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**Nadeem ur Rehman**[1] (Department of Mathematics, Aligarh Muslim University, India),
**Hafedh M. Alnoghashi** (Department of Computer Science, College of Engineering and Information Technology, Amran University, Yemen),
**Motoshi Hongan** (Maniwa, Okayama, Japan)

# ON GENERALIZED DERIVATIONS INVOLVING PRIME IDEALS WITH INVOLUTION

# ПРО УЗАГАЛЬНЕНІ ПОХІДНІ, ЩО ВКЛЮЧАЮТЬ ПЕРШІ ІДЕАЛИ З ІНВОЛЮЦІЄЮ

We study the structure of the quotient $A/P$, where $A$ is any ring with involution $*$ and $P$ is a prime ideal of $A$. With an aim to construct a ring with involution of this kind, we study the behavior of generalized derivations satisfying the algebraic identities involving prime ideals. As a consequence, currently existing results in this field are enhanced.

Основна мета статті — вивчити структуру фактор-кільця $A/P$, де $A$ — будь-яке кільце з інволюцією $*$, а $P$ — простий ідеал $A$. Для побудови кільця з інволюцією такого типу досліджено поведінку узагальнених похідних, що задовольняють алгебраїчні тотожності з простими ідеалами. Як наслідок, поточні результати досліджень у цій області були покращені.

**Introduction.** Throughout this paper, $A$ will represent an associative ring with center $Z(A)$. The symbol $(x \circ y)$ $[x, y]$, where $x, y \in A$, stands for the (anti)commutator $(xy+yx)$ $xy-yx$, respectively. An ideal $P$ is said to be a prime ideal of $A$ if $P \neq A$ and, for every $x, y \in A$, whenever $xAy \subseteq P$ implies $x \in P$ or $y \in P$ and $A$ is a prime ring if $(0)$ is a prime ideal. An additive map $\lambda\colon A \to A$ is called a derivation of $A$ if $\lambda(xy) = \lambda(x)y + x\lambda(y)$ holds for every $x, y \in A$. A generalized derivation of $A$ is an additive map $\psi : A \to A$ associated with a derivation $\lambda$ if $\psi(xy) = \psi(x)y + x\lambda(y)$ holds for every $x, y \in A$. An involution is an additive mapping $x \mapsto x^*$ that satisfies $(xy)^* = y^*x^*$ and $(x^*)^* = x$. A ring equipped with an involution is known as ring with involution or $*$-ring. If $x^* = x$, an element $x$ in a ring with involution $*$ is Hermitian, and skew-Hermitian if $x^* = -x$. $H(A)$ and $S(A)$ will represent the sets of all Hermitian and skew-Hermitian elements of $A$, respectively.

Over the last two decades, many scholars have investigated the commutativity of prime and semiprime rings admitting appropriately restricted additive mappings acting on suitable subsets of the rings. In addition, a number of the obtained findings are superior than those that were established just for the effect of the considered mapping on the entire ring. In addition, several findings on commutativity in prime and (semi)prime rings admitting restricted additive mappings, generalized derivations, and (skew) derivations, as well as automorphisms acting on suitable subsets of the rings, have been recently published in the field (see [1, 2, 4 – 12]).

In this paper, we will perform an unique research that extends and generalises current results from the scientific literature. We shall examine differential identities in a prime ideal of an arbitrary ring with involution $*$ using generalized derivation, without supposing the ring's primeness.

[1] Corresponding author, e-mails: rehman100@gmail.com, nu.rehman.mm@amu.ac.in.

**1. Preliminary.**

**Definition 1.1.** *We call the involution is of the $D$-second kind if $S(A) \cap Z(A) \nsubseteq D$ for some $D \subset A$, otherwise it is said to be of the $D$-first kind. In particular, if $D = \{0\}$, the involution is said to be of the second kind if $S(A) \cap Z(A) \neq \{0\}$, otherwise it is said to be of the first kind.*

S. Khan et al. [5] are the first to use the above condition. Now, we will give some examples.

***Example* 1.1.** Note that every the second kind is the $\{0\}$-second kind, but converse is not true in general, let $A$ be any ring with involution $*$ is of the first kind, a set $D \subseteq A \backslash \{0\}$. Then $S(A) \cap Z(A) = \{0\}$, therefore $S(A) \cap Z(A) \nsubseteq D$, that is, the involution is of the $D$-second kind but it is not the second kind.

Now we will look at some examples of involutions of the $P$-second ($P$-first) kind, where $P$ is a nonzero ideal of $A$.

***Example* 1.2.** (1) Let $A = \mathbb{C}[x]$ be a polynomial ring over the complex number $\mathbb{C}$ and $P = \langle x \rangle$ a nonzero ideal of $A$ is generated by $x$. Define $*\colon A \to A$ such that $f(x)^* = \sum_{k=0}^{n} \overline{a_k} x^k \in A$, where $\overline{a_k}$ is a conjugate of $a_k$ in $\mathbb{C}$. Then $S(A) = i\mathbb{R}[x]$ and so $*$ is an involution of the $P$-second kind.

(2) Let $A = M_2(\mathbb{Z})$ be a ring and $P = 2A$ a nonzero ideal of $A$. Define $*\colon A \to A$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then $S(A) \cap Z(A) = \{0\}$ and so $*$ is an involution of the $P$-first kind and so it is not of the $P$-second kind.

Here, we will start with some auxiliary lemmas.

**Lemma 1.1** [3, Lemma 2.1]. *Let $A$ be a ring with $P$ a prime ideal of $A$. If $A$ admits a derivation $\lambda$ such that $[x, \lambda(x)] \in P$ for every $x \in A$, then $\lambda(A) \subseteq P$ or $A/P$ is commutative.*

**Lemma 1.2.** *Let $A$ be a ring and $P$ a prime ideal. If $[x, y] \in P$ for every $x, y \in A$ or $x \circ y \in P$ for every $x, y \in A$, then $A/P$ is commutative.*

***Proof.*** Suppose that $[x, y] \in P$ and $\overline{A} = A/P$ and $\overline{[x, y]} = \overline{0}$ and hence $[\overline{x}, \overline{y}] = \overline{0}$. Thus, $\overline{A}$ is commutative.

Now, suppose that $x \circ y \in P$. Taking $y$ by $ys$ in the previous relation and using it, where $s \in A$, we get $y[x, s] \in P$, and since $P \neq A$, we obtain $[x, s] \in P$, the same as in the above, we find that $A/P$ is commutative.

