ of F in Y and observe that $x \notin \overline{F}$. By the countable compactness of Y, the infinite set F has an accumulation point $y \in \overline{F}$. Since Y is Hausdorff, there are two disjoint open sets $V, U \subset Y$ such that $x \in V$ and $y \in U$. Since y is an accumulation point of the set F, the intersection $F \cap U$ is infinite. Then $V \cap X$ and $U \cap X$ are two disjoint open sets in X such that $x \in V \cap X$ and $F \cap U \cap X$ is infinite, witnessing that the space X is weakly ∞-regular.

A subset D of a topological space X is called:

strictly discrete if each point $x \in D$ has a neighborhood $O_x \subset X$ such that the family $(O_x)_{x\in D}$ is disjoint in the sense that $O_x \cap O_y = \varnothing$ for any distinct points $x, y \in D$;

strongly discrete if each point $x \in D$ has a neighborhood $O_x \subset X$ such that the family $(O_x)_{x\in D}$ is disjoint and locally finite in X.

It is clear that for every subset $D \subset X$ we have the implications

$$\text{strongly discrete} \Rightarrow \text{strictly discrete} \Rightarrow \text{discrete}.$$

Theorem 3. *Let X be a subspace of a countably compact Hausdorff space Y. Then each infinite subset $I \subset X$ contains an infinite subset $D \subset I$ which is strictly discrete in X.*

Proof. By the countable compactness of Y, the set I has an accumulation point $y \in Y$. Choose any point $x_0 \in I \setminus \{y\}$ and using the Hausdorffness of Y, find a disjoint open neighborhoods V_0 and U_0 of the points x_0 and y, respectively. Choose any point $y_1 \in U_0 \cap I \setminus \{y\}$ and using the Hausdorffness of Y choose open disjoint neighborhoods $V_1 \subset U_0$ and $U_1 \subset U_0$ of the points x_1 and y, respectively. Proceeding by induction, we can construct a sequence $(x_n)_{n\in\omega}$ of points of X and sequences $(V_n)_{n\in\omega}$ and $(U_n)_{n\in\omega}$ of open sets in Y such that for every $n \in \mathbb{N}$ the following conditions are satisfied:

(1) $x_n \in V_n \subset U_{n-1}$;

(2) $y \in U_n \subset U_{n-1}$;

(3) $V_n \cap U_n = \varnothing$.

The inductive conditions imply that the sets V_n, $n \in \omega$, are pairwise disjoint, witnessing that the set $D = \{x_n\}_{n\in\omega} \subset I$ is strictly discrete in X.

Theorem 4. *Let X be a Lindelöf subspace of a countably compact Hausdorff space Y. Then each infinite closed discrete subset $I \subset X$ contains an infinite subset $D \subset I$ which is strongly discrete in X.*

Proof. By the countable compactness of Y, the set I has an accumulation point $y \in Y$. Since I is closed and discrete in X, the point y does not belong to the space X. Since Y is Hausdorff, for every $x \in X$, there are disjoint open sets $V_x, W_x \subset Y$ such that $x \in V_x$ and $y \in W_x$. Since the space X is Lindelöf, the open cover $\{V_x : x \in X\}$ has a countable subcover $\{V_{x_n}\}_{n\in\omega}$. For every $n \in \omega$, consider the open neighborhood $W_n = \bigcap_{k\le n} W_{x_k}$ of y.

Choose any point $y_0 \in I\setminus\{y\}$ and using the Hausdorffness of Y, find disjoint open neighborhoods V_0 and $U_0 \subset W_0$ of the points y_0 and y, respectively. Choose any point $y_1 \in U_0 \cap W_1 \cap I \setminus \{y\}$ and using the Hausdorffness of Y choose open disjoint neighborhoods $V_1 \subset U_0$ and $U_1 \subset U_0 \cap W_1$ of the points y_1 and y, respectively. Proceeding by induction, we can construct a sequence $(y_n)_{n\in\omega}$ of points of X and sequences $(V_n)_{n\in\omega}$ and $(U_n)_{n\in\omega}$ of open sets in Y such that for every $n \in \mathbb{N}$ the following conditions are satisfied:

(1) $y_n \in V_n \subset U_{n-1} \cap W_n$;

(2) $y \in U_n \subset U_{n-1} \cap W_n$;

(3) $V_n \cap U_n = \varnothing$.

