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SOME TAUBERIAN THEOREMS FOR THE WEIGHTED MEAN METHOD OF SUMMABILITY OF DOUBLE SEQUENCE

ДЕЯКІ ТАУБЕРОВІ ТЕОРЕМИ ДЛЯ МЕТОДУ ЗВАЖЕНОГО СЕРЕДНЬОГО ПІДСУМОВУВАННЯ ПОДВІЙНИХ ПОСЛІДОВНОСТЕЙ

Let $p = (p_j)$ and $q = (q_k)$ be real sequences of nonnegative numbers with the property that $P_m = \sum_{j=0}^m p_j \neq 0$ and $Q_n = \sum_{k=0}^n q_k \neq 0$ for all m and n. Let (P_m) and (Q_n) be regulary varying positive indices. Assume that (u_{mn}) is a double sequence of complex (real) numbers, which is $(\overline{N}, p, q; \alpha, \beta)$ summable with a finite limit, where $(\alpha, \beta) = (1, 1)$, $(1, 0)$, or $(0, 1)$. We present some conditions imposed on the weights under which (u_{mn}) converges in Pringsheim's sense. These results generalize and extend the results obtained by authors in [Comput. Math. Appl., **62**, № 6, 2609 – 2615 (2011)].

Нехай $p = (p_j)$ і $q = (q_k)$ — дійсні послідовності невід'ємних чисел такі, що $P_m = \sum_{j=0}^m p_j \neq 0$ і $Q_n = \sum_{k=0}^n q_k \neq 0$ для всіх m і n. Нехай (P_m) і (Q_n) — регулярно змінні додатні індекси. Припустимо, що (u_{mn}) — подвійна послідовність комплексних (або дійсних) чисел, яка є $(\overline{N}, p, q; \alpha, \beta)$ сумовною зі скінченною границею, де $(\alpha, \beta) = (1, 1)$, $(1, 0)$ або $(0, 1)$. Наведено деякі умови, що накладені на ваги, за яких (u_{mn}) збігається в розумінні Прінгсхейма. Ці результати узагальнюють і розширюють результати, отримані авторами в [Comput. Math. Appl., **62**, № 6, 2609 – 2615 (2011)].

1. Preliminary results for single sequences. Let $u = (u_n)$ be a single sequence of real numbers. Let $p = (p_j)$ be a sequence of nonnegative numbers $(p_0 > 0)$ with the property that

$$P_n := \sum_{j=0}^{n} p_j \to \infty, \quad n \to \infty. \tag{1}$$

Throughout this paper, $[\lambda n]$ denotes the integer part of the product λn.

We say that a positive sequence $P = (P_n)$ is regularly varying of index $\delta > 0$ if

$$\lim_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} = \lambda^{\delta}, \quad \lambda > 0. \tag{2}$$

The relation between sequences satisfying conditions

$$\liminf_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} > 1, \tag{3}$$

$$\limsup_{n\to\infty} \frac{P_n}{P_{[\lambda n]}} < 1 \tag{4}$$

and (2) are discussed in [5] and noted that (3) and (4) are clearly satisfied if (P_n) is regularly varying of positive index.

The nth weighted mean of a sequence $u = (u_n)$ is defined [13] by

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$$t_n^{(1)}(u) := \frac{1}{P_n}\sum_{k=0}^{n} p_k u_k.$$

For any sequence $u = (u_n)$ we use the notation $\Delta_n u_n = u_n - u_{n-1}$ for $n \ge 0$. Note that any sequence term with negative index is zero.

For a sequence (u_n), the identity

$$u_n - t_n^{(1)}(u) = V_n(\Delta u),$$

where $V_n(\Delta u) = \dfrac{1}{P_n}\sum_{k=1}^{n} P_{k-1}\Delta_k u_k$, is called weighted Kronecker identity.

A sequence (u_n) is called slowly oscillating [20] if

$$\lim_{\lambda\to 1^+}\limsup_{n\to\infty}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta_j u_j\right| = 0.$$

Under some conditions on the sequence $p = (p_n)$, Çanak and Totur [7] have proved that (u_n) is slowly oscillating if and only if $(V_n(\Delta u))$ is slowly oscillating and bounded.

