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# A CLASS OF FRACTIONAL INTEGRAL OPERATORS INVOLVING A CERTAIN GENERAL MULTIINDEX MITTAG-LEFFLER FUNCTION

# КЛАС ДРОБОВИХ ІНТЕГРАЛЬНИХ ОПЕРАТОРІВ, ЩО ВКЛЮЧАЮТЬ ДЕЯКУ УЗАГАЛЬНЕНУ БАГАТОІНДЕКСНУ ФУНКЦІЮ МІТТАГ-ЛЕФФЛЕРА

This paper is essentially motivated by the demonstrated potential for applications of the presented results in numerous widespread research areas, such as the mathematical, physical, engineering, and statistical sciences. The main object here is to introduce and investigate a class of fractional integral operators involving a certain general family of multiindex Mittag-Leffler functions in their kernel. Among other results obtained in the paper, we establish several interesting expressions for the composition of well-known fractional integral and fractional derivative operators, such as (e.g.) the Riemann – Liouville fractional integral and fractional derivative operators, the Hilfer fractional derivative operator, and the above-mentioned fractional integral operator involving the general family of multiindex Mittag-Leffler functions in its kernel. Our main result is a generalization of the results obtained in earlier investigations on this subject. We also present some potentially useful integral representations for the product of two members of the general family of multiindex Mittag-Leffler functions in terms of the well-known Fox – Wright hypergeometric function ${}_p\Psi_q$ with $p$ numerator and $q$ denominator parameters.

Ця стаття в основному мотивована продемонстрованим потенціалом для застосувань отриманих у ній результатів у багатьох популярних галузях досліджень, таких як математичні, фізичні, інженерні та статистичні науки. Основна мета полягає в тому, щоб ввести та дослідити клас дробових інтегральних операторів, що включають деяку загальну сім'ю багатоіндексних функцій Міттаг-Леффлера в своєму ядрі. Серед інших результатів, які отримані у цій статті, встановлено кілька цікавих виразів для композицій відомих операторів дробових інтегралів та дробових похідних, таких як (наприклад) оператори Рімана – Ліувілля дробових інтеграла та похідної, оператор Гілфера дробової похідної та оператор дробового інтеграла, який, як зазначено вище, включає загальну сім'ю багатоіндексних функцій Міттаг-Леффлера у своєму ядрі. Показано, що основні висновки, наведені у статті, узагальнюють результати досліджень, які були отримані раніше. Також наведено деякі потенційно корисні інтегральні зображення для добутку двох членів загальної сім'ї багатоіндексних функцій Міттаг-Леффлера в термінах відомої гіпергеометричної функції Фокса – Райта ${}_p\Psi_q$ з $p$ параметрами чисельника та $q$ параметрами знаменника.

**1. Introduction and motivation.** Over one century ago, in the year 1903, it was the Swedish mathematician, Magnus Gustaf (Gösta) Mittag-Leffler (1846 – 1927) [10] (see also [11]) who introduced and investigated what is popularly known today as the Mittag-Leffler function $E_\alpha(z)$ defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C}, \quad \Re(\alpha) > 0. \tag{1.1}$$

In a couple of sequels to [10, 11], Wiman (see [36, 37]) presented a generalization $E_{\alpha,\beta}(z)$ of $E_\alpha(z)$ given by

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$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad \Re(\alpha) > 0, \tag{1.2}$$

so that, obviously,

$$E_{\alpha,1}(z) = E_\alpha(z).$$

Since the publications of the aforementioned classical works by Mittag-Leffler (see [10, 11]) and Wiman (see [36, 37]), the Mittag-Leffler function $E_\alpha(z)$ in (1.1) and its generalization $E_{\alpha,\beta}(z)$ in (1.2) have been further generalized and extended in a number of different ways and in many different contexts. Together with its extensions and generalizations, Mittag-Leffler type functions have been applied in various research areas such as those in mathematical, physical, engineering and statistical sciences. The Mittag-Leffler type functions and their related distributions were studied in [12]. Furthermore, connections among various general families of the Mittag-Leffler type functions, pathway models, Tsallis statistics, superstatistics and power law and the corresponding entropy measures were established in [8].

Some of the *further* extensions of the Mittag-Leffler function $E_\alpha(z)$ and the general Mittag-Leffler function $E_{\alpha,\beta}(z)$ are worthy of note here. Indeed, apart from the extensions considered by Srivastava [20], Prabhakar [13] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ defined by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \quad \Re(\alpha) > 0, \tag{1.3}$$

which reduces immediately to $E_{\alpha,\beta}(z)$ when we set $\gamma = 1$. For a useful interpretation of the additional parameter $\gamma$ in the definition (1.3) and for the connection of the function $E^{\gamma}_{\alpha,\beta}(z)$ with the above-defined Mittag-Leffler function $E_{\alpha,\beta}(z)$ itself, one may see the recent work by Fernandez et al. [1].

An interesting further extension of the general Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ in (1.3) was given by Srivastava and Tomovski [33] as follows:

$$E^{\gamma,\kappa}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \quad z, \beta, \gamma, \kappa \in \mathbb{C}, \quad \Re(\kappa) > 0, \quad \Re(\alpha) > \max\{0, \Re(\kappa) - 1\}, \tag{1.4}$$

which, in the special case when

$$\kappa = q, \quad q \in (0,1) \cup \mathbb{N}, \quad \text{and} \quad \min\{\Re(\beta), \Re(\gamma)\} > 0,$$

was considered by Shukla and Prajapati [19].