The inductive conditions imply that the family $(V_n)_{n\in\omega}$ are pairwise disjoint, witnessing that the set $D = \{y_n\}_{n\in\omega} \subset I$ is strictly discrete in X. To show that D is strongly discrete, it remains to show that the family $(V_n)_{n\in\omega}$ is locally finite in X. Given any point $x \in X$, find $n \in \omega$ such that $x \in V_{x_n}$ and observe that for every $i > n$ we have $V_i \cap V_{x_n} \subset W_i \cap V_{x_n} \subset W_n \cap V_{x_n} = \varnothing$.

A topological space X is called *$\ddot{\omega}$-regular* if it is $\mathcal{F}$-regular for the family $\mathcal{F}$ of countable closed discrete subsets in X.

Proposition 2. *Each countable closed discrete subset D of a (Lindelöf) $\ddot{\omega}$-regular T_1-space X is strictly discrete (and strongly discrete) in X.*

Proof. The space X is Hausdorff, being a $\ddot{\omega}$-regular T_1-space. If the subset $D \subset X$ is finite, then D is strongly discrete, because X is Hausdorff. So, assume that D is infinite and hence $D = \{z_n\}_{n\in\omega}$ for some pairwise distinct points z_n. By the $\ddot{\omega}$-regularity there are two disjoint open sets $V_0, W_0 \subset X$ such that $z_0 \in V_0$ and $\{z_n\}_{n\ge1} \subset W_0$.

Proceeding by induction, we can construct sequences of open sets $(V_n)_{n\in\omega}$ and $(W_n)_{n\in\omega}$ in X such that for every $n \in \omega$ the following conditions are satisfied:

$z_n \in V_n \subset W_{n-1}$;

$\{z_k\}_{k>n} \subset W_n \subset W_{n-1}$;

$V_n \cap W_n = \varnothing$.

These conditions imply that the family $(V_n)_{n\in\omega}$ is disjoint, witnessing that the set D is strictly discrete in X.

Now assume that the space X is Lindelöf and $V = \bigcup_{n\in\omega} V_n$. By the $\ddot{\omega}$-regularity of X, each point $x \in X \setminus V$ has a neighborhood $O_x \subset X$ whose closure $\overline{O}_x$ does not intersect the closed discrete subset D of X. Since X is Lindelöf, there exists a countable set $\{x_n\}_{n\in\omega} \subset X \setminus V$ such that $X = V \cup \bigcup_{n\in\omega} O_{x_n}$. For every $n \in \omega$, consider the open neighborhood $U_n := V_n \setminus \bigcup_{k\le n} \overline{O}_{x_k}$ of z_n and observe that the family $(U_n)_{n\in\omega}$ is disjoint and locally finite in X, witnessing that the set D is strongly discrete in X.

The following proposition shows that the property described in Theorem 3 holds for $\ddot{\omega}$-regular spaces.

Proposition 3. *Every infinite subset I of an $\ddot{\omega}$-regular T_1-space X contains an infinite subset $D \subset I$, which is strictly discrete in X.*

Proof. If I has an accumulation point in X, then a strictly discrete infinite subset can be constructed repeating the argument of the proof of Theorem 3. So, we assume that I has no accumulation point in X and hence I is closed and discrete in X. Replacing I by a countable infinite subset of I, we can assume that I is countable. By Proposition 2, the set I is strictly discrete in X.

A topological space X is called *superconnected* [3] if for any nonempty open sets $U_1, \ldots, U_n$ the intersection $\overline{U}_1 \cap \cdots \cap \overline{U}_n$ is not empty. It is clear that a superconnected space containing more than one point is not regular. An example of a superconnected second-countable Hausdorff space can be found in [3].

Proposition 4. *Any first-countable superconnected Hausdorff space X with $|X| > 1$ contains an infinite set $I \subset X$ such that each infinite subset $D \subset I$ is not strictly discrete in X.*

Proof. For every point $x \in X$ fix a countable neighborhood base $\{U_{x,n}\}_{n\in\omega}$ at x such that $U_{x,n+1} \subset U_{x,n}$ for every $n \in \omega$.