A sequence (u_n) is said to be summable by the weighted mean method determined by the sequence p, in short, $(\overline{N}, p)$ summable to a finite number s if $\lim_{n\to\infty} t_n^{(1)}(u) = s$ and we write $u_n \to s\,(\overline{N}, p)$. The $(\overline{N}, p)$ summability method is regular under the condition (1) (see [13, p. 57] for more details). Namely, $u_n \to s$ implies $u_n \to s\,(\overline{N}, p)$ as $n \to \infty$. The converse implication is not necessarily true and it might be true under additional conditions called Tauberian conditions. Any theorem which states that convergence of a sequence follows from the $(\overline{N}, p)$ summability with some Tauberian conditions is said to be a Tauberian theorem.

Now, we give some well-known classical type Tauberian theorems for $(\overline{N}, p)$ summability method.

Theorem 1 [18]. *Suppose that the condition* (3) *is satisfied. If* (u_n) *is* $(\overline{N}, p)$ *summable to* s *and*

$$\frac{P_{n-1}}{p_n}\Delta_n u_n \ge -C, \quad n \ge 0,$$

for some $C > 0$, *then* (u_n) *is convergent to* s.

Theorem 2 [18]. *Suppose that the condition* (3) *is satisfied. If* (u_n) *is* $(\overline{N}, p)$ *summable to* s *and* (u_n) *is slowly oscillating*, *then* (u_n) *is convergent to* s.

Móricz and Rhoades [18] replaced Tauberian conditions in Theorems 1 and 2 to the following weaker conditions:

$$\sup_{\lambda>1}\liminf_{n\to\infty}\frac{1}{P_{[\lambda n]} - P_n}\sum_{j=n+1}^{[\lambda n]} p_j(s_j - s_n) \ge 0$$

and

$$\sup_{0<\lambda<1}\liminf_{n\to\infty}\frac{1}{P_n - P_{[\lambda n]}}\sum_{j=[\lambda n]+1}^{n} p_j(s_n - s_j) \ge 0.$$

Çanak and Totur [6] proved that the slow oscillation of $(V_n(\Delta u))$ is a Tauberian condition for $(\overline{N},p)$ summability. Moreover, they presented that if (u_n) is $(\overline{N},p)$ summable to s and $\dfrac{P_{n-1}}{p_n}\Delta V_n(\Delta u)\geq -C$ for some $C>0$, then (u_n) is convergent to s under certain conditions on (P_n).

In the literature, several Tauberian theorems have been investigated by Çanak and Totur [6, 8, 9], Borwein and Kratz [10], Móricz and Rhoades [18], Sezer and Çanak [21], Tietz and Zeller [23], Totur and Çanak [24] for the weighted mean method of summability. Recently, generalizations of weighted statistical convergence and equistatistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems have been studied by Belen and Mohiuddine [3] and Mohiuddine and Alamri [16], respectively.

2. The $(\overline{N},p,q;\alpha,\beta)$ summability method for double sequences. Let $p=(p_j)$ and $q=(q_k)$ be two sequences of nonnegative numbers $(p_0,q_0>0)$ with the property that

$$P_m:=\sum_{j=0}^{m}p_j\to\infty,\quad m\to\infty, \tag{5}$$

and

$$Q_n:=\sum_{k=0}^{n}q_k\to\infty,\quad n\to\infty. \tag{6}$$

A double sequence $u=(u_{mn})$ is called convergent in Pringsheim's sense (or P-convergent) if for a given $\varepsilon>0$, there exists a positive integer N_0 such that $|u_{mn}-L|<\varepsilon$ for all $m,n\geq N_0$ (see [19]). We note that throughout this paper we always mean convergence in Pringsheim's sense.

A double sequence (u_{mn}) is bounded if there exists a real number $C>0$ such that $|u_{mn}|\leq C$ for all $m,n\geq 0$, and is one-sided bounded if there exists a real number $C>0$ such that $u_{mn}\geq -C$ for all $m,n\geq 0$.

The symbols $u_{mn}=o(1)$ and $u_{mn}=O(1)$ represent that (u_{mn}) is P-convergent to 0 as $m,n\to\infty$ and (u_{mn}) is bounded as $m,n\to\infty$, respectively.