We now recall the following two generalizations of the Mittag-Leffler type functions (see [5]):

$$E^{\eta,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn}}, \quad p, q \in \mathbb{R}^+, \quad \alpha, \beta, \eta, \delta \in \mathbb{C}, \quad \Re(\alpha) > 0,$$

and

$$E^{\mu,\rho,\eta,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\eta)_{qn}}{(\nu)_{\sigma n}(\delta)_{pn}} \frac{z^n}{\Gamma(\alpha n + \beta)},$$

$$p, q \in \mathbb{R}^+, \quad q \leq \Re(\alpha) + p, \quad \alpha, \beta, \eta, \delta, \mu, \nu, \rho, \sigma \in \mathbb{C}, \quad \{\Re(\alpha), \Re(\rho), \Re(\sigma)\} > 0.$$

A general family of the multiindex Mittag-Leffler functions $E_{\gamma,\kappa,\delta,\epsilon}\Big[(\alpha_j, \beta_j)_{j=1}^m; z\Big]$, which is considered in this paper, was defined and studied by Saxena and Nishimoto (see, for details, [16] and [17] for the case when $\epsilon = 0$) in the following manner (see also [18]):

$$E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[z] = E_{\gamma,\kappa,\delta,\epsilon}\big[(\alpha_j, \beta_j)_{j=1}^m; z\big] := \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}(\delta)_{\epsilon n}}{\prod_{j=1}^m \Gamma(\alpha_j n + \beta_j)} \frac{z^n}{n!}, \tag{1.5}$$

$$\alpha_j, \beta_j, \gamma, \kappa, \delta, \epsilon \in \mathbb{C}, \quad \Re(\alpha_j) > 0, \quad j = 1, \ldots, m, \quad \Re\left(\sum_{j=1}^m \alpha_j\right) > \Re(\kappa + \epsilon) - 1.$$

Here (*and throughout our presentation*) $(\lambda)_\nu$ denotes the *general* Pochhammer symbol or the *shifted* factorial, since

$$(1)_n = n!, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{N} := \{1, 2, 3, \ldots\},$$

defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the familiar Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1, & \nu = 0, \quad \lambda \in \mathbb{C} \setminus \{0\}, \\ \lambda(\lambda + 1) \ldots (\lambda + n - 1), & \nu = n \in \mathbb{N}, \quad \lambda \in \mathbb{C}, \end{cases} \tag{1.6}$$

where we assume *conventionally* that $(0)_0 := 1$ and *tacitly* that the $\Gamma$-quotient in (1.6) exists (see, for details, [29, p. 16; 30, p. 22]; see also [24, p. 2 and 4–6] and [25, p. 2]).

In many of the recent investigations, the interest in the above-mentioned families of Mittag-Leffler type functions has grown considerably due chiefly to their potential for applications in some reaction-diffusion problems and their various generalizations appearing in the solutions of fractional-order differential and integral equations (see, for example, [22], see also [1, 26]).

Finally, we recall the familiar Fox–Wright hypergeometric function ${}_p\Psi_q(z)$ (with $p$ numerator and $q$ denominator parameters), which is given by the following series (see [2, 38, 39]; see also [7, p. 67, Eq. (1.12.68)] and [29, p. 21, Eq. 1.2(38)]):

$${}_p\Psi_q\left[\begin{array}{ll} (\alpha_1, A_1), \ldots, (\alpha_p, A_p); & \\ & z \\ (\beta_1, B_1), \ldots, (\beta_q, B_q); & \end{array}\right] := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^q \Gamma(\beta_j + B_j n)} \frac{z^n}{n!}$$

$$= \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_{A_j n}}{\prod_{j=1}^{q}(\beta_j)_{B_j n}} \frac{z^n}{n!}, \tag{1.7}$$

in which we have made use of the *general* Pochhammer symbol $(\lambda)_\nu,\ \lambda,\nu \in \mathbb{C}$, defined by (1.6), the parameters

$$\alpha_j, \beta_k \in \mathbb{C}, \quad j = 1, \ldots, p, \quad k = 1, \ldots, q,$$

and the coefficients

$$A_1, \ldots, A_p \in \mathbb{R}^+ \quad \text{and} \quad B_1, \ldots, B_p \in \mathbb{R}^+$$

such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0,$$

with the equality for appropriately constrained values of the argument $z$.

***Remark* 1.** Upon comparing the definition (1.5) of the general multiindex Mittag-Leffler function $E_{\gamma,\kappa,\delta,\epsilon}\Big[(\alpha_j,\beta_j)_{j=1}^m; z\Big]$ with the definition in (1.7), it is easily seen that

$$E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[z] = E_{\gamma,\kappa,\delta,\epsilon}\big[(\alpha_j,\beta_j)_{j=1}^m; z\big] = \frac{1}{\Gamma(\gamma)\Gamma(\delta)}\, {}_2\Psi_m\left[\begin{array}{r} (\gamma,\kappa),(\delta,\epsilon); \\ (\beta_1,\alpha_1),\ldots,(\beta_m,\alpha_m); \end{array} z\right], \tag{1.8}$$

which clearly exhibits the fact that, not only the general multiindex Mittag-Leffler function $E_{\gamma,\kappa,\delta,\epsilon}\Big[(\alpha_j,\beta_j)_{j=1}^m; z\Big]$ defined by (1.5), but indeed also all of the above-mentioned Mittag-Leffler type functions *and many more*, are contained, as special cases, in the the extensively- and widely-investigated Fox – Wright hypergeometric function ${}_p\Psi_q(z)$ defined by (1.7) (see also the work of Srivastava and Tomovski [33, p. 199] for similar remarks about the much more general nature of the Fox – Wright function ${}_p\Psi_q(z)$ than any of these Mittag-Leffler type functions).

Our main object in this paper, we make use of the Riemann – Liouville fractional integral operator $I^p_{a+}$ and the Riemann – Liouville fractional derivative operator $D^p_{a+}$, which are defined by (see, for details, [7, 9, 15])

$$(I^\mu_{a+} f)(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}}\, dt, \quad \Re(\mu) > 0, \tag{1.9}$$

and

$$\big(D^\mu_{a+} f\big)(x) = \left(\frac{d}{dx}\right)^n \Big(I^{n-\mu}_{a+} f\Big)(x), \quad \Re(\mu) > 0, \quad n = [\Re(\mu)] + 1, \tag{1.10}$$

where $[\xi]$ denotes the greatest integer in the real number $\xi$.

Recently, Hilfer [4] generalized the operator in (1.10) and defined a general fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $\mu$, $0<\mu<1$, and type $\nu$, $0\leq\nu\leq 1$, with respect to $x$ as follows:

$$\left(D_{a+}^{\mu,\nu}f\right)(x)=\left(I_{a+}^{\nu(1-\mu)}\frac{d}{dx}\left(I_{a+}^{(1-\nu)(1-\mu)}f\right)\right)(x). \tag{1.11}$$

The general fractional derivative operator (1.11) yields the classical Riemann–Liouville fractional derivative operator $D_{a+}^{\mu}$ when $\nu=0$. Moreover, when $\nu=1$, (1.11) reduces to the fractional derivative operator introduced essentially by Joseph Liouville (1809–1882) in the year 1832, which is often called now-a-days as the Liouville–Caputo fractional derivative operator (see [6, 34], see also [3, 27]).