**Lemma 1.3.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $[x, x^*] \in P$ or $x \circ x^* \in P$ for every $x \in A$, then one of the following holds*:

(i) $\operatorname{char}(A/P) = 2$,

(ii) *$A/P$ is commutative.*

***Proof.*** Assume that $\operatorname{char}(A/P) \neq 2$ and

$$[x, x^*] \in P \quad \text{for every} \quad x \in A. \tag{1.1}$$

By linearizing (1.1) and used it, we find that

$$[x, y^*] + [y, x^*] \in P \quad \text{for every} \quad x, y \in A. \tag{1.2}$$

Replacing $x$ by $xk$ in (1.2), for every $k \in S(A) \cap Z(A) \backslash P$, we get that $k\big([x, y^*] - [y, x^*]\big) \in P$ and so $kA\big([x, y^*] - [y, x^*]\big) \subseteq P$ but $k \notin P$. Thus,

$$[x, y^*] - [y, x^*] \in P. \tag{1.3}$$

Comparing (1.2) and (1.3), we obtain that $2[x,y^*]\in P$ and so $k[x,y^*]\in P$. Putting $y$ by $y^*$ in the last relation, we see that $[x,y]\in P$ and, by Lemma 1.2, $A/P$ is commutative. Similarly in case $x\circ x^*\in P$.

**2. The main results.**

**Theorem 2.1.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi,\lambda)$ is a generalized derivation of $A$ satisfies $\big[\psi(x),x^*\big]\in P$ for every $x\in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P)=2$,

(ii) $\lambda(A)\subseteq P$,

(iii) *$A/P$ is commutative.*

***Proof.*** Assume that $\mathrm{char}(A/P)\neq 2$ and

$$\big[\psi(x),x^*\big]\in P \text{ for every } x\in A. \tag{2.1}$$

By linearizing (2.1), we have

$$\big[\psi(x),y^*\big]+\big[\psi(y),x^*\big]\in P \text{ for every } x,y\in A. \tag{2.2}$$

In (2.2), replace $x$ with $xh$ and use it, where $0\neq h\in H(A)\cap Z(A)$, we get $[x,y^*]\lambda(h)\in P$. When we use $y^*$ instead of $y$ in the previous relation, we have $[x,y]\lambda(h)\in P$, and since $\lambda(h)\in Z(A)$, we find that $[x,y]A\lambda(h)\subseteq P$ but $P$ is a prime ideal. Thus, $[x,y]\in P$ or $\lambda(h)\in P$. In case $[x,y]\in P$ and by Lemma 1.2, we see that $A/P$ is commutative. Now, suppose that $\lambda(h)\in P$. Substituting $k^2$ for $h$ in the last relation, for every $k\in S(A)\cap Z(A)\backslash P$ (note that $S(A)\cap Z(A)\backslash P\neq\varnothing$ because $S(A)\cap Z(A)\nsubseteq P$), we conclude that $2k\lambda(k)\in P$, and since $\mathrm{char}(A/P)\neq 2$, we get $k\lambda(k)\in P$, and since $k\in Z(A)$, we obtain $kA\lambda(k)\subseteq P$, but $k\notin P$ and, hence,

$$\lambda(k)\in P \text{ for every } k\in S(A)\cap Z(A)\backslash P. \tag{2.3}$$

Replacing $x$ by $xk$ in (2.2) and using (2.3), we have $k\big(\big[\psi(x),y^*\big]-\big[\psi(y),x^*\big]\big)\in P$ and, hence, $kA\big(\big[\psi(x),y^*\big]-\big[\psi(y),x^*\big]\big)\subseteq P$ and

$$\big[\psi(x),y^*\big]-\big[\psi(y),x^*\big]\in P. \tag{2.4}$$

Comparing (2.2) and (2.4), we get $2\big[\psi(x),y^*\big]\in P$, that is, $\big[\psi(x),y^*\big]\in P$. Thus, $[\psi(x),y]\in P$. Writing $xy$ instead of $x$ in the last relation and using it, we obtain $[x\lambda(y),y]\in P$. We can replace $x$ by $k\in S(A)\cap Z(A)\backslash P$ to have $[\lambda(y),y]\in P$ for every $y\in A$. Therefore, the desired result follows from Lemma 1.1.

***Corollary* 2.1.** *Let $A$ be a prime ring and $\mathrm{char}(A)\neq 2$ with involution $*$ of the second kind. If $(\psi,\lambda)$ is a generalized derivation of $A$ satisfies $\big[\psi(x),x^*\big]=0$ for every $x\in A$, then $A$ is commutative.*

**Theorem 2.2.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi,\lambda)$ and $(\phi,\mu)$ are generalized derivations of $A$ satisfy $\big[\psi(x),\phi(x^*)\big]\pm[x,x^*]\in P$ for every $x\in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P)=2$,

(ii) *$\lambda(A)\subseteq P$ and $\mu(A)\subseteq P$,*

(iii) *$A/P$ is commutative.*

***Proof.*** Assume that $\mathrm{char}(A/P) \neq 2$ and

$$\big[\psi(x), \phi(x^*)\big] \pm [x, x^*] \in P \ \text{ for every } \ x \in A. \tag{2.5}$$

By linearizing (2.5), we have

$$\big[\psi(x), \phi(y^*)\big] + \big[\psi(y), \phi(x^*)\big] \pm [x, y^*] \pm [y, x^*] \in P \ \text{ for every } \quad x, y \in A. \tag{2.6}$$

Replacing $x$ by $xh$ in (2.6) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)[x, \phi(y^*)] + \mu(h)\big[\psi(y), x^*\big] \in P. \tag{2.7}$$

Substituting $xk$ for $x$ in (2.7), for every $k \in S(A) \cap Z(A)\backslash P$, we obtain

$$\lambda(h)[x, \phi(y^*)] - \mu(h)\big[\psi(y), x^*\big] \in P. \tag{2.8}$$

From (2.7) and (2.8), we have $\lambda(h)[x, \phi(y)] \in P$ and $\mu(h)\big[\psi(y), x^*\big] \in P$. Thus, $A/P$ is commutative, or $\lambda(A) \subseteq P$ and $\mu(A) \subseteq P$, or $\lambda(h)$ and $\mu(h) \in P$. In the latter case, we get

$$\lambda(k) \ \text{ and } \ \mu(k) \in P \ \text{ for every } \ k \in S(A) \cap Z(A)\backslash P. \tag{2.9}$$

Writing $xk$ instead of $x$ in (2.6) and using (2.9), for every $k \in S(A) \cap Z(A)\backslash P$, we see that

$$\big[\psi(x), \phi(y^*)\big] - \big[\psi(y), \phi(x^*)\big] \pm [x, y^*] \mp [y, x^*] \in P. \tag{2.10}$$

Comparing (2.6) and (2.10), we find that

$$\big[\psi(x), \phi(y^*)\big] \pm [x, y^*] \in P.$$

Hence, $\big[\psi(x), \phi(y)\big] \pm [x, y] \in P$. Therefore, the result follows by [6, Theorem 1.4].