Choose any two distinct points $x_0, x_1 \in X$ and for every $n \ge 2$ choose a point $x_n \in \bigcap_{k<n} \overline{U}_{x_k,n}$. We claim that the set $I = \{x_n\}_{n\in\omega}$ is infinite. In the opposite case, we use the Hausdorffness and find a neighborhood V of x_0 such that $\overline{V} \cap I = \{x_0\}$. Find $m \in \omega$ such that $U_{x_0,m} \subset V$ and $x_0 \notin \overline{U}_{x_1,m}$. Observe that

$$x_m \in I \cap \overline{U}_{x_0,m} \cap \overline{U}_{x_1,m} = \varnothing,$$

which is a desired contradiction showing that the set I is infinite.

Next, we show that any infinite subset $D \subset I$ is not strictly discrete in X. To derive a contradiction, assume that D is strictly discrete. Then each point $x \in D$ has a neighborhood $O_x \subset X$ such that the family $(O_x)_{x\in D}$ is disjoint. Choose any point $x_k \in D$ and find $m \in \omega$ such that $U_{x_k,m} \subset O_{x_k}$. Replacing m by a larger number, we can assume that $m > k$ and $x_m \in D$. Since

$x_m \in \overline{U}_{x_k,m} \subset \overline{O}_{x_k}$, the intersection $O_{x_m} \cap O_{x,k}$ is not empty, which contradicts the choice of the neighborhoods $O_x,\ x \in D$.

Next, we establish one property of subspaces of functionally Hausdorff countably compact spaces. We recall that a topological space X is *functionally Hausdorff* if for any distinct points $x, y \in X$ there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(x) = 1$.

A subset U of a topological space X is called *functionally open* if $U = f^{-1}(V)$ for some continuous function $f : X \to \mathbb{R}$ and some open set $V \subset \mathbb{R}$.

A subset K of a topological space X is called *functionally compact* if each open cover of K by functionally open subsets of X has a finite subcover.

Proposition 5. *If X is a subspace of a functionally Hausdorff countably compact space Y, then no infinite closed discrete subspace $D \subset X$ is contained in a functionally compact subset of X.*

Proof. To derive a contradiction, assume that D is contained in a functionally compact subset K of X. By the countable compactness of Y, the set D has an accumulation point $y \in Y$. Since D is closed and discrete in X, the point y does not belong to X and hence $y \notin K$. Since Y is functionally Hausdorff, for every $x \in K$ there exists a continuous function $f_x : Y \to [0,1]$ such that $f_x(x) = 0$ and $f_x(y) = 1$. By the functional compactness of K, the cover $\left\{ f_x^{-1}\left(\left[0, \frac{1}{2}\right)\right) : x \in K \right\}$ contains a finite subcover $\left\{ f_x^{-1}\left(\left[0, \frac{1}{2}\right)\right) : x \in E \right\}$ where E is a finite subset of K. Then $D \subset K \subset f^{-1}\left(\left[0, \frac{1}{2}\right)\right)$ for the continuous function $f = \max_{x \in E} f_x : Y \to [0,1]$, and $f^{-1}\left(\left(\frac{1}{2}, 1\right]\right)$ is a neighborhood of y, which is disjoint with the set D. But this is not possible as y is an accumulation point of D.

Finally, we construct an example of a regular separable first-countable scattered space that embeds into a Hausdorff countably compact space but does not embed into Urysohn countably compact spaces.

***Example* 3.** There exists a topological space X such that:

(1) X is regular, separable, and first-countable;

(2) X cannot be embedded as an open subspace into a Hausdorff countably compact space;

(3) X cannot be embedded into a Urysohn countably compact space;

(4) X can be embedded into a Hausdorff totally countably compact space.

Proof. In the construction of the space X we shall use almost disjoint dominating subsets of ω^ω. Let us recall [5] that a subset $D \subset \omega^\omega$ is called *dominating* if for any $x \in \omega^\omega$ there exists $y \in D$ such that $x \le^* y$, which means that $x(n) \le y(n)$ for all but finitely many numbers $n \in \omega$. By $\mathfrak{d}$ we denote the smallest cardinality of a dominating subset $D \subset \omega^\omega$. It is clear that $\omega_1 \le \mathfrak{d} \le \mathfrak{c}$.

We say that a family of functions $D \subset \omega^\omega$ is *almost disjoint* if for any distinct $x, y \in D$ the intersection $x \cap y$ is finite. Here we identify a function $x \in \omega^\omega$ with its graph $\{(n, x(n)) : n \in \omega\}$ and hence identify the set of functions ω^ω with a subset of the family $[\omega \times \omega]^\omega$ of all infinite subsets of $\omega \times \omega$.