**2. A family of integral operators associated with the general multiindex Mittag-Leffler type function.** In this section, we investigate the developments of several interesting properties of the fractional integral operator (2.3) below, which is associated with the general multiindex Mittag-Leffler function defined by (1.5).

Our first result in this section is the following theorem.

**Theorem 1.** *The general multiindex Mittag-Leffler function* $E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa,\delta,\epsilon}[z]$ *defined by* (1.5) *is an entire* (*integral*) *function of order* $\rho$ *and of type* $\sigma$ *in the complex* $z$*-plane*, *which are given by*

$$\rho=\frac{1}{\sum_{j=1}^{m}\{\Re(\alpha_j-\kappa-\epsilon)\}+1} \quad \textit{and} \quad \sigma=\frac{1}{\rho}\left(\frac{\{\Re(\kappa)\}^{\Re(\kappa)}\{\Re(\epsilon)\}^{\Re(\epsilon)}}{\prod_{j=1}^{m}\{\Re(\alpha_j)\}^{\Re(\alpha_j)}}\right)^{\rho}.$$

*Furthermore*, *the infinite series in the definition* (1.5) *is absolutely convergent also when*

$$\sum_{j=1}^{m}\Re(\alpha_j)=\Re(\kappa+\epsilon)-1>0 \quad \textit{and} \quad |z|<\frac{\prod_{j=1}^{m}\{\Re(\alpha_j)\}^{\Re(\alpha_j)}}{\{\Re(\kappa)\}^{\Re(\kappa)}\{\Re(\epsilon)\}^{\Re(\epsilon)}}. \tag{2.1}$$

***Proof.*** In our demonstration of Theorem 1, we apply the following asymptotic expansions for the Gamma function [33, p. 4, Eqs. (2.3) to (2.7)] (see also [7]):

$$|\Gamma(x+iy)|=\sqrt{2\pi}|x|^{x-\frac{1}{2}}\exp\left(-x-\frac{\pi[1-\operatorname{sgn}(x)y]}{2}\right)\left[1+O\left(\frac{1}{x}\right)\right], \quad x,y\in\mathbb{R}, \quad x\to\infty,$$

$$\log\Gamma(z+a)=\left(z+a-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log(2\pi)+O\left(\frac{1}{z}\right),$$

$$|z|\to\infty, \quad |\arg(z)|\leq\pi-\epsilon, \quad |\arg(z+a)|\leq\pi-\epsilon, \quad 0<\epsilon<\pi,$$

and

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=z^{a-b}\left[1+\frac{(a-b)(a+b-1)}{2z}+O\left(\frac{1}{z^2}\right)\right],$$

$$|z|\to\infty, \quad |\arg(z)|\leq\pi-\epsilon, \quad |\arg(z+a)|\leq\pi-\epsilon, \quad 0<\epsilon<\pi,$$

where $a$ and $b$ are bounded complex numbers.

First of all, we denote by $R$ denote the radius of convergence of the infinite series in (1.5), which we rewrite as follows:

$$E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[z] = \sum_{n=0}^{\infty} c_n z^n,$$

where, for convenience,

$$c_n = \frac{(\gamma)_{\kappa n}(\delta)_{\epsilon n}}{n! \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)}, \quad n \in \mathbb{N}_0, \tag{2.2}$$

so that

$$R = \limsup_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|.$$

Then, by virtue of the following hypotheses:

$$\alpha_j, \beta_j, \gamma, \kappa, \delta, \epsilon, z \in \mathbb{C}, \quad \Re(\alpha_j) > 0, \quad j = 1, \ldots, m, \quad \text{and} \quad \Re\left( \sum_{j=1}^{m} \alpha_j \right) > \Re(\kappa + \epsilon) - 1,$$

we can easily see from (2.2) in conjunction with the above asymptotic expansions for the Gamma function that

$$c_n = \frac{\Gamma(\gamma + \kappa n)\Gamma(\delta + \epsilon n)}{n!\Gamma(\gamma)\Gamma(\delta) \prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)} \to 0, \quad n \to \infty,$$

and

$$\left| \frac{c_n}{c_{n+1}} \right| = \left| (n+1) \left( \frac{\Gamma(\gamma + \kappa n)}{\Gamma(\kappa n + \gamma + \kappa)} \, \frac{\Gamma(\delta + \epsilon n)}{\Gamma(\epsilon n + \delta + \epsilon)} \right) \left( \frac{\prod_{j=1}^{m} \Gamma(\alpha_j n + \alpha_j + \beta_j)}{\prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)} \right) \right|$$

$$\sim \left( \frac{\prod_{j=1}^{m} \{\Re(\alpha_j)\}^{\Re(\alpha_j)}}{\{\Re(\kappa)\}^{\Re(\kappa)} \{\Re(\epsilon)\}^{\Re(\epsilon)}} \right) n^{\sum\limits_{j=1}^{m} \Re(\alpha_j - \kappa - \epsilon) + 1} \to R$$

$$= \begin{cases} \infty, & n \to \infty, \quad \sum_{j=1}^{m} \Re(\alpha_j) > \Re(\kappa + \epsilon) - 1, \\ \dfrac{\prod_{j=1}^{m} \{\Re(\alpha_j)\}^{\Re(\alpha_j)}}{\{\Re(\kappa)\}^{\Re(\kappa)} \{\Re(\epsilon)\}^{\Re(\epsilon)}} & n \to \infty, \quad \sum_{j=1}^{m} \Re(\alpha_j) = \Re(\kappa + \epsilon) - 1 > 0, \end{cases}$$

which leads us to the first assertion of Theorem 1 pertaining to the radius $R$ of (*absolute*) convergence of the power series in (1.5) involving (2.1) and also to the assertion that the general multiindex Mittag-Leffler function $E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[z]$ defined by (1.5) is an entire function in the complex $z$-plane.

With a view to determining the order $\rho$ and the type $\sigma$ of the entire function $E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[z]$, we simply apply the following definitions involving similar limits:

$$\rho = \limsup_{n\to\infty}\left\{\frac{n\log n}{\log\left(\dfrac{1}{|c_n|}\right)}\right\} \quad \text{and} \quad \sigma = \limsup_{n\to\infty}\left\{n|c_n|^{\rho/n}\right\}.$$

The involved details in each of these limit evaluations are being skipped here.

