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STOCHASTIC BERNOULLI EQUATION ON THE ALGEBRA OF GENERALIZED FUNCTIONS

СТОХАСТИЧНЕ РІВНЯННЯ БЕРНУЛЛІ З АЛГЕБРИ УЗАГАЛЬНЕНИХ ФУНКЦІЙ

Based on the topological dual space $\mathcal{F}_\theta^*(\mathcal{S}'_\mathbb{C})$ of the space of entire functions with θ-exponential growth of finite type, we introduce the generalized stochastic Bernoulli – Wick differential equation (or the stochastic Bernoulli equation on the algebra of generalized functions) by using the Wick product of elements in $\mathcal{F}_\theta^*(\mathcal{S}'_\mathbb{C})$. This equation is an infinite-dimensional stochastic distributions analog of the classical Bernoulli differential equation. This stochastic differential equation is solved and exemplified by several examples.

На основі топологічного простору $\mathcal{F}_\theta^*(\mathcal{S}'_\mathbb{C})$, спряженого до простору цілих функцій з θ-експоненціальним зростанням скінченного типу, введено узагальнене стохастичне диференціальне рівняння Бернуллі – Віка (або стохастичне рівняння Бернуллі на алгебрі узагальнених функцій) за допомогою добутку Віка елементів простору $\mathcal{F}_\theta^*(\mathcal{S}'_\mathbb{C})$. Таке рівняння є нескінченновимірним аналогом класичного диференціального рівняння Бернуллі для стохастичних розподілів. Ми розв'язуємо це стохастичне диференціальне рівняння та наводимо кілька прикладів.

1. Introduction. In 1695, Jacob Bernoulli proposed a new type of equation (called later Bernoulli differential equation) which was solved later after one year by Leibniz using a change variable which brings back to a linear differential equation. More precisely, a Bernoulli differential equation is an ordinary differential equation of the form

$$y' + P(x)y = Q(x)y^n, \tag{1}$$

where $P(x)$ and $Q(x)$ are continuous functions and n is any real number such that $n \neq 0$ and $n \neq 1$. It is clear that the Bernoulli equations are special case of nonlinear differential equations which are widely used to depict a large varieties of physical, chemical and biological phenomena. A famous special case of the Bernoulli equation is the logistic differential equation. Equation (1) is extended in infinite-dimensional distribution case [1], using a suitable product, called Wick product and denoted by $\diamond$. The Wick product was introduced by Hida and Ikeda [6] and it has been used extensively in the study of white noise integral equations (see [1, 7, 11, 12] and references therein).

On the other hand, the mathematical theory of stochastic differential equations was developed in the 1940s thanks to the important work of the mathematician Kiyosi Itô, who initiated the study of nonlinear stochastic differential equations. Many areas of applied mathematics require efficient computation in infinite dimensions. This is most apparent in quantum physics and in all scientific disciplines which describe natural phenomena by equations involving stochasticity. In recent years, there has been an increasing interest in the study of stochastic differential equations on the infinite dimension which have a great impact on current quantum field theory, hydrodynamics and statistical mechanics.

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In this paper, we introduce a new class of nonlinear stochastic differential equations in infinite dimensions which is flexible enough to be applicable in many fields, using the Wick product.

The paper is organized as follows. In Section 2, we briefly recall some basic notations in quantum white noise calculus. Namely, we give definitions and properties of the test functions space of entire functions with θ-exponential growth condition of minimal type and the associated generalized functions space. Section 3 is devoted to study the generalized stochastic Bernoulli – Wick differential equation. In Section 4, we study the important examples of the generalized stochastic Bernoulli – Wick differential equation.

2. Preliminaries. In this section we shall briefly recall some of the concepts, notations and known results on nuclear algebras of entire functions and Wick calculus which can be found also in [1, 2, 4, 5, 9 – 12].

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space which can be reconstructed in a standard way: $\mathcal{S}(\mathbb{R}) = \text{proj}\lim_{p\to\infty} \mathcal{S}_p$ (see [8]) and its topological dual space is given by $\mathcal{S}'(\mathbb{R}) = \text{ind}\lim_{p\to\infty} \mathcal{S}_{-p}$, where, for $p \geq 0$, $\mathcal{S}_p$ is the completion of $\mathcal{S}(\mathbb{R})$ with respect to some norm $|\cdot|_p$ and $\mathcal{S}_{-p}$ is the topological dual space of $\mathcal{S}_p$. We denote by $\mathcal{S}_\mathbb{C}$ and $\mathcal{S}_{\mathbb{C},-p}$ the complexification of $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}_{-p}$, respectively. Let $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ be a Young function. The projective system $\{\text{Exp}(\mathcal{S}_{\mathbb{C},-p}, \theta, m);\ p \in \mathbb{N},\ m > 0\}$ give the space