For a double sequence (u_{mn}), we use the notations following notations: $\Delta_n u_{mn}=u_{mn}-u_{m,n-1}$ and $\Delta_m u_{mn}=u_{mn}-u_{m-1,n}$ for all $m,n\geq 1$. From these, we easily get $\Delta_{m,n}u_{mn}=\Delta_m\Delta_n u_{mn}=\Delta_m(\Delta_n u_{mn})=\Delta_n(\Delta_m u_{mn})$ for all $m,n\geq 1$.

The weighted means of a double sequence $u=(u_{mn})$ are the sequence $\big(t_{mn}^{(11)}(u)\big)$, which are defined by

$$t_{mn}^{(11)}(u)=\frac{1}{P_mQ_n}\sum_{j=0}^{m}\sum_{k=0}^{n}p_jq_ku_{jk}$$

for all $m,\ n\geq 0$.

A double sequence (u_{mn}) is said to be $(\overline{N},p,q;1,1)$ summable to a finite number s if $\big(t_{mn}^{(11)}(u)\big)$ converges to s; in symbols, $u_{mn}\to s\,(\overline{N},p,q;1,1)$.

The $(\overline{N},p,*;1,0)$ and $(\overline{N},*,q;0,1)$ means of the sequence (u_{mn}) are defined respectively by

$$t_{mn}^{(10)}(u)=\frac{1}{P_m}\sum_{j=0}^{m}p_ju_{jn}\qquad\text{and}\qquad t_{mn}^{(01)}(u)=\frac{1}{Q_n}\sum_{k=0}^{n}q_ku_{mk}$$

for all $m,n\geq 0$.

A double sequence (u_{mn}) is said to be $(\overline{N}, p, *; 1, 0)$ summable to a finite number s if $\lim_{m,n\to\infty} t_{mn}^{(10)}(u) = s$. In the light of the discussion above, the $(\overline{N}, *, q; 0, 1)$ summability is defined analogously.

Every P-convergent and bounded double sequence is $(\overline{N}, p, q; 1, 1)$ summable to its P-convergence, but the converse of this implication is not true in general. Namely, bounded and $(\overline{N}, p, q; 1, 1)$ summable a double sequence may not be P-convergent. We can give an example of a double sequence which is $(\overline{N}, p, q; 1, 1)$ summable and bounded, but not P-convergent as follows:

***Example* 1.** The sequence $(u_{mn}) = ((-1)^{m+n})$ is bounded and is not P-convergent. But it is $(\overline{N}, p, q; 1, 1)$ summable to 0. Indeed, from the definition of $(\overline{N}, p, q; 1, 1)$ means, if we take $p = q = 1$, then we easily get $\lim_{m,n\to\infty} t_{mn}^{(11)}(u) = 0$.

In [5], Chen and Hsu obtained necessary and sufficient conditions under which $u_{mn} \to s$ follows from $u_{mn} \to s\,(\overline{N}, p, q; \alpha, \beta)$, where $(\alpha, \beta) = (1, 1), (1, 0)$, and $(0, 1)$.

Several particular cases of the weighted mean methods of double sequences have been investigated by Móricz [17], Baron and Stadtmüller [2], and Stadtmüller [22].

Móricz [17] obtained necessary and sufficient conditions under which convergence of (u_{mn}) in Pringsheim's sense follows from (C, α, β) summability of (u_{mn}), where $(\alpha, \beta) = (1, 1)$, $(1, 0)$, and $(0, 1)$.

Baron and Stadtmüller [2] studied the relations between power series methods, weighted mean methods and convergence in ordinary sense for double sequences. In particular, they proved that analogues of Landau's two-sided conditions for double sequences are Tauberian conditions for the weighted mean method $(\overline{N}, p, q; 1, 1)$, where P and Q are regularly varying sequences.

In [22], Stadtmüller established the relations between weighted mean methods and convergence in ordinary sense for double sequences and he obtained a Tauberian theorem for $(C, 1, 1)$ summability method which includes a classical Tauberian theorem of Knopp [14] as a special case of his results and generalized theorems due to Móricz [17].