***Corollary* 2.2.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\big[\psi(x), \phi(x^*)\big] = \pm[x, x^*]$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.3.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy one of the following holds*:

(1) $\psi(x) \circ \phi(x^*) \pm (x \circ x^*) \in P$,

(2) $\big[\psi(x), \phi(x^*)\big] \pm (x \circ x^*) \in P$,

(3) $\psi(x) \circ \phi(x^*) \pm [x, x^*] \in P$

*for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) $\lambda(A) \subseteq P$ *and* $\mu(A) \subseteq P$,

(iii) $A/P$ *is commutative.*

***Proof.*** The same arguments as used in the proof of Theorem 2.2 and then using [6, Theorems 1.6, 1.8, and 1.10] for (1), (2) and (3), respectively.

***Corollary* 2.3.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy one of the following holds*:

(1) $\psi(x) \circ \phi(x^*) = \pm(x \circ x^*)$,

(2) $\big[\psi(x), \phi(x^*)\big] = \pm(x \circ x^*)$,

(3) $\psi(x) \circ \phi(x^*) = \pm[x, x^*]$

*for every* $x \in A$, *then* $A$ *is commutative.*

**Theorem 2.4.** *Let* $A$ *be a ring and* $P$ *be a prime ideal with* $P$*-second involution* $*$. *If* $(\psi, \lambda)$ *and* $(\phi, \mu)$ *are generalized derivations of* $A$ *satisfy* $\big[\psi(x), x^*\big] \pm \big[x, \phi(x^*)\big] \in P$ *for every* $x \in A$, *then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) $\lambda(A) \subseteq P$ *and* $\mu(A) \subseteq P$,

(iii) $A/P$ *is commutative.*

***Proof.*** Assume that $\mathrm{char}(A/P) \neq 2$ and

$$\big[\psi(x), x^*\big] \pm \big[x, \phi(x^*)\big] \in P \text{ for every } x \in A. \tag{2.11}$$

By linearizing (2.11) and using it, we have

$$\big[\psi(x), y^*\big] + \big[\psi(y), x^*\big] \pm [x, \phi(y^*)] \pm [y, \phi(x^*)] \in P \text{ for every } \quad x, y \in A. \tag{2.12}$$

Replacing $x$ by $xh$ in (2.12) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)[x, y^*] \pm \mu(h)[y, x^*] \in P. \tag{2.13}$$

Writing $xk$ instead of $x$ in (2.13), for every $k \in S(A) \cap Z(A) \backslash P$, we obtain

$$\lambda(h)[x, y^*] \mp \mu(h)[y, x^*] \in P. \tag{2.14}$$

From (2.13) and (2.14), we see that $\lambda(h)[x, y] \in P$ and $\mu(h)[y, x] \in P$. Thus, $A/P$ is commutative or $\lambda(h) \in P$ and $\mu(h) \in P$. In the latter case, we have

$$\lambda(k) \in P \text{ and } \quad \mu(k) \in P \text{ for every } k \in S(A) \cap Z(A) \backslash P. \tag{2.15}$$

Substituting $xk$ for $x$ in (2.12) and using (2.15), for every $k \in S(A) \cap Z(A) \backslash P$, we obtain

$$\big[\psi(x), y^*\big] - \big[\psi(y), x^*\big] \pm [x, \phi(y^*)] \mp [y, \phi(x^*)] \in P. \tag{2.16}$$

Comparing (2.12) and (2.16), we conclude that $[\psi(x), y] \pm [x, \phi(y)] \in P$ and so the result follows by [6, Theorem 1.12].

***Corollary* 2.4.** *Let* $A$ *be a prime ring and* $\mathrm{char}(A) \neq 2$ *with involution* $*$ *of the second kind. If* $(\psi, \lambda)$ *and* $(\phi, \mu)$ *are generalized derivations of* $A$ *satisfy* $\big[\psi(x), x^*\big] = \pm\big[x, \phi(x^*)\big]$ *for every* $x \in A$, *then* $A$ *is commutative.*

**Theorem 2.5.** *Let* $A$ *be a ring and* $P$ *be a prime ideal with* $P$*-second involution* $*$. *If* $(\psi, \lambda)$ *and* $(\phi, \mu)$ *are generalized derivations of* $A$ *satisfy one of the following holds*:

(1) $\big[\psi(x), x^*\big] + \big[x, \phi(x^*)\big] \pm [x, x^*] \in P$,

(2) $\big[\psi(x), x^*\big] + \big[x, \phi(x^*)\big] \pm (x \circ x^*) \in P$,

(3) $\psi(x) \circ x^* \pm x \circ \phi(x^*) \in P$

*for every* $x \in A$, *then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) $\lambda(A) \subseteq P$ *and* $\mu(A) \subseteq P$,

(iii) $A/P$ *is commutative.*

***Proof.*** The same arguments as used in the proof of Theorem 2.4 and then using [6, Theorems 1.14, 1.16, and 1.18] for (1), (2) and (3), respectively.

***Corollary* 2.5.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy one of the following holds*:

(1) $\big[\psi(x), x^*\big] + \big[x, \phi(x^*)\big] = \pm[x, x^*]$,

(2) $\big[\psi(x), x^*\big] + \big[x, \phi(x^*)\big] = \pm(x \circ x^*)$,

(3) $\psi(x) \circ x^* = \pm x \circ \phi(x^*)$

*for every $x \in A$, then $A$ is commutative.*

**Theorem 2.6.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $[\psi(x), x^*] \pm x \circ \phi(x^*) \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) *$\lambda(A) \subseteq P$ and $\mu(A) \subseteq P$,*

(iii) *$A/P$ is commutative.*

***Proof.*** Assume that $\mathrm{char}(A/P) \neq 2$ and

$$[\psi(x), x^*] \pm x \circ \phi(x^*) \in P \text{ for every } x \in A. \tag{2.17}$$

By linearizing (2.17) and using it, we have

$$\big[\psi(x), y^*\big] + \big[\psi(y), x^*\big] \pm (x \circ \phi(y^*)) \pm (y \circ \phi(x^*)) \in P \text{ for every } x, y \in A. \tag{2.18}$$