$$\mathcal{F}_\theta(\mathcal{S}'_\mathbb{C}) = \underset{p\to\infty, m\downarrow 0}{\text{proj}\lim} \text{Exp}(\mathcal{S}_{\mathbb{C},-p}, \theta, m)\ ,$$

where $\text{Exp}(\mathcal{S}_{\mathbb{C},-p}, \theta, m)$ the space of all entire functions on $\mathcal{S}_{\mathbb{C},-p}$ with θ-exponential growth of finite type m. The space $\mathcal{F}_\theta(\mathcal{S}'_\mathbb{C})$ is called the space of *test functions* on $\mathcal{S}'_\mathbb{C}$. Its topological dual space $\mathcal{F}^*_\theta(\mathcal{S}'_\mathbb{C})$, equipped with the strong topology, is called the space of *distributions* on $\mathcal{S}'_\mathbb{C}$ or nuclear algebra of generalized functions. It is easy to see that, for each $\xi \in \mathcal{S}_\mathbb{C}$, the exponential function

$$e_\xi(z) = e^{\langle z, \xi\rangle}, \quad z \in \mathcal{S}'_\mathbb{C},$$

is a test function in the space $\mathcal{F}_\theta(\mathcal{S}'_\mathbb{C})$ for any Young function θ. Thus, we can define the *Laplace transform* of a distribution $\Phi \in \mathcal{F}^*_\theta(\mathcal{S}'_\mathbb{C})$ by

$$\mathcal{L}\Phi(\xi) = \langle\langle \Phi, e_\xi \rangle\rangle, \quad \xi \in \mathcal{S}_\mathbb{C}.$$

The Laplace transform $\mathcal{L}$ realizes a topological isomorphism from $\mathcal{F}^*_\theta(\mathcal{S}'_\mathbb{C})$ onto $\mathcal{G}_{\theta^*}(\mathcal{S}_\mathbb{C})$, where

$$\mathcal{G}_{\theta^*}(\mathcal{S}_\mathbb{C}) = \underset{p\to\infty, m\downarrow\infty}{\text{ind}\lim} \text{Exp}(\mathcal{S}_{\mathbb{C},p}, \theta^*, m)$$

and θ^* is the polar function associated to θ.

For $\Phi_1, \Phi_2 \in \mathcal{F}^*_\theta(\mathcal{S}'_\mathbb{C})$, the Wick product of Φ_1 and Φ_2 denoted by $\Phi_1 \diamond \Phi_2$ is the unique element of $\mathcal{F}^*_\theta(\mathcal{S}'_\mathbb{C})$ satisfying (see [9])

$$\mathcal{L}(\Phi_1 \diamond \Phi_2)(\xi) = \mathcal{L}(\Phi_1)(\xi)\mathcal{L}(\Phi_2)(\xi), \quad \xi \in \mathcal{S}_\mathbb{C}.$$

Using this definition, one can easily show that the Wick product is associative and commutative. Moreover, for $\Phi \in \mathcal{F}^*_\theta(\mathcal{S}'_\mathbb{C})$, we have

$$\delta_0 \diamond \Phi = \Phi \diamond \delta_0 = \Phi,$$

where δ_0 denoting the Dirac distribution at 0 (which is also the unique distribution satisfying $\mathcal{L}(\delta_0)(\xi) = 1$).

Let $\mathcal{G}^0_{\theta^*}$ be the space of generalized functions g in $\mathcal{G}_{\theta^*}(\mathcal{S}_{\mathbb{C}})$ such that $g(\xi)$ has no zero, i.e.,

$$\mathcal{G}^0_{\theta^*} := \{g \in \mathcal{G}_{\theta^*}(\mathcal{S}_{\mathbb{C}}) \mid g(\xi) \neq 0 \quad \forall \xi \in \mathcal{S}_{\mathbb{C}}\}.$$

Let $\Phi \in \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}})$. If there exists $\psi \in \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}})$ such that $\psi \diamond \Phi = \delta_0$, then we say that Φ is Wick invertible and its Wick inverse is equal to ψ which will be denoted by $\Phi^{\diamond(-1)}$ (see [1]). Let $\mathcal{E}^0_\theta$ be the set of all Wick invertible elements on $\mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}})$. The Laplace transform realizes a topological isomorphism from the space $\mathcal{E}^0_\theta$ onto the space $\mathcal{G}^0_{\theta^*}$ (see [1]).