Móricz [17] introduced the following slow oscillations in different senses with respect to the indices: A double sequence (u_{mn}) is said to be slowly oscillating in sense (1,1) if

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{\substack{m+1\le j\le[\lambda m]\\ n+1\le j\le[\lambda n]}} \left| \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{r,s} u_{rs} \right| = 0.$$

A double sequence (u_{mn}) is said to be slowly oscillating in sense $(1, 0)$ if

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{m+1\le j\le[\lambda m]} \left| \sum_{r=m+1}^{j} \Delta_r u_{rn} \right| = 0.$$

A double sequence (u_{mn}) is said to be slowly oscillating in sense $(0, 1)$ if

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{n+1\le j\le[\lambda n]} \left| \sum_{s=n+1}^{k} \Delta_s u_{ms} \right| = 0.$$

Note that every P-convergent sequence is slowly oscillating in senses $(1, 1)$, $(1, 0)$, and $(0, 1)$. However, the converse is not necessarily true. For example, the sequence $(u_{mn}) = (\log m \log n)$ is slowly oscillating in sense $(1, 1)$ and is not P-convergent. Indeed, we have

$$\left|\sum_{r=m+1}^{j}\sum_{s=n+1}^{k}\Delta_{r,s}u_{rs}\right| = |\log j\log k - \log j\log n - \log m\log k + \log m\log n| = \log\left(\frac{j}{m}\right)\log\left(\frac{k}{n}\right).$$

Hence we get

$$\max_{\substack{m+1\le j\le[\lambda m]\\ n+1\le k\le[\lambda n]}}\left|\sum_{r=m+1}^{j}\sum_{s=n+1}^{k}\Delta_{r,s}u_{rs}\right| = \log\left(\frac{[\lambda m]}{m}\right)\log\left(\frac{[\lambda n]}{n}\right).$$

It follows from the last line that $(\log m\log n)$ is slowly oscillating in sense $(1,1)$.

We say that (u_{mn}) satisfies weighted analogues of Landau's condition in senses $(1,0)$ and $(0,1)$ if there exists $m_1 > 0$ such that

$$\frac{P_{m-1}}{p_m}\Delta_m u_{mn} = O(1), \quad m,n > m_1, \tag{7}$$

$$\frac{Q_{n-1}}{q_n}\Delta_n u_{mn} = O(1), \quad m,n > m_1, \tag{8}$$

respectively. It is clear that if (7) and (8) hold and (P_m) and (Q_n) are regularly varying of positive indices, then (u_{mn}) is slowly oscillating in senses $(1,0)$ and $(0,1)$, respectively.

We define the weighted de la Vallée Poussin means of a sequence (u_{mn}) as follows: if $\lambda > 1$,

$$\tau^{>}_{mn}(u) = \frac{1}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]} p_j q_k u_{jk},$$

and if $0 < \lambda < 1$,

$$\tau^{<}_{mn}(u) = \frac{1}{\big(P_m - P_{[\lambda m]}\big)\big(Q_n - Q_{[\lambda n]}\big)}\sum_{j=[\lambda m]+1}^{m}\sum_{k=[\lambda n]+1}^{n} p_j q_k u_{jk}$$

for sufficiently large $m,n \ge 0$.

In this paper, we extend the results obtained by Çanak and Totur in [6] to double sequences and prove that classical type Tauberian theorems are corollaries of our results.

We should mention the following novelties of the present paper. Certain conditions on the sequence (u_{mn}) or some other sequence related to (u_{mn}) are sufficient conditions for $(\overline{N},p,q;1,1)$ summable of (u_{mn}) to be P-convergent. We prove some classical type Tauberian theorems for the $(\overline{N},p,q;1,1)$ summability method. Therefore, classical type Tauberian theorems such as Hardy – Landau's theorem [12, 15] and Schmidt's theorem [20] for the $(\overline{N},p,q;1,1)$ summability method are corollaries of our two main theorems.

3. Lemmas. In this section, we present the following lemmas which will be used in the proof of our main theorems.

The weighted Kronecker identity for single sequences is extended to double sequences in the following form.

Lemma 1.

$$u_{mn} - t^{(10)}_{mn}(u) - t^{(01)}_{mn}(u) + t^{(11)}_{mn}(u) = V^{(11)}_{mn}(\Delta u),$$

where $V^{(11)}_{mn}(\Delta u) = \dfrac{1}{P_mQ_n}\sum_{j=1}^{m}\sum_{k=1}^{n} P_{j-1}Q_{k-1}\Delta_{j,k}u_{jk}.$

Note that the sequence $\big(V_{mn}^{(11)}(\Delta u)\big)$ is the sequence of $(\overline{N},p,q;1,1)$ means of the sequence $\left(\dfrac{P_{m-1}}{p_m}\dfrac{Q_{n-1}}{q_n}\Delta_{m,n}u_{mn}\right)$.