Replacing $x$ by $xh$ in (2.18) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)[x, y^*] \pm \mu(h)(y \circ x^*) \in P. \tag{2.19}$$

Writing $xk$ instead of $x$ in (2.19), for every $k \in S(A) \cap Z(A) \backslash P$, we obtain

$$\lambda(h)[x, y^*] \mp \mu(h)(y \circ x^*) \in P. \tag{2.20}$$

Now, using the same arguments as used in the proof of Theorem 2.4 in (2.13) and (2.14), we have that $A/P$ is commutative or

$$\lambda(k) \in P \text{ and} \mu(k) \in P \text{ for every } k \in S(A) \cap Z(A) \backslash P. \tag{2.21}$$

Substituting $xk$ for $x$ in (2.18) and using (2.21), for every $k \in S(A) \cap Z(A) \backslash P$,

$$\big[\psi(x), y^*\big] - \big[\psi(y), x^*\big] \pm (x \circ \phi(y^*)) \mp (y \circ \phi(x^*)) \in P. \tag{2.22}$$

Comparing (2.18) and (2.22), we conclude that $[\psi(x), y] \pm (x \circ \phi(y)) \in P$ and so the result follows by [6, Theorem 1.20].

***Corollary* 2.6.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $[\psi(x), x^*] = \pm x \circ \phi(x^*)$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.7.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(x)x^* \pm x^*\phi(x) \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) *$\lambda(A) \subseteq P$ and $\mu(A) \subseteq P$,*

(iii) *$A/P$ is commutative.*

***Proof.*** Assume that $\mathrm{char}(A/P) \neq 2$ and

$$\psi(x)x^* \pm x^*\phi(x) \in P \text{ for every } x \in A. \tag{2.23}$$

By linearizing (2.23) and using it, we have

$$\psi(x)y^* + \psi(y)x^* \pm x^*\phi(y) \pm y^*\phi(x) \in P \text{ for every } x, y \in A. \tag{2.24}$$

Replacing $x$ by $xh$ in (2.24) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)xy \pm \mu(h)yx \in P. \tag{2.25}$$

Writing $k$ instead of $y$ in (2.25), for every $k \in S(A) \cap Z(A) \backslash P$, we obtain $(\lambda(h) \pm \mu(h))x \in P$, and since $P \neq A$, we get $\lambda(h) \pm \mu(h) \in P$. Right multiplying the last relation by $yx$, we see that

$$\lambda(h)yx \pm \mu(h)yx \in P \text{ for every } x, y \in A. \tag{2.26}$$

From (2.25) and (2.26), we find that $\lambda(h)[x, y] \in P$ and so $\lambda(h) \in P$ or $[x, y] \in P$. If $[x, y] \in P$, then, by Lemma 1.2, we get that $A/P$ is commutative. If

$$\lambda(h) \in P \text{ for every } 0 \neq h \in H(A) \cap Z(A), \tag{2.27}$$

then

$$\lambda(k) \in P \text{ for every } k \in S(A) \cap Z(A) \backslash P. \tag{2.28}$$

By using (2.27) in (2.25), we have $\mu(h)yx \in P$ and $\mu(h) \in P$. Hence,

$$\mu(k) \in P \text{ for every } k \in S(A) \cap Z(A) \backslash P. \tag{2.29}$$

Substituting $xk$ for $x$ in (2.24) and using (2.28) and (2.29), we get

$$\psi(x)y^* - \psi(y)x^* \mp x^*\phi(y) \pm y^*\phi(x) \in P. \tag{2.30}$$

From (2.24) and (2.30), we find that $\psi(x)y \pm y\phi(x) \in P$. Putting $y = k$ in the last relation, for every $k \in S(A) \cap Z(A) \backslash P$, we obtain

$$\psi(x) \pm \phi(x) \in P. \tag{2.31}$$

Left multiplying (2.31) by $x^*$, we conclude that

$$x^*\psi(x) \pm x^*\phi(x) \in P. \tag{2.32}$$

Comparing (2.23) and (2.32), we have that $\left[\psi(x), x^*\right] \in P$ and, by Theorem 2.1, $\lambda(A) \subseteq P$ or $A/P$ is commutative. Right multiplying (2.31) by $x^*$, we see that $\psi(x)x^* \pm \phi(x)x^* \in P$ and, from the last relation and (2.23), we get $\left[\phi(x), x^*\right] \in P$ and, by Theorem 2.1, $\mu(A) \subseteq P$ or $A/P$ is commutative.

***Corollary* 2.7.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(x)x^* \pm x^*\phi(x) = 0$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.8.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi,\lambda)$ and $(\phi,\mu)$ are generalized derivations of $A$ satisfy $\psi(x)x \pm x^*\phi(x^*) \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) $\lambda(A) \subseteq P$ *and* $\mu(A) \subseteq P$,

(iii) $A/P$ *is commutative.*

***Proof.*** Assume that $\mathrm{char}(A/P) \neq 2$ and

$$\psi(x)x \pm x^*\phi(x^*) \in P \ \text{ for every } \ x \in A. \tag{2.33}$$

By linearizing (2.33) and using it, we have

$$\psi(x)y + \psi(y)x \pm x^*\phi(y^*) \pm y^*\phi(x^*) \in P \ \text{ for every } \ x, y \in A. \tag{2.34}$$

Replacing $x$ by $xh$ in (2.34) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)xy \pm \mu(h)y^*x^* \in P. \tag{2.35}$$

Writing $k$ instead of $xk$ in (2.35), for every $k \in S(A) \cap Z(A) \backslash P$, we get

$$\lambda(h)xy \mp \mu(h)y^*x^* \in P. \tag{2.36}$$

Now, using the same arguments as used in the proof of Theorem 2.4 in (2.13) and (2.14), we get that $A/P$ is commutative or

$$\lambda(k) \in P \ \text{ and } \ \mu(k) \in P \ \text{ for every } \ k \in S(A) \cap Z(A) \backslash P. \tag{2.37}$$