***Claim* 1.** *There exists an almost disjoint dominating subset $D \subset \omega^\omega$ of cardinality $|D| = \mathfrak{d}$.*

Proof. By the definition of $\mathfrak{d}$, there exists a dominating family $\{x_\alpha\}_{\alpha \in \mathfrak{d}} \subset \omega^\omega$. It is well-known that $[\omega]^\omega$ contains an almost disjoint family $\{A_\alpha\}_{\alpha \in \mathfrak{c}}$ of cardinality continuum. For every $\alpha < \mathfrak{d}$

choose a strictly increasing function $y_\alpha : \omega \to A_\alpha$ such that $x_\alpha \le y_\alpha$. Then the set $D = \{y_\alpha\}_{\alpha\in\mathfrak{d}}$ is dominating and almost disjoint.

By Claim 1, there exists an almost disjoint dominating subset $D \subset \omega^\omega \subset [\omega\times\omega]^\omega$. For every $n\in\omega$, consider the set $\lambda_n = \{n\}\times\omega$ and observe that the family $L = \{\lambda_n\}_{n\in\omega}$ is disjoint and the family $D\cup L \subset [\omega\times\omega]^\omega$ is almost disjoint.

Consider the space $Y = (D\cup L)\cup(\omega\times\omega)$ endowed with the topology consisting of the sets $U\subset Y$ such that, for every $y\in(D\cup L)\cap U$, the set $y\setminus U\subset\omega\times\omega$ is finite. Observe that all points from $\omega\times\omega$ are isolated in Y. Using the almost disjointness of the family $D\cup L$, it can be shown that the space Y is regular, separable, locally countable, scattered and locally compact.

Choose any point $\infty\notin\omega\times Y$ and consider the space $Z = \{\infty\}\cup(\omega\times Y)$ endowed with the topology consisting of the sets $W\subset Z$ such that

for every $n\in\omega$ the set $\{y\in Y\colon (n,y)\in W\}$ is open in Y

and

if $\infty\in W$, then there exists $n\in\omega$ such that $\bigcup_{m\ge n}\{m\}\times Y\subset W$.

It is easy to see $Z = \{\infty\}\cup(\omega\times Y)$ is first-countable, separable, scattered and regular.

Let $\sim$ be the smallest equivalence relation on Z such that

$$(2n,\lambda)\sim(2n+1,\lambda) \qquad \text{and} \qquad (2n+1,d)\sim(2n+2,d)$$

for any $n\in\omega,\ \lambda\in L$ and $d\in D$.

Let X be the quotient space $Z/_\sim$ of Z by the equivalence relation $\sim$. It is easy to see that the equivalence relation $\sim$ has at most two-element equivalence classes and the quotient map $q\colon Z\to X$ is closed and hence perfect. Applying [6, Theorem 3.7.20], we conclude that the space X is regular. It is easy to see that X is separable, scattered and first-countable. Observe that X is not locally countably compact at the point ∞. Hence Theorem 2 implies that X cannot be embedded as an open subspace into a Hausdorff countably compact space. It remains to show that X has the properties (3), (4) of Example 3. This is proved in the following two claims.

***Claim* 2.** *The space X does not admit an embedding into a Urysohn countably compact space.*

Proof. To derive a contradiction, assume that $X = q(Z)$ is a subspace of a Urysohn countably compact space C. By the countable compactness of C, the set $q(\{0\}\times L)\subset X\subset C$ has an accumulation point $c_0\in C$. The point c_0 is distinct from $q(\infty)$, as $q(\infty)$ is not an accumulation point of the set $q(\{0\}\times L)$ in X. Let $l\in\omega$ be the largest number such that c_0 is an accumulation point of the set $q(\{l\}\times L)$ in C.

Let us show that the number l is well-defined. Indeed, by the Hausdorffness of the space C, there exists a neighborhood $W\subset C$ of $q(\infty)$ such that $c_0\not\subset\overline{W}$. By the definition of the topology of the space Z, there exists $m\in\omega$ such that $\bigcup_{k\ge m}\{k\}\times Y\subset q^{-1}(W)$. Then c_0 is not an accumulation point of the set $\bigcup_{k\ge m} q(\{k\}\times L)$ and hence the number l is well-defined and $l<m$.