We now introduce our general fractional integral operator (2.3) defined below. Indeed, in the existing literature on the subject, various families of operators of fractional integration (involving, for example, those with such general classes of functions as the Fox – Wright function ${}_p\Psi_q(z)$ in their kernels) were investigated rather systematically by (for example) Srivastava and Saxena [32]. Here we consider the following integral operator:

$$\left(\mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\, E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[\omega(x-t)^{\alpha}]\,\varphi(t)\,dt, \quad x > a, \tag{2.3}$$

$$\alpha,\beta,\alpha_j,\beta_j,\gamma,\kappa,\delta,\epsilon,\omega \in \mathbb{C}, \quad \Re(\alpha_j) > 0, \quad j = 1,\ldots,m, \quad \min\{\Re(\beta),\Re(\kappa)\} > 0,$$

$$\Re\left(\sum_{j=1}^{m}\alpha_j\right) > \Re(\kappa+\epsilon) - 1,$$

which contains the general multiindex Mittag-Leffler function $E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[z]$ defined by (1.5) in its kernel. In the special case when $\epsilon = 0$ in the definition (2.3), we get the integral operator $\mathcal{E}^{\omega;\gamma,\kappa;\alpha}_{a+;(\alpha_j,\beta_j)_{m;\beta}}$, which was studied by Srivastava et al. [23]. Moreover, upon setting $m = 1$ and $\epsilon = 0$ in the definition (2.3), we get the integral operator $\mathcal{E}^{\omega;\gamma,\kappa}_{a+;\alpha,\beta}$, which was studied by Srivastava and Tomovski [33].

**Theorem 2.** *Under the various parametric constraints stated already with the definition* (2.3), *let the function* $\varphi$ *be in the space* $L(\mathfrak{a},\mathfrak{b})$ *of Lebesgue measurable functions on a finite interval* $[\mathfrak{a},\mathfrak{b}]$, $\mathfrak{b} > \mathfrak{a}$, *of the real line* $\mathbb{R}$ *given by*

$$L(\mathfrak{a},\mathfrak{b}) = \left\{\mathfrak{f}\colon \|\mathfrak{f}\|_1 := \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathfrak{f}(x)|\,dx < \infty\right\}. \tag{2.4}$$

*Then the integral operator* $\mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta}$ *is bounded on* $L(\mathfrak{a},\mathfrak{b})$ *and*

$$\left\|\mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta}\,\varphi\right\|_1 \leq \mathfrak{M}\,\|\varphi\|_1,$$

*where the constant* $\mathfrak{M}$, $0 < \mathfrak{M} < \infty$, *is given by*

$$\mathfrak{M} := (\mathfrak{b}-\mathfrak{a})^{\Re(\beta)} \sum_{n=0}^{\infty} \frac{|(\gamma)_{\kappa n}|\,|(\delta)_{\epsilon n}|}{\{\Re(\alpha)n+\beta\}\prod_{j=1}^{m}\Gamma(\Re(\alpha_j)n+\Re(\beta_j))}\,\frac{\left|\omega(b-a)^{\Re(\alpha)}\right|^n}{n!}. \tag{2.5}$$

***Proof.*** Following the arguments used by Srivastava and Tomovski [33], it is sufficient to prove that

$$\left\|\mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta}\,\varphi\right\|_1 = \int\limits_{\mathfrak{a}}^{\mathfrak{b}}\left|\int\limits_{a}^{x}(x-t)^{\beta-1}E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}\left[\omega(x-t)^{\alpha}\right]\varphi(t)dt\right|dx<\infty,$$

$$\alpha,\beta,\alpha_j,\beta_j,\gamma,\kappa,\delta,\epsilon,\omega\in\mathbb{C},\quad \Re(\beta_j)>0,\quad j=1,\ldots,m,\quad \min\{\Re(\beta),\Re(\kappa)\}>0,$$

$$\Re\left(\sum_{j=1}^{m}\alpha_j\right)>\Re(\kappa+\epsilon)-1.$$

We apply the definitions (2.3) and (2.4) in conjunction with the definition (1.5) of the general multiindex Mittag-Leffler function $E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[z]$. Upon interchanging the order of integration by means of the Dirichlet formula [9, p. 56] (or, alternatively, Fubini's theorem), we thus find that

$$\left\|\mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta}\,\varphi\right\|_1 \le \int\limits_{\mathfrak{a}}^{\mathfrak{b}}|\varphi(t)|\left(\int\limits_{t}^{\mathfrak{b}}(x-t)^{\Re(\beta)-1}\left|E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}\left[\omega(x-t)^{\alpha}\right]\right|dx\right)dt$$

$$=\int\limits_{\mathfrak{a}}^{\mathfrak{b}}|\varphi(t)|\left(\int\limits_{0}^{\mathfrak{b}-t}\tau^{\Re(\beta)-1}\left|E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}(\omega\tau^{\alpha})\right|d\tau\right)dt$$

$$\le\int\limits_{\mathfrak{a}}^{\mathfrak{b}}|\varphi(t)|\left(\int\limits_{0}^{\mathfrak{b}-a}\tau^{\Re(\beta)-1}\left|E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}(\omega\tau^{\alpha})\right|d\tau\right)dt$$

$$\le\left(\sum_{n=0}^{\infty}\frac{|(\gamma)_{\kappa n}|\,|(\delta)_{\epsilon n}|}{\prod_{j=1}^{m}\Gamma(\Re(\alpha_j)n+\Re(\beta_j))}\,\frac{|\omega|^n}{n!}\int\limits_{0}^{b-a}\tau^{\Re(\alpha)n+\Re(\beta)-1}d\tau\right)\|\varphi\|_1$$

$$=\mathfrak{M}\,\|\varphi\|_1,\quad \Re(\beta)>0,$$

where, in view of Theorem 1, the constant $\mathfrak{M}$ is finite and is given by (2.5). This completes the proof of the boundedness property of the integral operator $\mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta}$, just as asserted by Theorem 2.

***Remark*** **2.** Throughout our investigation, it is *tacitly* assumed that, in such situations as those occurring in the definitions (1.9), (1.10) and (2.3), the number $\mathfrak{a}$ in the function space $L(\mathfrak{a},\mathfrak{b})$ coincides precisely with the *lower* terminal $a$ in the integrals involved in the definitions (1.9), (1.10) and (2.3).

***Remark*** **3.** The results obtained by Kilbas et al. [6] as well as the results obtained by Srivastava and Tomovski [33] can be deduced as special cases of Theorem 2.