Substituting $xk$ for $x$ in (2.34) and using (2.37), we find that

$$\psi(x)y + \psi(y)x \mp x^*\phi(y^*) \mp y^*\phi(x^*) \in P. \tag{2.38}$$

Comparing (2.34) and (2.38), we see that

$$\psi(x)y + \psi(y)x \in P. \tag{2.39}$$

Replacing $x$ by $xt$ in (2.39), where $t \in A$, we have

$$\psi(x)ty + x\lambda(t)y + \psi(y)xt \in P. \tag{2.40}$$

Writing $ty$ instead of $y$ in (2.39), we get

$$\psi(x)ty + \psi(t)yx + t\lambda(y)x \in P. \tag{2.41}$$

From (2.40) and (2.41), we obtain

$$x\lambda(t)y + \psi(y)xt - \psi(t)yx - t\lambda(y)x \in P. \tag{2.42}$$

Substituting $xt$ for $x$ in (2.42), we see that

$$xt\lambda(t)y + \psi(y)xt^2 - \psi(t)yxt - t\lambda(y)xt \in P. \tag{2.43}$$

Right multiplying (2.42) by $t$, we find that

$$x\lambda(t)yt + \psi(y)xt^2 - \psi(t)yxt - t\lambda(y)xt \in P. \tag{2.44}$$

Comparing (2.43) and (2.44), we conclude that $x(t\lambda(t)y - \lambda(t)yt) \in P$. Taking $x = y = k$ in the last relation, for every $k \in S(A) \cap Z(A) \backslash P$, we have that $[t, \lambda(t)] \in P$ and, by Lemma 1.1, $\lambda(A) \subseteq P$ or $A/P$ is commutative. Now, by using (2.39) in (2.38), we get

$$x\phi(y) + y\phi(x) \in P. \tag{2.45}$$

Replacing $y$ by $ty$ in (2.45), where $t \in A$, we obtain

$$x\phi(t)y + xt\mu(y) + ty\phi(x) \in P. \tag{2.46}$$

Left multiplying (2.45) by $t$, we see that

$$tx\phi(y) + ty\phi(x) \in P. \tag{2.47}$$

From (2.46) and (2.47), we find that

$$x\phi(t)y + xt\mu(y) - tx\phi(y) \in P. \tag{2.48}$$

Writing $rx$ instead of $x$ in (2.48), where $r \in A$, we get

$$rx\phi(t)y + rxt\mu(y) - trx\phi(y) \in P. \tag{2.49}$$

Left multiplying (2.48) by $r$, we have

$$rx\phi(t)y + rxt\mu(y) - rtx\phi(y) \in P. \tag{2.50}$$

Comparing (2.49) and (2.50), we get $[t, r]x\phi(y) \in P$ and so $[t, r] \in P$ or $\phi(y) \in P$. If $[t, r] \in P$, then, by Lemma 1.2, $A/P$ is commutative. In case $\phi(y) \in P$, by the relation (2.46), we obtain $xt\mu(y) \in P$ and so $\mu(y) \in P$. Hence, $\mu(A) \subseteq P$.

***Corollary* 2.8.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(x)x \pm x^*\phi(x^*) = 0$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.9.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(x)\phi(x^*) \pm [x, x^*] \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) *$A/P$ is commutative.*

***Proof.*** Assume that $\mathrm{char}(A/P) \neq 2$ and

$$\psi(x)\phi(x^*) \pm [x, x^*] \in P \ \text{ for every } \ x \in A. \tag{2.51}$$

By linearizing (2.51) and using it, we have

$$\psi(x)\phi(y^*) + \psi(y)\phi(x^*) \pm [x, y^*] \pm [y, x^*] \in P \ \text{ for every } \ x, y \in A. \tag{2.52}$$

Replacing $x$ by $xh$ in (2.52) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)x\phi(y^*) + \mu(h)\psi(y)x^* \in P. \tag{2.53}$$

Substitution $xk$ for $k$ in (2.53), for every $k \in S(A) \cap Z(A) \backslash P$,

$$\lambda(h)x\phi(y^*) - \mu(h)\psi(y)x^* \in P. \tag{2.54}$$

From (2.53) and (2.54), we obtain $\lambda(h)x\phi(y) \in P$ and $\mu(h)\psi(y)x \in P$. Now, in case $\phi(y) \in P$ or $\psi(y) \in P$ and, by (2.51), we get that $[x, x^*] \in P$ and so $A/P$ is commutative. Now, suppose that $\lambda(h) \in P$ and $\mu(h) \in P$ and so

$$\lambda(k) \in P \text{ and } \mu(k) \in P \text{ for every } k \in S(A) \cap Z(A) \backslash P. \tag{2.55}$$

Substituting $xk$ for $x$ in (2.52) and using (2.55), we find that

$$\psi(x)\phi(y^*) - \psi(y)\phi(x^*) \pm [x, y^*] \mp [y, x^*] \in P. \tag{2.56}$$

Comparing (2.52) and (2.56), we get

$$\psi(x)\phi(y) \pm [x, y] \in P. \tag{2.57}$$

Replacing $y$ by $yt$ in (2.57) and using it, where $t \in A$, we have

$$\psi(x)y\mu(t) \pm y[x, t] \in P. \tag{2.58}$$

Taking $t = x$ in (2.58), we get $\psi(x)y\mu(x) \in P$ and so $\psi(x) \in P$ or $\mu(x) \in P$. Therefore, we have $A = \{x \in A \mid \psi(x) \in P\} \cup \{x \in A \mid \mu(x) \in P\}$. By Brauer's trick, we get $A = \{x \in A \mid \psi(x) \in P\}$ or $A = \{x \in A \mid \mu(x) \in P\}$. In both cases, using any one of them in (2.58), we obtain $y[x, t] \in P$. Writing $k$ instead of $y$ in the last relation, for every $k \in S(A) \cap Z(A) \backslash P$, we see that $[x, t] \in P$ and, by Lemma 1.2, $A/P$ is commutative.

***Corollary* 2.9.** *Let $A$ be a prime ring and* $\mathrm{char}(A) \neq 2$ *with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(x)\phi(x^*) = \pm[x, x^*]$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.10.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(x)\phi(x^*) \pm (x \circ x^*) \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) *$A/P$ is commutative.*

***Proof.*** By using the same arguments as used in the proof of Theorem 2.9, we get that $A/P$ is commutative or $\psi(x)\phi(y) \pm (x \circ y) \in P$ for every $x, y \in A$. Replacing $y$ by $yt$ in the last relation and using it, where $t \in A$, we find that

$$\psi(x)y\mu(t) \mp y[x, t] \in P. \tag{2.59}$$

Now, using the same arguments as used in the proof of Theorem 2.9 in (2.58), we get that $A/P$ is commutative.