For $n \in \mathbb{N}$, by recurrence one can easily show that the Wick product $\Phi \diamond \Phi \diamond \cdots \diamond \Phi$ n-times (denoted by $\Phi^{\diamond n}$) is given via

$$\mathcal{L}(\Phi^{\diamond n}) = (\mathcal{L}(\Phi))^n.$$

By convention we take $\Phi^{\diamond 0} = \delta_0$.

Let $r \in \mathbb{R}^*_+ = (0, \infty)$ and $\Phi \in \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}})$. Then, using Lemma 3.1 in [1], we get that $(\mathcal{L}(\Phi))^r \in \mathcal{G}_{\theta^*}(\mathcal{S}_{\mathbb{C}})$. Therefore, by Definition 3.1 in [1], the element $\Phi^{\diamond r}$ is defined by

$$\mathcal{L}(\Phi^{\diamond r}) = (\mathcal{L}(\Phi))^r.$$

Now, for $r \in \mathbb{R}^*_+ = (0, \infty)$ and $\Phi \in \mathcal{E}^0_\theta$, the element $\Phi^{\diamond(-r)}$ (see Lemma 3.2 in [1]) is given by

$$\Phi^{\diamond(-r)} = \left(\Phi^{\diamond(-1)}\right)^{\diamond r}.$$

Recall that (see [2]), for $\Phi \in \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}})$, the Wick exponential is defined by

$$e^{\diamond\Phi} = \sum_{n=0}^{\infty} \frac{1}{n!}\Phi^{\diamond n} = \delta_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\Phi^{\diamond n}$$

which is an element of $\mathcal{F}^*_{(e^{\theta^*})^*}(\mathcal{S}'_{\mathbb{C}})$.

3. Generalized stochastic Bernoulli equation. From [2], a one parameter generalized stochastic process with values in $\mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}})$ (or generalized stochastic $\mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}})$-process) is a family of distributions

$$\{\Phi_t,\ t \in I\} \subset \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}}),$$

where I is an interval containing zero ($0 \in I$). The process Φ_t is said to be continuous if the map $t \to \Phi_t$ is continuous.

For a given continuous generalized stochastic process $\{\Phi_t\}_{t \in I}$, the stochastic generalized process $S_t = \displaystyle\int_0^t \Phi_s ds$ is defined as the unique element of $\mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}})$ satisfying

$$\mathcal{L}\left(\int_0^t \Phi_s ds\right)(\xi) = \int_0^t \mathcal{L}(\Phi_s)(\xi) ds \quad \forall \xi \in \mathcal{S}_{\mathbb{C}}.$$

Note that the process $S_t = \displaystyle\int_0^t \Phi_s ds$ is differentiable in $\mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}})$ and $\dfrac{\partial}{\partial t} S_t = \Phi_t$.

Proposition 1. *Let A_t and B_t be two continuous generalized stochastic $\mathcal{F}_\theta^*(\mathcal{S}'_\mathbb{C})$-processes. Then the solution of*

$$\frac{\partial}{\partial t}\Phi_t = A_t \diamond \Phi_t + B_t, \qquad \Phi_0 = F \in \mathcal{F}_\theta^*(\mathcal{S}'_\mathbb{C}) \tag{2}$$

is given by

$$\Phi_t = \left(F + \int_0^t B_s \diamond e^{\diamond(-\int_0^s A_u du)} ds \right) \diamond e^{\diamond \int_0^t A_s ds}. \tag{3}$$

Proof. Let U_t and V_t be two $\mathcal{F}_\theta^*(\mathcal{S}'_\mathbb{C})$-processes. Applying the Laplace transform, we get