**Theorem 3.** *Let*

$$x>a,\quad a\in\mathbb{R}^+=[0,\infty),\quad 0<\mu<1,\quad 0\le\nu\le 1.$$

*Suppose also that*

$$\alpha,\beta,\alpha_j,\beta_j,\gamma,\kappa,\omega,\delta,\epsilon\in\mathbb{C},\quad j=1,\ldots,m,\quad \Re(\alpha_j)>0,\quad j=1,\ldots,m,\quad \min\{\Re(\beta),\Re(\kappa)\}>0$$

*and*

$$\Re\left(\sum_{j=1}^{m}\alpha_j\right)>\Re(\kappa+\epsilon)-1.$$

*Then*

$$\Big(I^{\lambda}_{a+}\Big[(t-a)^{\beta-1}\,E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[\omega(t-a)^{\alpha}]\Big]\Big)(x)$$

$$=\frac{(x-a)^{\beta+\lambda-1}}{\Gamma(\gamma)\Gamma(\delta)}\,{}_3\Psi_{m+1}\left[\begin{matrix}(\gamma,\kappa),(\delta,\epsilon),(\beta,\alpha);\\ (\beta_j,\alpha_j)_m,(\beta+\lambda,\alpha);\end{matrix}\ \omega(x-a)^{\alpha}\right], \tag{2.6}$$

$$\Big(D^{\lambda}_{a+}\Big[(t-a)^{\beta-1}\,E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[\omega(t-a)^{\alpha}]\Big]\Big)(x)$$

$$=\frac{(x-a)^{\beta-\lambda-1}}{\Gamma(\gamma)\Gamma(\delta)}\,{}_3\Psi_{m+1}\left[\begin{matrix}(\gamma,\kappa),(\delta,\epsilon),(\beta,\alpha);\\ (\beta_j,\alpha_j)_m,(\beta-\lambda,\alpha);\end{matrix}\ \omega(x-a)^{\alpha}\right] \tag{2.7}$$

*and*

$$\Big(D^{\mu,\nu}_{a+}\Big[(t-a)^{\beta-1}\,E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[\omega(t-a)^{\alpha}]\Big]\Big)(x)$$

$$=\frac{(x-a)^{\beta-\mu-1}}{\Gamma(\gamma)\Gamma(\delta)}\,{}_3\Psi_{m+1}\left[\begin{matrix}(\gamma,\kappa),(\delta,\epsilon),(\beta,\alpha);\\ (\beta_j,\alpha_j)_m,(\beta-\mu,\alpha);\end{matrix}\ \omega(x-a)^{\alpha}\right], \tag{2.8}$$

*provided that each member of the assertions* (2.6), (2.7) *and* (2.8) *exists.*

***Proof.*** Our demonstration of each of the assertions (2.6) and (2.7) runs parallel to those of the corresponding known results [6, p. 38 – 39, Theorem 3] (see also [33]), so we shall prove here only the assertion (2.8). To this end, we have

$$\Big(D^{\mu,\nu}_{a+}\Big[(t-a)^{\beta-1}\,E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[\omega(t-a)^{\alpha}]\Big]\Big)(x)$$

$$=\left(D^{\mu,\nu}_{a+}\left[\sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}(\delta)_{\epsilon n}}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j)}\,\frac{\omega^n}{n!}\,(t-a)^{\alpha n+\beta-1}\right]\right)(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}(\delta)_{\epsilon n}}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j)}\,\frac{\omega^n}{n!}\Big(D^{\mu,\nu}_{a+}\Big[(t-a)^{\alpha n+\beta-1}\Big]\Big)(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}(\delta)_{\epsilon n}}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j)}\,\frac{\omega^n}{n!}\,\frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta-\mu)}\,(x-a)^{\alpha n+\beta-\mu-1}$$

$$= \frac{(x-a)^{\beta-\mu-1}}{\Gamma(\gamma)\Gamma(\delta)} \, {}_3\Psi_{m+1}\left[\begin{array}{c} (\gamma,\kappa),(\delta,\epsilon),(\beta,\alpha); \\ (\beta_j,\alpha_j)_m,(\beta-\mu,\alpha); \end{array} \omega(x-a)^{\alpha}\right].$$

This evidently completes the proof of Theorem 3.

**Theorem 4.** *Under the various parametric constraints and conditions, which are listed already with the definition* (2.3), *each of the following composition relationships:*

$$I^{\lambda}_{a+}\mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta}\,\varphi = \mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta} I^{\lambda}_{a+}\,\varphi \tag{2.9}$$

*and*

$$D^{\lambda}_{a+}\mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta}\,\varphi = \mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta} D^{\lambda}_{a+}\,\varphi \tag{2.10}$$

*holds true for any Lebesgue measurable function* $\varphi \in L(\mathfrak{a},\mathfrak{b})$.

***Proof.*** For convenience, we denote by $\Delta$ the first member of the equation (2.9). Then, if we make use of (1.9) and (2.3), we have

$$\Delta = \frac{1}{\Gamma(\lambda)}\int_a^x (x-u)^{\lambda-1}\int_a^u (u-t)^{\beta-1}\,E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[\omega(u-t)^{\alpha}]\varphi(t)\,dt\,du.$$

We now interchange the order of the $t$-integral and the $u$-integral, which is permissible under the conditions stated already. We thus easily arrive at the following equation after a little simplication:

$$\Delta = \frac{1}{\Gamma(\lambda)}\int_a^x \left[\int_t^x (x-u)^{\lambda-1}\,(u-t)^{\beta-1}\,E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[\omega(u-t)^{\alpha}]\,du\right]\varphi(t)\,dt. \tag{2.11}$$

Making the substitution $u-t=\tau$ in the equation (2.11), we obtain

$$\Delta = \frac{1}{\Gamma(\lambda)}\int_a^x \left[\int_0^{x-t} (x-t-\tau)^{\lambda-1}\,\tau^{\beta-1}\,E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[\omega\tau^{\alpha}]d\tau\right]\varphi(t)\,dt. \tag{2.12}$$

Applying the assertion (2.6) to the right-hand side of (2.12), we get

$$\Delta = \int_a^x \frac{(x-t)^{\beta+\lambda-1}}{\Gamma(\gamma)\Gamma(\delta)} \, {}_3\Psi_{m+1}\left[\begin{array}{c} (\gamma,\kappa),(\delta,\epsilon),(\beta,\alpha); \\ (\beta_j,\alpha_j)_m,(\beta+\lambda,\alpha); \end{array} \omega(x-t)^{\alpha}\right]\varphi(t)\,dt. \tag{2.13}$$