***Corollary* 2.10.** *Let $A$ be a prime ring and* $\mathrm{char}(A) \neq 2$ *with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(x)\phi(x^*) = \pm(x \circ x^*)$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.11.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(xx^*) \pm \psi(x)\phi(x^*) \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) $\lambda(A) \subseteq P$,

(iii) *$A/P$ is commutative.*

***Proof.*** Assume that $\mathrm{char}(A/P) \neq 2$ and

$$\psi(xx^*) \pm \psi(x)\phi(x^*) \in P \text{ for every } x \in A. \tag{2.60}$$

By linearizing (2.60) and using it, we have

$$\psi(xy^*) + \psi(yx^*) \pm \psi(x)\phi(y^*) \pm \psi(y)\phi(x^*) \in P \text{ for every } x, y \in A. \tag{2.61}$$

Replacing $x$ by $xh$ in (2.61) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)xy^* + \lambda(h)yx^* \pm \lambda(h)x\phi(y^*) \pm \mu(h)\psi(y)x^* \in P. \tag{2.62}$$

Writing $k$ instead of $xk$ in (2.62), for every $k \in S(A) \cap Z(A)\backslash P$, we obtain

$$\lambda(h)xy^* - \lambda(h)yx^* \pm \lambda(h)x\phi(y^*) \mp \mu(h)\psi(y)x^* \in P. \tag{2.63}$$

From (2.62) and (2.63), we obtain $\lambda(h)x(y \pm \phi(y)) \in P$ and so $\lambda(h) \in P$ or $y \pm \phi(y) \in P$.

Firstly, in case

$$y \pm \phi(y) \in P, \tag{2.64}$$

putting $y$ by $yx$ in (2.64) and using it, we have $y\mu(x) \in P$ and so $\mu(x) \in P$. Hence,

$$\mu(A) \subseteq P. \tag{2.65}$$

By using (2.65) in (2.63), we get

$$\lambda(h)xy^* - \lambda(h)yx^* \pm \lambda(h)x\phi(y^*) \in P. \tag{2.66}$$

Taking $y$ by $yk$ in (2.66) and using it and (2.65), for every $k \in S(A) \cap Z(A)\backslash P$, we see that

$$-\lambda(h)xy^* - \lambda(h)yx^* \mp \lambda(h)x\phi(y^*) \in P. \tag{2.67}$$

From (2.66) and (2.67), we get $\lambda(h)yx \in P$ and so $\lambda(h) \in P$.

Secondly, suppose that

$$\lambda(h) \in P \text{ for every } 0 \neq h \in H(A) \cap Z(A). \tag{2.68}$$

Hence,

$$\lambda(k) \in P \text{ for every } k \in S(A) \cap Z(A)\backslash P. \tag{2.69}$$

By using (2.68) in (2.63), we find that $\mu(h)\psi(y)x \in P$ and so $\mu(h)\psi(y) \in P$. Hence, $\mu(h) \in P$ or $\psi(y) \in P$. If $\psi(y) \in P$, then $\psi(A) \subseteq P$. Hence, $\psi(xy) \in P$ and so $x\lambda(y) \in P$. Thus, $\lambda(y) \in P$, that is, $\lambda(A) \subseteq P$, as desired. Now, if $\mu(h) \in P$, then

$$\mu(k) \in P \text{ for every } k \in S(A) \cap Z(A) \backslash P. \tag{2.70}$$

Substituting $xk$ for $x$ in (2.61) and using (2.69) and (2.70), we find that

$$\psi(xy^*) - \psi(yx^*) \pm \psi(x)\phi(y^*) \mp \psi(y)\phi(x^*) \in P. \tag{2.71}$$

Comparing (2.61) and (2.71), we have that $\psi(xy) \pm \psi(x)\phi(y) \in P$ and, by [8, Theorem 1.9], we get that $\lambda(A) \subseteq P$ or $A/P$ is commutative.

***Corollary* 2.11.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(xx^*) \pm \psi(x)\phi(x^*) = 0$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.12.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(xx^*) \pm \psi(x^*)\phi(x) \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) $\lambda(A) \subseteq P$,

(iii) *$A/P$ is commutative.*

***Proof.*** Assume that $\mathrm{char}(A/P) \neq 2$ and

$$\psi(xx^*) \pm \psi(x^*)\phi(x) \in P \text{ for every } x \in A. \tag{2.72}$$

By linearizing (2.72) and using it, we have

$$\psi(xy^*) + \psi(yx^*) \pm \psi(x^*)\phi(y) \pm \psi(y^*)\phi(x) \in P \text{ for every } x, y \in A. \tag{2.73}$$

Replacing $x$ by $xh$ in (2.73) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)xy^* + \lambda(h)yx^* \pm \lambda(h)x^*\phi(y) \pm \mu(h)\psi(y^*)x \in P. \tag{2.74}$$

Writing $k$ instead of $xk$ in (2.74), for every $k \in S(A) \cap Z(A) \backslash P$, we have

$$\lambda(h)xy^* - \lambda(h)yx^* \mp \lambda(h)x^*\phi(y) \pm \mu(h)\psi(y^*)x \in P. \tag{2.75}$$

From (2.74) and (2.75), we obtain $\lambda(h)(yx \pm x\phi(y)) \in P$. Taking $x$ by $k$ in the last relation, for every $k \in S(A) \cap Z(A) \backslash P$, we get $\lambda(h)(y \pm \phi(y)) \in P$ and so $\lambda(h) \in P$ or $y \pm \phi(y) \in P$.

Firstly, in case

$$y \pm \phi(y) \in P, \tag{2.76}$$

putting $y$ by $yx$ in (2.76) and using it, we have $y\mu(x) \in P$ and so $\mu(x) \in P$. Hence,

$$\mu(A) \subseteq P. \tag{2.77}$$

By using (2.77) in (2.75), we get

$$\lambda(h)xy^* - \lambda(h)yx^* \pm \lambda(h)x^*\phi(y) \in P. \tag{2.78}$$

Taking $x$ by $xk$ in (2.78) and using it and (2.77), for every $k \in S(A) \cap Z(A) \backslash P$, we see that

$$\lambda(h)xy^* + \lambda(h)yx^* \mp \lambda(h)x^*\phi(y) \in P. \tag{2.79}$$

From (2.78) and (2.79), we conclude that $\lambda(h)xy \in P$ and so $\lambda(h) \in P$.