$$\begin{aligned}\mathcal{L}\left(\frac{\partial}{\partial t}\{U_t \diamond V_t\}\right)(\xi) &= \left\langle\!\!\left\langle \frac{\partial}{\partial t}\{U_t \diamond V_t\}, e_\xi \right\rangle\!\!\right\rangle = \frac{\partial}{\partial t}\langle\langle U_t \diamond V_t, e_\xi\rangle\rangle \\ &= \frac{\partial}{\partial t}\mathcal{L}(U_t \diamond V_t)(\xi) = \frac{\partial}{\partial t}(\mathcal{L}(U_t)(\xi)\mathcal{L}(V_t)(\xi)) \\ &= \mathcal{L}(U_t)(\xi)\frac{\partial}{\partial t}\mathcal{L}(V_t)(\xi) + \mathcal{L}(V_t)(\xi)\frac{\partial}{\partial t}\mathcal{L}(U_t)(\xi) \\ &= \mathcal{L}(U_t)(\xi)\mathcal{L}\left(\frac{\partial}{\partial t}V_t\right)(\xi) + \mathcal{L}(V_t)(\xi)\mathcal{L}\left(\frac{\partial}{\partial t}U_t\right)(\xi) \\ &= \mathcal{L}\left(U_t \diamond \frac{\partial}{\partial t}V_t + V_t \diamond \frac{\partial}{\partial t}U_t\right)(\xi)\cdot\end{aligned}$$

Then we obtain

$$\frac{\partial}{\partial t}\{U_t \diamond V_t\} = U_t \diamond \frac{\partial}{\partial t}V_t + V_t \diamond \frac{\partial}{\partial t}U_t,$$

from which we deduce that $\frac{\partial}{\partial t}$ is a Wick derivation and from [1] we have

$$\frac{\partial}{\partial t}(e^{\diamond U_t}) = \frac{\partial}{\partial t}(U_t) \diamond e^{\diamond U_t}.$$

Now, denoting by Y_t and Z_t as follows:

$$Y_t = \int_0^t A_s ds,$$

$$Z_t = \int_0^t B_s \diamond e^{\diamond(-Y_s)} ds = \int_0^t B_s \diamond e^{\diamond(-\int_0^s A_u du)} ds,$$

and let Φ_t satisfies equation (3). Then we get

$$\frac{\partial}{\partial t}(\Phi_t) = \frac{\partial}{\partial t}(Z_t) \diamond e^{\diamond Y_t} + (Z_t + F) \diamond \frac{\partial}{\partial t}(e^{\diamond Y_t})$$

$$= B_t \diamond e^{\diamond(-Y_t)} \diamond e^{\diamond Y_t} + \frac{\partial}{\partial t}(Y_t) \diamond (Z_t + F) \diamond e^{\diamond Y_t}$$

$$= B_t + A_t \diamond \Phi_t,$$

which shows that Φ_t is the solution of (2). Conversely, suppose that Φ_t solution of (2). We note that

$$\frac{\partial}{\partial t}\big(\Phi_t \diamond e^{\diamond(-Y_t)}\big) = \frac{\partial}{\partial t}(\Phi_t) \diamond e^{\diamond(-Y_t)} - \frac{\partial}{\partial t}(Y_t) \diamond \Phi_t \diamond e^{\diamond(-Y_t)}$$

$$= (A_t \diamond \Phi_t + B_t) \diamond e^{\diamond(-Y_t)} + \Phi_t \diamond (-A_t) \diamond e^{\diamond(-Y_t)}$$

$$= B_t \diamond e^{\diamond(-Y_t)} = \frac{\partial}{\partial t}(Z_t).$$

Then we have

$$\Phi_t \diamond e^{\diamond(-Y_t)} = Z_t + F.$$

Therefore, we obtain

$$\Phi_t = (Z_t + F) \diamond e^{\diamond Y_t},$$

which is equivalent to

$$\Phi_t = \left(F + \int\limits_0^t B_s \diamond e^{\diamond(-\int_0^s A_u du)} ds \right) \diamond e^{\diamond \int_0^t A_s ds}.$$

The proposition is proved.

A one parameter generalized stochastic process with values in $\mathcal{E}_\theta^0$ is a family of distributions

$$\{\Phi_t,\ t \in I\} \subset \mathcal{E}_\theta^0.$$

Let $(\Phi_n)_{n\geq 0}$ be a sequence in $\mathcal{E}_\theta^0$. Then (Φ_n) converges in $\mathcal{E}_\theta^0$ if and only if the following conditions hold:

there exist $p \geq 0,\ m > 0$ and $c \geq 0$ such that, for every integer n,

$$|\mathcal{L}(\Phi_n)(\xi)| \leq c e^{\theta^*(m|\xi|_p)} \quad \forall \xi \in \mathcal{S}_\mathbb{C};$$

the sequence $\mathcal{L}(\Phi_n)(\xi)$ converges in $\mathbb{C}$ for each $\xi \in \mathcal{S}_\mathbb{C}$;

for every integer $n,\ \mathcal{L}(\Phi_n)(\xi)$ has no zero.