For convenience, let the right-hand side of (2.9) be denoted by $\Omega$. Then, by using the definitions (2.3) and (1.9), we have

$$\Omega = \int_a^u (u-t)^{\beta-1}\,E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[\omega(u-t)^{\alpha}]\,\frac{1}{\Gamma(\lambda)}\int_a^t (t-x)^{\lambda-1}\,\varphi(x)\,dx\,dt.$$

Further, upon first interchanging the order of the $x$-integral and the $t$-integral (which is permissible under the conditions stated already) and then making the substitution $u-t=\tau$, we easily arrive at the following equation after a little simplification:

$$\Omega = \frac{1}{\Gamma(\lambda)} \int\limits_a^u \left[ \int\limits_0^{u-x} (u-\tau-x)^{\lambda-1} \, \tau^{\beta-1} \, E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[\omega\tau^{\alpha}] \, d\tau \right] \varphi(x) \, dx. \tag{2.14}$$

In concluding proof of Theorem 4, we apply the assertion (2.6) in the second member of the equation (2.14). We thus obtain an expression similar to that in the equation (2.13). This implies that

$$I^{\lambda}_{a+} \left( \mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta} \, \varphi \right)(u) = \int\limits_a^u \frac{(u-x)^{\beta+\lambda-1}}{\Gamma(\gamma)\Gamma(\delta)}$$

$$\times \, {}_3\Psi_{m+1} \left[ \begin{matrix} (\gamma,\kappa), (\delta,\epsilon), (\beta,\alpha); \\ (\beta_j,\alpha_j)_m, (\beta+\lambda,\alpha); \end{matrix} \; \omega(u-x)^{\alpha} \right] \varphi(x) \, dx$$

$$= \left( \mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta} I^{\lambda}_{a+} \, \varphi \right)(u). \tag{2.15}$$

This proves the assertion (2.9).

Our demonstration of the assertion (2.10) runs parallel to that of the assertion (2.9), so we choose omit the details involved. We thus have completed the proof of Theorem 4.

**Theorem 5.** *Under the various parametric constraints and conditions, which are listed already with the definition* (2.3), *each of the following composition relationships:*

$$D^{\mu,\nu}_{a+} \left( \mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta} \, \varphi \right) = I^{(1-\nu)(1-\mu)}_{a+} \, \mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta} \, \varphi, \quad 0 < \mu < 1, \quad 0 \leq \nu \leq 1, \tag{2.16}$$

*holds true for any Lebesgue measurable function* $\varphi \in L(\mathfrak{a}, \mathfrak{b})$.

***Proof.*** We start with the left-hand side of (2.16). Indeed, if we make use of the definition (1.11), we get

$$D^{\mu,\nu}_{a+} \left( \mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta} \, \varphi \right) = I^{\nu(1-\mu)}_{a+} \, \frac{d}{dx} \left( I^{(1-\nu)(1-\mu)}_{a+} \mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta} \, \varphi \right).$$

Moreover, if we make use of the result (2.15), we obtain

$$\left( I^{(1-\nu)(1-\mu)}_{a+} \mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta} \, \varphi \right)(x) = \int\limits_a^x \frac{(x-t)^{\beta+(1-\nu)(1-\mu)-1}}{\Gamma(\gamma)\Gamma(\delta)}$$

$$\times \, {}_3\Psi_{m+1} \left[ \begin{matrix} (\gamma,\kappa), (\delta,\epsilon), (\beta,\alpha); \\ (\beta_j,\alpha_j)_m, (\beta+(1-\nu)(1-\mu),\alpha); \end{matrix} \; \omega(x-t)^{\alpha} \right] \varphi(t) \, dt. \tag{2.17}$$

Now, taking the first derivative of each member of the equation (2.17) with respect to $x$, we find that

$$\frac{d}{dx} \left( I^{(1-\nu)(1-\mu)}_{a+} \mathcal{E}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}_{a+;(\alpha_j,\beta_j)_m;\beta} \, \varphi \right)(x) = \int\limits_a^x \frac{(x-t)^{\beta+(1-\nu)(1-\mu)-2}}{\Gamma(\gamma)\Gamma(\delta)}$$

$$\times \, {}_3\Psi_{m+1} \left[ \begin{matrix} (\gamma,\kappa), (\delta,\epsilon), (\beta,\alpha); \\ (\beta_j,\alpha_j)_m, (\beta+(1-\nu)(1-\mu)-1,\alpha); \end{matrix} \; \omega(x-t)^{\alpha} \right] \varphi(t) \, dt. \tag{2.18}$$

Next, by applying the operator $I^{\nu(1-\mu)}_{a^+}$ to both members of the equation (2.18) with the help of the definition (1.9), we have

$$\left(I_{a+}^{\nu(1-\mu)}\frac{d}{dx}\left(I_{a+}^{(1-\nu)(1-\mu)}\mathcal{E}_{a+;(\alpha_j,\beta_j)_m;\beta}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}\varphi\right)(x)\right)(u)$$

$$=\frac{1}{\Gamma(\nu(1-\mu))}\int\limits_a^u(u-s)^{\nu(1-\mu)-1}\,ds\int\limits_a^s\frac{(s-t)^{\beta+(1-\nu)(1-\mu)-2}}{\Gamma(\gamma)\Gamma(\delta)}$$

$$\times\,{}_3\Psi_{m+1}\left[\begin{matrix}(\gamma,\kappa),(\delta,\epsilon),(\beta,\alpha);\\(\beta_j,\alpha_j)_m,(\beta+(1-\nu)(1-\mu)-1,\alpha);\end{matrix}\;\omega(s-t)^{\alpha}\right]\varphi(t)\,dt. \qquad (2.19)$$