Secondly, in case

$$\lambda(h) \in P \ \text{ for every } \ 0 \neq h \in H(A) \cap Z(A), \tag{2.80}$$

we get

$$\lambda(k) \in P \ \text{ for every } \ k \in S(A) \cap Z(A)\backslash P. \tag{2.81}$$

By using (2.80) in (2.75), we find that $\mu(h)\psi(y)x \in P$ and so $\mu(h)\psi(y) \in P$. Hence, $\mu(h) \in P$ or $\psi(y) \in P$. If $\psi(y) \in P$, then $\psi(xy) \in P$ and so $x\lambda(y) \in P$. Hence, $\lambda(y) \in P$, that is, $\lambda(A) \subseteq P$. If $\mu(h) \in P$, then

$$\mu(k) \in P \ \text{ for every } \ k \in S(A) \cap Z(A)\backslash P. \tag{2.82}$$

Substituting $xk$ for $x$ in (2.73) and using (2.81) and (2.82), we find that

$$\psi(xy^*) - \psi(yx^*) \mp \psi(x^*)\phi(y) \pm \psi(y^*)\phi(x) \in P. \tag{2.83}$$

Comparing (2.73) and (2.83), we get $\psi(xy) \pm \psi(y)\phi(x) \in P$ and, by [8, Theorem 1.11], we have that $\lambda(A) \subseteq P$ or $A/P$ is commutative.

***Corollary* 2.12.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(xx^*) \pm \psi(x^*)\phi(x) = 0$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.13.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(xx^*) \pm \phi(x)\phi(x^*) \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) $\lambda(A) \subseteq P$ *and* $\mu(A) \subseteq P$.

***Proof.*** Assume that $\mathrm{char}(A/P) \neq 2$ and

$$\psi(xx^*) \pm \phi(x)\phi(x^*) \in P \ \text{ for every } \ x \in A. \tag{2.84}$$

By linearizing (2.84) and using it, we have

$$\psi(xy^*) + \psi(yx^*) \pm \phi(x)\phi(y^*) \pm \phi(y)\phi(x^*) \in P \ \text{ for every } \ x, y \in A. \tag{2.85}$$

Replacing $x$ by $xh$ in (2.85) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)xy^* + \lambda(h)yx^* \pm \mu(h)x\phi(y^*) \pm \mu(h)\phi(y)x^* \in P. \tag{2.86}$$

Writing $k$ instead of $xk$ in (2.86), for every $k \in S(A) \cap Z(A)\backslash P$, we obtain

$$\lambda(h)xy^* - \lambda(h)yx^* \pm \mu(h)x\phi(y^*) \mp \mu(h)\phi(y)x^* \in P. \tag{2.87}$$

From (2.86) and (2.87), we have $\lambda(h)xy \pm \mu(h)x\phi(y) \in P$. Taking $x$ by $k$ in the last relation, for every $k \in S(A) \cap Z(A)\backslash P$, we get

$$\lambda(h)y \pm \mu(h)\phi(y) \in P. \tag{2.88}$$

Putting $y$ by $yx$ in (2.88) and using it, we obtain $\mu(h)y\mu(x) \in P$ and so $\mu(h) \in P$ or $\mu(x) \in P$. In both cases, we have

$$\mu(h) \in P. \tag{2.89}$$

Using (2.89) in (2.88), we see that $\lambda(h)y \in P$ and so $\lambda(h) \in P$. Thus,

$$\lambda(k) \in P \text{ for every } k \in S(A) \cap Z(A)\backslash P. \tag{2.90}$$

From (2.89), we find that

$$\mu(k) \in P \text{ for every } k \in S(A) \cap Z(A)\backslash P. \tag{2.91}$$

Substituting $xk$ for $x$ in (2.85) and using (2.90) and (2.91), we conclude that

$$\psi(xy^*) - \psi(yx^*) \pm \phi(x)\phi(y^*) \mp \phi(y)\phi(x^*) \in P. \tag{2.92}$$

Comparing (2.85) and (2.92), we see that $\psi(xy) \pm \phi(x)\phi(y) \in P$ and, by [8, Theorem 1.13], we get $\lambda(A) \subseteq P$ and $\mu(A) \subseteq P$

***Corollary* 2.13.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(xx^*) \pm \phi(x)\phi(x^*) = 0$ for every $x \in A$, then $\lambda = 0$ and $\mu = 0$.*

**Theorem 2.14.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(xx^*) \pm \phi(x^*)\phi(x) \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) $\mu(A) \subseteq P$,

(iii) *$A/P$ is commutative.*

***Proof.*** Assume that $\mathrm{char}(A/P) \neq 2$ and

$$\psi(xx^*) \pm \phi(x^*)\phi(x) \in P \text{ for every } x \in A. \tag{2.93}$$

By linearizing (2.93) and using it, we have

$$\psi(xy^*) + \psi(yx^*) \pm \phi(x^*)\phi(y) \pm \phi(y^*)\phi(x) \in P \text{ for every } x, y \in A. \tag{2.94}$$

Replacing $x$ by $xh$ in (2.94) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)xy^* + \lambda(h)yx^* \pm \mu(h)x^*\phi(y) \pm \mu(h)\phi(y^*)x \in P. \tag{2.95}$$

Writing $k$ instead of $xk$ in (2.95), for every $k \in S(A) \cap Z(A)\backslash P$, we obtain

$$\lambda(h)xy^* - \lambda(h)yx^* \mp \mu(h)x^*\phi(y) \pm \mu(h)\phi(y^*)x \in P. \tag{2.96}$$

From (2.95) and (2.96), we have $\lambda(h)xy \pm \mu(h)\phi(y)x \in P$. Taking $x$ by $k$ in the last relation, for every $k \in S(A) \cap Z(A)\backslash P$, we get $\lambda(h)y \pm \mu(h)\phi(y) \in P$. Now, using the same arguments as used in the proof of Theorem 2.13 in (2.88), we obtain that $\psi(xy) \pm \phi(y)\phi(x) \in P$ and, by [8, Theorem 1.15], we get that $\mu(A) \subseteq P$ or $A/P$ is commutative.