Let $\{\Phi_t\}_{t\in I}$ be a continuous $\mathcal{E}_\theta^0$-process. Since the map $s \mapsto \mathcal{L}(\Phi_s) \in \mathcal{G}_{\theta^*}^0$ is continuous, $\{\mathcal{L}(\Phi_s),$ $s \in [0,t]\}$, becomes a compact set, in particular it is bounded in $\mathcal{G}_{\theta^*}^0$, i.e., there exist $p \in \mathbb{N},\ m > 0$ and C_t such that, for every $\xi \in \mathcal{S}_{\mathbb{C},p}$, we obtain

$$|\mathcal{L}(\Phi_s)(\xi)| \leq C_t e^{\theta^*(m|\xi|_p)} \quad \forall s \in [0,t].$$

Which shows that the function $\xi \mapsto \displaystyle\int_0^t \mathcal{L}(\Phi_s)(\xi) ds$ belongs to $\mathcal{G}_{\theta^*}^0$. Then we define $S_t = \int_0^t \Phi_s ds$ as the unique element of $\mathcal{E}_\theta^0$ satisfying

$$\mathcal{L}\left(\int\limits_0^t \Phi_s ds\right)(\xi) = \int\limits_0^t \mathcal{L}(\Phi_s)(\xi) ds \quad \forall \xi \in \mathcal{S}_\mathbb{C}.$$

Moreover, the process $S_t = \int_0^t \Phi_s ds$ is differentiable in $\mathcal{E}_\theta^0$ and

$$\frac{\partial S_t}{\partial t} = \Phi_t \cdot$$

Note that, using the Laplace transform, one can verify that, for $\big(r \geq 1$ and $\{\Phi_t\}_{t \in I}$ a $\mathcal{F}_\theta^*(\mathcal{S}'_{\mathbb{C}})$-process$\big)$ or for $\big(r < 1$ and $\{\Phi_t\}_{t \in I}$ a $\mathcal{E}_\theta^0$-process$\big)$, we have

$$\frac{\partial}{\partial t}(\Phi_t^{\diamond r}) = r\frac{\partial}{\partial t}(\Phi_t) \diamond \Phi_t^{\diamond(r-1)}.$$

We are going to study the differential equation in the form

$$\begin{aligned} &A_t \diamond \frac{\partial}{\partial t}\Phi_t + B_t \diamond \Phi_t = G_t \diamond \Phi_t^{\diamond k}, \quad k \geq 2, \\ &\Phi_0 = f \in \mathcal{E}_\theta^0, \end{aligned} \tag{4}$$

where A_t is a continuous $\mathcal{E}_\theta^0$-process, B_t and G_t are two continuous $\mathcal{F}_\theta^*(\mathcal{S}'_{\mathbb{C}})$-processes and $\Phi_0 = f \in \mathcal{E}_\theta^0$. The stochastic Wick differential equation of the form (4) will be called the generalized stochastic Bernoulli – Wick differential equation or stochastic Bernoulli equation on the algebra of generalized functions which is the analog of the classical Bernoulli differential equation.

Theorem 1. *The equation* (4) *has a unique solution given by*

$$\Phi_t = \left\{ f^{\diamond(1-k)} + (1-k)\int\limits_0^t \Big(G_s \diamond A_s^{\diamond(-1)} \diamond e^{\diamond \int_0^s (1-k) B_u \diamond A_u^{\diamond(-1)} du} \Big) ds \right\}^{\diamond\left(\frac{1}{1-k}\right)} \diamond e^{\diamond\left(-\int_0^t B_s \diamond A_s^{\diamond(-1)} ds\right)}.$$

Proof. Let T_t given by $T_t = \Phi_t^{\diamond(1-k)}$. Therefore, we get

$$\frac{\partial}{\partial t}T_t = (1-k)\frac{\partial}{\partial t}(\Phi_t) \diamond \Phi_t^{\diamond(-k)}.$$

This implies that

$$\frac{\partial}{\partial t}(\Phi_t) = \frac{1}{1-k}\frac{\partial}{\partial t}(T_t) \diamond \Phi_t^{\diamond k}.$$

Therefore, by using equation (4), we obtain

$$\frac{1}{1-k}A_t \diamond \frac{\partial}{\partial t}(T_t) + B_t \diamond T_t = G_t,$$

which is equivalent to

$$A_t \diamond \frac{\partial}{\partial t}(T_t) = (k-1)B_t \diamond T_t + (1-k)G_t.$$

This gives

$$\frac{\partial}{\partial t}(T_t) = (k-1)B_t \diamond A_t^{\diamond(-1)} \diamond T_t + (1-k)A_t^{\diamond(-1)} \diamond G_t.$$