The definition of the equivalence relation $\sim$ implies that the number l is odd. By the countable compactness of C, the infinite set $q(\{l+1\}\times L)$ has an accumulation point $c_1\in C$. The maximality of l ensures that $c_1\ne c_0$. Since C is Urysohn, the points c_0, c_1 have open neighborhoods $U_0, U_1\subset C$ with disjoint closures in C.

For every $i\in\{0,1\}$, consider the set $J_i = \big\{n\in\omega : q(l+i,\lambda_n)\in U_i\big\}$, which is infinite, because c_i is an accumulation point of the set $q(\{l+i\}\times L) = \big\{q(l+i,\lambda_n) : n\in\omega\big\}$. For every $n\in J_i$, the open set $q^{-1}(U_i) \subset Z$ contains the pair $(l+i, \lambda_n)$. By the definition of the topology at $(l+i, \lambda_n)$, the set $(\{l+i\} \times \lambda_n) \setminus q^{-1}(U_i) \subset \{l+i\} \times \{n\} \times \omega$ is finite and hence is contained in the set $\{l+i\} \times \{n\} \times [0, f_i(n)]$ for some number $f_i(n) \in \omega$. Using the dominating property of the family D, choose a function $f \in D$ such that $f(n) \geq f_i(n)$ for any $i \in \{0,1\}$ and $n \in J_i$. It follows that, for every $i \in \{1,2\}$, the set $\{l+i\} \times f \subset \{l+i\} \times (\omega \times \omega)$ has infinite intersections with the preimage $q^{-1}(U_i)$ and hence $\{(l+i, f)\} \in \overline{q^{-1}(U_i)} \subset q^{-1}(\overline{U}_i)$. Taking into account that the number l is odd, we conclude that

$$q(l,f) = q(l+1,f) \in \overline{U}_0 \cap \overline{U}_1 = \varnothing,$$

which is a desired contradiction completing the proof of the claim.

***Claim* 3.** *The space X admits an embedding into a Hausdorff totally countably compact space.*

Proof. Using the Kuratowski – Zorn lemma, enlarge the almost disjoint family $D \cup L$ to a maximal almost disjoint family $M \subset [\omega \times \omega]^{\omega}$. Consider the space $Y_M = M \cup (\omega \times \omega)$ endowed with the topology consisting of the sets $U \subset Y_M$ such that for every $y \in M \cap U$ the set $y \setminus U \subset \omega \times \omega$ is finite. It follows that Y_M is a regular locally compact first-countable space, containing Y as an open dense subspace. The maximality of M implies that each sequence in $\omega \times \omega$ contains a subsequence that converges to some point of the space Y_M. This property implies that the subspace $\tilde{Y} := (W_\omega M) \cup (\omega \times \omega)$ of the Wallman extension $W(Y_M)$ is totally countably compact. Repeating the argument from Example 1, one can show that the space $\tilde{Y}$ is Hausdorff.

Let $\tilde{Z} = \{\infty\} \cup (\omega \times \tilde{Y})$ where $\infty \notin \omega \times \tilde{Y}$. The space $\tilde{Z}$ is endowed with the topology consisting of the sets $W \subset \tilde{Z}$ such that

for every $n \in \omega$ the set $\{y \in \tilde{Y} : (n, y) \in W\}$ is open in $\tilde{Y}$

and

if $\infty \in W$, then there exists $n \in \omega$ such that $\bigcup_{m \geq n} \{m\} \times \tilde{Y} \subset W$.

Taking into account that the space $\tilde{Y}$ is Hausdorff and totally countably compact, we can prove that so is the the space $\tilde{Z}$.

Let $\sim$ be the smallest equivalence relation on $\tilde{Z}$ such that

$$(2n, \lambda) \sim (2n+1, \lambda) \qquad \text{and} \qquad (2n+1, d) \sim (2n+2, d)$$

for any $n \in \omega$, $\lambda \in W_\omega L$ and $d \in W_\omega D$.

Let $\tilde{X}$ be the quotient space $\tilde{Z}/_\sim$ of $\tilde{Z}$ by the equivalence relation $\sim$. It is easy to see that the space $\tilde{X}$ is Hausdorff, totally countably compact and contains the space X as a dense subspace.

However, we do not know the answer on the following intriguing problem (from Lviv Scottish Book [4]).

Problem 2. *Is it true that each* (*scattered, functionally Hausdorff*) *regular topological space can be embedded into a Hausdorff countably compact topological space*?

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