***Corollary* 2.14.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ and $(\phi, \mu)$ are generalized derivations of $A$ satisfy $\psi(xx^*) \pm \phi(x^*)\phi(x) = 0$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.15.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ is a generalized derivation of $A$ satisfies $\psi(xx^*) \pm [x, x^*] \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) *$A/P$ is commutative.*

***Proof.*** Assume that $\mathrm{char}(A/P) \neq 2$ and

$$\psi(xx^*) \pm [x, x^*] \in P \text{ for every } x \in A. \tag{2.97}$$

By linearizing (2.97) and using it, we have

$$\psi(xy^*) + \psi(yx^*) \pm [x, y^*] \pm [y, x^*] \in P \text{ for every } x, y \in A. \tag{2.98}$$

Replacing $x$ by $xh$ in (2.98) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)(xy^* + yx^*) \in P. \tag{2.99}$$

Writing $k$ instead of $xk$ in (2.99), for every $k \in S(A) \cap Z(A) \backslash P$, we obtain

$$\lambda(h)(xy^* - yx^*) \in P. \tag{2.100}$$

From (2.99) and (2.100), we have $\lambda(h)xy \in P$ and so $\lambda(h) \in P$. Thus,

$$\lambda(k) \in P \text{ for every } k \in S(A) \cap Z(A) \backslash P. \tag{2.101}$$

Substituting $xk$ for $x$ in (2.98) and using (2.101), we conclude that

$$\psi(xy^*) - \psi(yx^*) \pm [x, y^*] \mp [y, x^*] \in P. \tag{2.102}$$

Comparing (2.98) and (2.102), we find that $\psi(xy) \pm [x, y] \in P$ for every $x, y \in A$. Replacing $y$ by $yr$ in the last relation and using it, where $r \in A$, we have

$$xy\lambda(r) \pm y[x, r] \in P. \tag{2.103}$$

Putting $r = x$ in (2.103), we get $xy\lambda(x) \in P$ and so $\lambda(x) \in P$. By using the last relation in (2.103), we see that $y[x, r] \in P$. So, we have that $[x, r] \in P$ and, by Lemma 1.2, $A/P$ is commutative.

***Corollary* 2.15.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ is a generalized derivation of $A$ satisfies $\psi(xx^*) = \pm[x, x^*]$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.16.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ is a generalized derivation of $A$ satisfies $\psi(xx^*) \pm (x \circ x^*) \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) *$A/P$ is commutative.*

***Proof.*** By using the same arguments as used in the proof of Theorem 2.15, we get $\psi(xy) \pm (x \circ y) \in P$. Replacing $y$ by $yr$ in the last relation and using it, we have

$$xy\lambda(r) \pm y[x,r] \in P \text{ for every } x,y,r \in A. \tag{2.104}$$

Now, using the same arguments as used in the proof of Theorem 2.15 in (2.103), we have that $A/P$ is commutative.

***Corollary* 2.16.** *Let $A$ be a prime ring and $\operatorname{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi,\lambda)$ is a generalized derivation of $A$ satisfies $\psi(xx^*) = \pm(x \circ x^*)$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.17.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi,\lambda)$ is a generalized derivation of $A$ satisfies $\psi(x)\psi(x^*) \pm [x,x^*] \in P$ for every $x \in A$, then one of the following holds*:

(i) $\operatorname{char}(A/P) = 2$,

(ii) *$A/P$ is commutative.*

***Proof.*** Assume that $\operatorname{char}(A/P) \neq 2$ and

$$\psi(x)\psi(x^*) \pm [x,x^*] \in P \text{ for every } x \in A. \tag{2.105}$$

By linearizing (2.105) and using it, we have

$$\psi(x)\psi(y^*) + \psi(y)\psi(x^*) \pm [x,y^*] \pm [y,x^*] \in P \text{ for every } x,y \in A. \tag{2.106}$$

Replacing $x$ by $xh$ in (2.106) and using it, where $0 \neq h \in H(A) \cap Z(A)$, we get

$$\lambda(h)(x\psi(y^*) + \psi(y)x^*) \in P. \tag{2.107}$$

Writing $k$ instead of $xk$ in (2.107), for every $k \in S(A) \cap Z(A) \backslash P$, we obtain

$$\lambda(h)(x\psi(y^*) - \psi(y)x^*) \in P. \tag{2.108}$$

From (2.107) and (2.108), we have $\lambda(h)x\psi(y) \in P$ and so $\lambda(h) \in P$ or $\psi(y) \in P$. If $\psi(y) \in P$, then $\psi(xy) \in P$. Thus, $x\lambda(y) \in P$ and hence $\lambda(y) \in P$, that is, $\lambda(h) \in P$, $0 \neq h \in H(A) \cap Z(A)$, and so

$$\lambda(k) \in P \text{ for every } k \in S(A) \cap Z(A) \backslash P. \tag{2.109}$$

Substituting $xk$ for $x$ in (2.106) and using (2.109), we conclude that

$$\psi(x)\psi(y^*) - \psi(y)\psi(x^*) \pm [x,y^*] \mp [y,x^*] \in P. \tag{2.110}$$

Comparing (2.106) and (2.110), we get $\psi(x)\psi(y) \pm [x,y] \in P$. Replacing $y$ by $yr$ in the last relation and using it, where $r \in A$, we have

$$\psi(x)y\lambda(r) + y[x,r] \in P. \tag{2.111}$$

So, we get $\psi(x)y\lambda(x) \in P$ and so $\lambda(x) \in P$ or $\psi(x) \in P$. The same as in the above, if $\psi(x) \in P$, then $\lambda(x) \in P$. Now, by using the last relation in (2.111), we get that $y[x,r] \in P$. Thus, we find that $[x,r] \in P$ and, by Lemma 1.2, $A/P$ is commutative.

***Corollary* 2.17.** *Let $A$ be a prime ring and $\operatorname{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi,\lambda)$ is a generalized derivation of $A$ satisfies $\psi(x)\psi(x^*) = \pm[x,x^*]$ for every $x \in A$, then $A$ is commutative.*

**Theorem 2.18.** *Let $A$ be a ring and $P$ be a prime ideal with $P$-second involution $*$. If $(\psi, \lambda)$ is a generalized derivation of $A$ satisfies $\psi(x)\psi(x^*) \pm (x \circ x^*) \in P$ for every $x \in A$, then one of the following holds*:

(i) $\mathrm{char}(A/P) = 2$,

(ii) *$A/P$ is commutative.*

***Proof.*** By using the same arguments as used in the proof of Theorem 2.17, we get $\psi(x)\psi(y) \pm (x \circ y) \in P$. Replacing $y$ by $yr$ in the last relation and using it, where $r \in A$, we have

$$\psi(x)y\lambda(r) - y[x, r] \in P \text{ for every } x, y, r \in A. \qquad (2.112)$$

Now, using the same as in the proof of Theorem 2.17 in (2.111), we have that $A/P$ is commutative.

***Corollary* 2.18.** *Let $A$ be a prime ring and $\mathrm{char}(A) \neq 2$ with involution $*$ of the second kind. If $(\psi, \lambda)$ is a generalized derivation of $A$ satisfies $\psi(x)\psi(x^*) = \pm(x \circ x^*)$ for every $x \in A$, then $A$ is commutative.*

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