Then, by using Proposition 1, we have

$$T_t = (Z_t + T_0) \diamond e^{\diamond Y_t} = (Z_t + \Phi_0^{\diamond(1-k)}) \diamond e^{\diamond Y_t} = (Z_t + f^{\diamond(1-k)}) \diamond e^{\diamond Y_t},$$

where Y_t and Z_t are given by

$$Y_t = (k-1) \int_0^t B_s \diamond A_s^{\diamond(-1)} ds$$

and

$$Z_t = (1-k) \int_0^t A_s^{\diamond(-1)} \diamond G_s \diamond e^{\diamond(-Y_s)} ds$$

$$= (1-k) \int_0^t A_s^{\diamond(-1)} \diamond G_s \diamond e^{\diamond(-(k-1)\int_0^s B_u \diamond A_u^{\diamond(-1)} du)} ds.$$

Then we get

$$\Phi_t^{\diamond(1-k)} = (Z_t + F) \diamond e^{\diamond Y_t},$$

where $F = f^{\diamond(1-k)}$, which gives

$$\Phi_t = \left((Z_t + F) \diamond e^{\diamond Y_t}\right)^{\diamond(\frac{1}{1-k})} = (Z_t + F)^{\diamond(\frac{1}{1-k})} \diamond e^{\diamond(\frac{1}{1-k} Y_t)}.$$

Hence, we obtain

$$\Phi_t = \left\{ F + (1-k) \int_0^t \left(G_s \diamond A_s^{\diamond(-1)} \diamond e^{\diamond \int_0^s (1-k) B_u \diamond A_u^{\diamond(-1)} du} \right) ds \right\}^{\diamond(\frac{1}{1-k})} \diamond e^{\diamond\left(-\int_0^t B_s \diamond A_s^{\diamond(-1)} ds\right)}.$$

The theorem is proved.

4. Examples of the generalized stochastic Bernoulli equation. In this section, we study the important examples of the generalized stochastic Bernoulli equation.

***Example* 1.** The generalized stochastic Bernoulli – Wick differential equation

$$\frac{\partial}{\partial t}\Phi_t + \Phi_t = \Phi_t^{\diamond k}, \quad k \geq 2,$$

$$\Phi_0 = f \in \mathcal{E}_\theta^0$$

admits (by Theorem 1 for $A_t = \delta_0,\ B_t = \delta_0$ and $G_t = \delta_0$) a unique solution given by

$$\Phi_t = e^{-t}\left\{ f^{\diamond(1-k)} + (e^{t(1-k)} - 1)\delta_0 \right\}^{\diamond(\frac{1}{1-k})}.$$

***Example* 2** (Generalized stochastic logistic Wick differential equation)**.** We introduce the generalized stochastic logistic Wick differential equation as follows:

$$\frac{\partial}{\partial t}\Phi_t = \Phi_t \diamond (\delta_0 - \Phi_t),$$
$$\Phi_0 = f \in \mathcal{E}^0_\theta. \tag{5}$$

This equation is equivalent to

$$\frac{\partial}{\partial t}\Phi_t - \Phi_t = -\Phi_t^{\diamond 2},$$
$$\Phi_0 = f \in \mathcal{E}^0_\theta,$$

which is a particular case of the generalized stochastic Bernoulli – Wick differential equation (4) by taking $k = 2$ and $A_t = \delta_0,\ B_t = -\delta_0,\ G_t = -\delta_0$. Then, by Theorem 1, the solution of the generalized stochastic logistic Wick differential equation (5) is given by

$$\Phi_t = e^t\Big\{f^{\diamond(-1)} + (e^t - 1)\delta_0\Big\}^{\diamond(-1)}.$$

***Example* 3.** Let $A_t = a\delta_0,\ B_t = b\delta_0$ and $G_t = ce^{-t\frac{b}{a}}\delta_0$ such that $ab < 0$ and $c \neq 0$. Then the solution of (4) with $f = \left(\dfrac{c(k-1)}{bk}\right)^{\frac{1}{1-k}}\delta_0$ is given by

$$\Phi_t = \mathbb{E}\left[e^{-t}\left(\frac{(k-1)c}{kb}\right)^{\frac{1}{1-k}}\left(\frac{a(1-k)}{b}\right)^{N_{\frac{b}{a(1-k)}t}}\right]\delta_0$$

or, equivalently,

$$\mathcal{L}(\Phi_t) = \mathbb{E}\left(e^{-t}\left(\frac{(k-1)c}{kb}\right)^{\frac{1}{1-k}}\left(\frac{a(1-k)}{b}\right)^{N_{\frac{b}{a(1-k)}t}}\right),$$

where $N_{\frac{bt}{a(1-k)}}$ is a random variable with Poisson distribution with intensity $\dfrac{bt}{a(1-k)}$ and $\mathbb{E}$ is the expectation.