We first invert the order of the $t$-integral and the $s$-integral (which is permissible under already stated conditions) and then substitute $s-t=\tau$ in the equation (2.19). We thus find that

$$\left(I_{a+}^{\nu(1-\mu)}\frac{d}{dx}\left(I_{a+}^{(1-\nu)(1-\mu)}\mathcal{E}_{a+;(\alpha_j,\beta_j)_m;\beta}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}\varphi\right)(x)\right)(u)$$

$$=\frac{1}{\Gamma(\gamma)\Gamma(\delta)}\frac{1}{\Gamma(\nu(1-\mu))}\int\limits_a^u\varphi(t)\,dt\int\limits_0^{u-t}(u-\tau-t)^{\nu(1-\mu)-1}\tau^{\beta+(1-\nu)(1-\mu)-2}$$

$$\times\,{}_3\Psi_{m+1}\left[\begin{matrix}(\gamma,\kappa),(\delta,\epsilon),(\beta,\alpha);\\(\beta_j,\alpha_j)_m,(\beta+(1-\nu)(1-\mu)-1,\alpha);\end{matrix}\;\omega\tau^{\alpha}\right]d\tau=\frac{1}{\Gamma(\gamma)\Gamma(\delta)}\int\limits_a^u I_{0+}^{\nu(1-\mu)}$$

$$\times\left(\tau^{\beta+(1-\nu)(1-\mu)-2}{}_3\Psi_{m+1}\left[\begin{matrix}(\gamma,\kappa),(\delta,\epsilon),(\beta,\alpha);\\(\beta_j,\alpha_j)_m,(\beta+(1-\nu)(1-\mu)-1,\alpha);\end{matrix}\;\omega\tau^{\alpha}\right]\right)\varphi(t)\,dt. \qquad (2.20)$$

In concluding our proof of Theorem 5, we apply the result (2.15) to the right-hand side of the equation (2.20). We thus obtain

$$\left(I_{a+}^{\nu(1-\mu)}\frac{d}{dx}\left(I_{a+}^{(1-\nu)(1-\mu)}\mathcal{E}_{a+;(\alpha_j,\beta_j)_m;\beta}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}\varphi\right)(x)\right)(u)$$

$$=\frac{1}{\Gamma(\gamma)\Gamma(\delta)\Gamma(\nu(1-\mu))}\int\limits_a^u(u-t)^{\beta+(1-\nu)(1-\mu)-1}$$

$$\times\,{}_3\Psi_{m+1}\left[\begin{matrix}(\gamma,\kappa),(\delta,\epsilon),(\beta,\alpha),\\(\beta_j,\alpha_j)_m,(\beta+(1-\nu)(1-\mu),\alpha),\end{matrix}\;\omega(u-t)^{\alpha}\right]\varphi(t)\,dt$$

$$=\left(I_{a+}^{(1-\nu)(1-\mu)}\mathcal{E}_{a+;(\alpha_j,\beta_j)_m;\beta}^{\omega;\gamma,\kappa,\delta,\epsilon;\alpha}\varphi\right)(u).$$

The proof of Theorem 5 is thus completed.

***Remark* 4.** If, on the left-hand sides of the equations (2.6), (2.7) and (2.8), we reduce the general multiindex Mittag-Leffler function to the Mittag-Leffler type function $E_{\alpha,\beta}^{\gamma,\kappa}(z)$ in (1.4), which was introduced by Srivastava and Tomovski [33], we get the results obtained by Srivastava and Tomovski et al. [33]. Moreover, if we reduce the general multiindex Mittag-Leffler function to the Mittag-Leffler type function $E_{\alpha,\beta}^{\gamma,\kappa}(z)$ in (1.4), which was also introduced by Srivastava and Tomovski [33], in the integral operator on the left-hand sides of the equations (2.9), (2.10) and (2.16), we are led to the results obtained by Srivastava and Tomovski [33].

**3. A set of integral representations for the product of two general multiindex Mittag-Leffler functions.** A number of earlier authors have developed integral representations for the product of two orthogonal and other related hypergeometric polynomials. For example, Srivastava [21] gave an integral representation for the product of two generalized Rice polynomials. In a sequel to the investigation by Srivastava [21], Srivastava and Joshi [28] presented a general integral representation for the product of two generalized hypergeometric polynomials. More recently, Srivastava and Panda [31] derived integral representations for the product of two Jacobi polynomials as well as for the Kampé de Fériet polynomials in two variables.

In our investigation, we make use of each of the following integral formulas for the Gamma function $\Gamma(z)$ and the Beta function $B(\alpha,\beta)$, especially in demonstration of Theorem 6 below:

$$\Gamma(z) = \int_0^\infty e^{-\tau}\,\tau^{z-1}\,d\tau, \quad \Re(z) > 0, \tag{3.1}$$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\,dt, \quad \min\{\Re(\alpha),\Re(\beta)\} > 0, \tag{3.2}$$

and

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{(0+)} e^{\zeta}\zeta^{-z}\,d\zeta, \quad |\arg(\zeta)| \leq \pi, \tag{3.3}$$

where the contour of integration is a Hankel's loop which starts at $-\infty$ on the real axis in the complex $\zeta$-plane, encircles the origin $(\zeta = 0)$ once in the positive (counter-clockwise) direction, and then returns to $-\infty$ (see, for details, [35, p. 244–246]).

**Theorem 6.** *Each of the following integral representations holds true:*

$$E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa,\delta,\epsilon}(z)\,E_{(\lambda_j,\mu_j)_m}^{\sigma,\kappa,\rho,\epsilon}(z) = -\frac{1}{4\pi^2\Gamma(\gamma)\Gamma(\sigma)\Gamma(\rho)\Gamma(\delta)} \int_0^1 t^{\gamma-1}(1-t)^{\sigma-1} \int_0^1 u^{\delta-1}(1-u)^{\rho-1}$$

$$\times \Bigg( \int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} e^{\zeta+\omega}\,\zeta^{-\prod_{j=1}^{m}\mu_j}\,\omega^{-\prod_{j=1}^{m}\beta_j}$$

$$\times {}_2\Psi_0\left[\begin{array}{l}(\gamma+\sigma,\kappa),(\rho+\delta,\epsilon);\\ \overline{\qquad\qquad\qquad};\end{array}\left(t^{\kappa}u^{\epsilon}\,\omega^{-\prod\limits_{j=1}^{m}\alpha_j}+(1-t)^{\kappa}(1-u)^{\epsilon}\,\zeta^{-\prod\limits_{j=1}^{m}\lambda_j}\right)z\right]d\omega\,d\zeta\Bigg)dtdu, \tag{3.4}$$

$$\max\{|\arg(\omega)|,|\arg(\zeta)|\}\leq\pi,\quad \min\{\Re(\kappa),\Re(\gamma),\Re(\sigma)\}>0,$$

$$\min\left\{\Re\left(\sum_{j=1}^{m}\alpha_j\right),\Re\left(\sum_{j=1}^{m}\lambda_j\right)\right\}>\Re(\kappa+\epsilon)-1,$$

*where "————" exhibits the fact that there are no denominator parameters in the Fox–Wright function* ${}_2\Psi_0$.