Proof. Using Theorem 1, the solution of (4) is given by

$$\Phi_t = (Z_t + F)^{\diamond\frac{1}{1-k}} \diamond e^{\diamond(\frac{1}{1-k}Y_t)},$$

where Y_t and Z_t are given as follows:

$$Y_t = (k-1)\int_0^t \left(\frac{b}{a}\right) d_s\delta_0 = \frac{b}{a}(k-1)t\delta_0 \tag{6}$$

and

$$Z_t = (1-k)\int_0^t \left(\frac{c}{a}\right) e^{-s\frac{b}{a}}\delta_0 \diamond e^{\diamond(\frac{b}{a}(1-k)s\delta_0)}ds$$

$$= (1-k)\frac{c}{a}\int_0^t e^{-s\frac{b}{a}} e^{\diamond(\frac{b}{a}(1-k)s)\delta_0}ds = (1-k)\frac{c}{a}\int_0^t e^{\frac{-kb}{a}s}d_s\delta_0$$

$$= (1-k)\frac{c}{a}\left(\frac{-a}{bk}\right)\left(e^{\frac{-kb}{a}t}-1\right)\delta_0 = \frac{(k-1)c}{kb}\left(e^{\frac{-kb}{a}t}-1\right)\delta_0. \tag{7}$$

Then we get

$$Z_t + F = \frac{(k-1)c}{kb}\left(e^{\frac{-kb}{a}t}-1\right)\delta_0 + f^{\diamond(1-k)}$$
$$= \frac{(k-1)c}{kb}\left(e^{\frac{-kb}{a}t}-1\right)\delta_0 + \frac{c(k-1)}{bk}\delta_0 = \frac{(k-1)c}{kb}e^{\frac{-kb}{a}t}\delta_0.$$

Therefore, we obtain

$$\Phi_t = \left(\frac{(k-1)c}{kb}\right)^{\frac{1}{1-k}} e^{\frac{-kb}{(1-k)a}t}\delta_0 \diamond e^{\diamond(-\frac{b}{a}t\delta_0)}$$
$$= \left(\frac{(k-1)c}{kb}\right)^{\frac{1}{1-k}} e^{\frac{-kb}{(1-k)a}t} e^{-\frac{b}{a}t}\delta_0 = \left(\frac{(k-1)c}{kb}\right)^{\frac{1}{1-k}} e^{-\frac{b}{a(1-k)}t}\delta_0.$$

But, we know that (see [3])

$$e^{-\beta+\alpha} = \mathbb{E}\left[\left(\frac{\alpha}{\beta}\right)^{N_\beta}\right]$$

for $\alpha \in \mathbb{R}$ and $\beta > 0$. Then we have

$$\Phi_t = \mathbb{E}\left[e^{-t}\left(\frac{(k-1)c}{kb}\right)^{\frac{1}{1-k}}\left(\frac{a(1-k)}{b}\right)^{N_{\frac{b}{a(1-k)}t}}\right]\delta_0$$

or, equivalently,

$$\mathcal{L}(\Phi_t) = \mathbb{E}\left(e^{-t}\left(\frac{(k-1)c}{kb}\right)^{\frac{1}{1-k}}\left(\frac{a(1-k)}{b}\right)^{N_{\frac{b}{a(1-k)}t}}\right).$$

***Example* 4.** Let $A_t = a\delta_0$, $B_t = b\delta_0$ and $G_t = c\delta_0$ such that $ab > 0$ and $c \neq 0$. Then the solution of (4) with $f = 0$ is given by

$$\Phi_t = \left(\mathbb{E}\left[-\frac{c}{b}e^{-t(1+2\frac{b}{a}(1-k))}\left(\frac{a}{b(k-1)}\right)^{N_{\frac{b(k-1)}{a}t}}\chi_{\{\Lambda(\frac{b}{a}(k-1))<t\}}\right]\right)^{\frac{1}{1-k}}\delta_0$$