***Proof.*** By using the definition (1.5), the first member of the equation (3.4) can be written as follows:

$$E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}(z)\,E^{\sigma,\kappa,\rho,\epsilon}_{(\lambda_j,\mu_j)_m}(z)=\sum_{r=0}^{\infty}\frac{(\gamma)_{\kappa r}(\delta)_{\epsilon r}}{\prod_{j=1}^{m}\Gamma(\alpha_j r+\beta_j)}\frac{z^r}{r!}\sum_{n=0}^{\infty}\frac{(\sigma)_{\kappa n}(\rho)_{\epsilon n}}{\prod_{j=1}^{m}\Gamma(\lambda_j n+\mu_j)}\frac{z^n}{n!}. \tag{3.5}$$

Next, in light of a known result [14, p. 56, Eq. (1)], the right-hand side of the equation (3.5) would lead us to the following equation after a little simplification:

$$E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}(z)\,E^{\sigma,\kappa,\rho,\epsilon}_{(\lambda_j,\mu_j)_m}(z)=\frac{1}{\Gamma(\gamma)\Gamma(\sigma)\Gamma(\delta)\Gamma(\rho)}$$

$$\times\sum_{n=0}^{\infty}\frac{z^n}{n!}\left(\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(\gamma+\kappa r)\Gamma(\delta+\epsilon r)\Gamma(\sigma+\kappa(n-r))\Gamma(\rho+\epsilon(n-r))}{\prod_{j=1}^{m}\Gamma(\alpha_j r+\beta_j)\prod_{j=1}^{m}\Gamma(\lambda_j(n-r)+\mu_j)}\right)$$

$$=\frac{1}{\Gamma(\gamma)\Gamma(\sigma)\Gamma(\delta)\Gamma(\rho)}\sum_{n=0}^{\infty}\Gamma(\gamma+\sigma+\kappa n)\Gamma(\rho+\delta+\epsilon n)\frac{z^n}{n!}$$

$$\times\left(\sum_{r=0}^{n}\binom{n}{r}\frac{B(\gamma+\kappa r,\sigma+\kappa(n-r))B(\delta+\epsilon r,\rho+\epsilon(n-r))}{\prod_{j=1}^{m}\Gamma(\alpha_j r+\beta_j)\prod_{j=1}^{m}\Gamma(\lambda_j(n-r)+\mu_j)}\right). \tag{3.6}$$

In order to conclude proof of Theorem 6, we suitably apply the integral formulas (3.2) and (3.3) in the right-hand side of (3.6). We thus find that

$$E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}(z)\,E^{\sigma,\kappa,\rho,\epsilon}_{(\lambda_j,\mu_j)_m}(z)=-\frac{1}{4\pi^2\Gamma(\gamma)\Gamma(\sigma)\Gamma(\delta)\Gamma(\rho)}\int_0^1 t^{\gamma-1}(1-t)^{\sigma-1}\int_0^1 u^{\delta-1}(1-u)^{\rho-1}$$

$$\times\int_{-\infty}^{(0+)}\int_{-\infty}^{(0+)}e^{\zeta+\omega}\,\zeta^{-\prod\limits_{j=1}^{m}\mu_j}\,\omega^{-\prod\limits_{j=1}^{m}\beta_j}$$

$$\times \sum_{n=0}^{\infty} \Gamma(\gamma+\sigma+\kappa n)\Gamma(\rho+\delta+\epsilon n)\,\frac{z^n}{n!}\sum_{r=0}^{n}\binom{n}{r}\omega^{-\prod\limits_{j=1}^{m}\alpha_j r}\,\zeta^{-\prod\limits_{j=1}^{m}\lambda_j(n-r)}$$

$$\times\, t^{\kappa r}\,(1-t)^{\kappa(n-r)}u^{\epsilon r}(1-u)^{\epsilon(n-r)}\,d\omega\,d\zeta\,dt\,du$$

$$= -\frac{1}{4\pi^2\Gamma(\gamma)\Gamma(\sigma)\Gamma(\delta)\Gamma(\rho)}\int\limits_0^1 t^{\gamma-1}\,(1-t)^{\sigma-1}\int\limits_0^1 u^{\delta-1}(1-u)^{\rho-1}$$

$$\times \int\limits_{-\infty}^{(0+)}\int\limits_{-\infty}^{(0+)} e^{\zeta+\omega}\,\zeta^{-\prod\limits_{j=1}^{m}\mu_j}\,\omega^{-\prod\limits_{j=1}^{m}\beta_j}$$

$$\times \sum_{n=0}^{\infty}\Gamma(\gamma+\sigma+\kappa n)\Gamma(\rho+\delta+\epsilon n)\,\frac{z^n}{n!}$$

$$\times \left(t^{\kappa}u^{\epsilon}\,\omega^{-\prod\limits_{j=1}^{m}\alpha_j} + (1-t)^{\kappa}(1-u)^{\epsilon}\,\zeta^{-\prod\limits_{j=1}^{m}\lambda_j}\right)^n d\zeta\,d\omega\,dt\,du. \tag{3.7}$$

The second member of this last equation (3.7) can be interpreted in terms of the Fox – Wright hypergeometric function with the help of the definition (1.7). This evidently completes the proof of Theorem 6.

**4. Conclusion.** The main investigation in this paper involves an introduction and a systematic study of a general fractional integral operator which contains (in its kernel) the general multi-index Mittag-Leffler type function $E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}(z)$ defined by (1.5). Although this general multiindex Mittag-Leffler function $E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}(z)$ is contained in the familiar Fox – Wright hypergeometric function ${}_p\Psi_q(z)$ with $p$ numerator and $q$ denominator parameters, as we have clearly mentioned in the relationship (1.8) (see Remark 1), it is observed to be an elegant unification of various known Mittag-Leffler type functions. Some of the main results of our present investigation are shown to provide generalizations and extensions of the results which were derived earlier by Kilbas et al. [6] as well as by Srivastava and Tomovski [33].

We conclude our investigation by remarking further that the properties and results, which are systematically presented here, have the potential to motivate interesting further researches on the subject of this paper.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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