or, equivalently,

$$\mathcal{L}(\Phi_t) = \left(\mathbb{E}\left[-\frac{c}{b}e^{-t(1+2\frac{b}{a}(1-k))}\left(\frac{a}{b(k-1)}\right)^{N_{\frac{b(k-1)}{a}t}}\chi_{\{\Lambda(\frac{b}{a}(k-1))<t\}}\right]\right)^{\frac{1}{1-k}},$$

where $N_{\frac{b(k-1)t}{a}}$ is a random variable with Poisson distribution with intensity $\dfrac{b(k-1)t}{a}$, $\Lambda\left(\dfrac{b}{a}(k-1)\right)$ is a random variable which has an exponential distribution of parameter $\dfrac{b(k-1)}{a}$ and independent of $N_{\frac{b(k-1)}{a}t}$ and $\mathbb{E}$ is the expectation.

Proof. The solution of (4) is given by

$$\Phi_t = (Z_t + F)^{\diamond(\frac{1}{1-k})} \diamond e^{\diamond(\frac{1}{1-k}Y_t)} = Z_t^{\diamond(\frac{1}{1-k})} \diamond e^{\diamond(\frac{1}{1-k}Y_t)},$$

where Y_t and Z_t are given (similarly to (6) and (7)) as follows:

$$Y_t = \frac{b}{a}(k-1)t\delta_0$$

and

$$Z_t = (1-k)\frac{c}{a}\int_0^t e^{\frac{b}{a}(1-k)s} ds\delta_0.$$

Similarly to the proof of Example 3, we get

$$e^{\diamond Y_t} = e^{\frac{b}{a}(k-1)t}\delta_0 = e^{-2\frac{b}{a}(1-k)t}e^{-\frac{b}{a}(k-1)t}\delta_0$$

$$= \mathbb{E}\left(e^{-t(1+2\frac{b}{a}(1-k))}\left(\frac{a}{b(k-1)}\right)^{N_{\frac{b(k-1)}{a}t}}\right)\delta_0.$$

Now, let $\Lambda\left(\frac{b}{a}(k-1)\right)$ be a random variable which has an exponential distribution of parameter $\frac{b}{a}(k-1)$ and independent of $N_{\frac{b(k-1)}{a}t}$. Since we have

$$\mathbb{E}\Big(\chi_{\{\Lambda(\frac{b}{a}(k-1))<t\}}\Big) = \int_0^t \frac{b}{a}(k-1)e^{-\frac{b}{a}(k-1)s}ds,$$

then we obtain

$$Z_t = (1-k)\frac{c}{a}\frac{a}{b(k-1)}\int_0^t \frac{a}{b}(k-1)e^{-\frac{b}{a}(k-1)s}ds\delta_0$$

$$= -\frac{c}{b}\mathbb{E}\Big(\chi_{\{\Lambda(\frac{b}{a}(k-1))<t\}}\Big)\delta_0 = \mathbb{E}\Big(-\frac{c}{b}\chi_{\{\Lambda(\frac{b}{a}(k-1))<t\}}\Big)\delta_0.$$

But, we have $\Phi_t = Z_t^{\diamond(\frac{1}{1-k})} \diamond e^{\diamond(\frac{1}{1-k}Y_t)} = (Z_t \diamond e^{\diamond Y_t})^{\diamond\frac{1}{1-k}}$. Then we get

$$\Phi_t = \left[\mathbb{E}\Big(-\frac{c}{b}\chi_{\{\Lambda(\frac{b}{a}(k-1))<t\}}\Big)\mathbb{E}\left(e^{-t(1+2\frac{b}{a}(1-k))}\left(\frac{a}{b(k-1)}\right)^{N_{\frac{b(k-1)}{a}t}}\right)\right]^{\frac{1}{1-k}}\delta_0$$

$$= \left(\mathbb{E}\left[-\frac{c}{b}e^{-t(1+2\frac{b}{a}(1-k))}\left(\frac{a}{b(k-1)}\right)^{N_{\frac{b(k-1)}{a}t}}\chi_{\{\Lambda(\frac{b}{a}(k-1))<t\}}\right]\right)^{\frac{1}{1-k}}\delta_0$$

or, equivalently,

$$\mathcal{L}(\Phi_t) = \left(\mathbb{E}\left[-\frac{c}{b}e^{-t(1+2\frac{b}{a}(1-k))}\left(\frac{a}{b(k-1)}\right)^{N_{\frac{b(k-1)}{a}t}}\chi_{\{\Lambda(\frac{b}{a}(k-1))<t\}}\right]\right)^{\frac{1}{1-k}}.$$

The author states that there is no conflict of interest.

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