Proof. By applying the Abel transformation for double sequences (see [1]), we have

$$
\begin{aligned}
&u_{mn}-t_{mn}^{(10)}(u)-t_{mn}^{(01)}(u)+t_{mn}^{(11)}(u)\\
&\quad=\frac{1}{P_mQ_n}\Big(P_mQ_nu_{mn}-Q_n\sum_{j=0}^{m}p_ju_{jn}-P_m\sum_{k=0}^{n}q_ku_{mk}+\sum_{j=0}^{m}\sum_{k=0}^{n}u_{jk}\Big)\\
&\quad=\frac{1}{P_mQ_n}\sum_{j=0}^{m}\sum_{k=0}^{n}p_jq_k(u_{mn}-u_{jn}-u_{mk}+u_{jk})\\
&\quad=\frac{1}{P_mQ_n}\sum_{j=0}^{m}\sum_{k=0}^{n}p_jq_k\big[(u_{mn}-u_{mk})-(u_{jn}-u_{jk})\big]\\
&\quad=\frac{1}{P_mQ_n}\sum_{j=1}^{m}\sum_{k=0}^{n}P_{j-1}\Delta_jq_k(u_{mk}-u_{jk})\\
&\quad=\frac{1}{P_mQ_n}\sum_{j=1}^{m}\sum_{k=1}^{n}P_{j-1}Q_{k-1}\Delta_{j,k}u_{jk}.
\end{aligned}
$$

***Remark* 1.** Similar to the weighted Kronecker identity, we can obtain

$$u_{mn}-t_{mn}^{(10)}(u)=V_{mn}^{(10)}(\Delta u), \tag{9}$$

where $V_{mn}^{(10)}(\Delta u)=\dfrac{1}{P_m}\sum_{j=0}^{m}P_{j-1}\Delta_ju_{jn}$, and

$$u_{mn}-t_{mn}^{(01)}(u)=V_{mn}^{(01)}(\Delta u), \tag{10}$$

where $V_{mn}^{(01)}(\Delta u)=\dfrac{1}{Q_n}\sum_{k=0}^{n}Q_{k-1}\Delta_ku_{mk}$.

Proofs of (9) and (10) are similar to that of Lemma 1. We omit it here. See [4] for the different proof.

Note that the sequences $\big(V_{mn}^{(10)}(\Delta u)\big)$ and $\big(V_{mn}^{(01)}(\Delta u)\big)$ are the sequence of $(\overline{N},p,*;1,0)$ means of the sequence $\left(\dfrac{P_{m-1}}{p_m}\Delta_mu_{mn}\right)$ and the sequence of $(\overline{N},*,q;0,1)$ means of the sequence $\left(\dfrac{Q_{n-1}}{q_n}\Delta_nu_{mn}\right)$, respectively.

In the next lemma, we give relations between the sequences $\big(V_{mn}^{(10)}(\Delta u)\big)$, $\big(V_{mn}^{(01)}(\Delta u)\big)$ and $\big(V_{mn}^{(11)}(\Delta u)\big)$.

Lemma 2. *The following identities are satisfied*:

$$V_{mn}^{(10)}\big(\Delta V^{(01)}(\Delta u)\big)=V_{mn}^{(11)}(\Delta u)$$

and

$$V_{mn}^{(01)}\big(\Delta V^{(10)}(\Delta u)\big)=V_{mn}^{(11)}(\Delta u).$$

Proof. From the definition of the sequence $\big(V_{mn}^{(10)}(\Delta u)\big)$, we have

$$\begin{aligned} V_{mn}^{(10)}\big(\Delta V^{(01)}(\Delta u)\big) &= V_{mn}^{(01)}(\Delta u) - t_{mn}^{(10)}\big(V^{(01)}(\Delta u)\big) \\ &= u_{mn} - t_{mn}^{(01)}(u) - t_{mn}^{(10)}(u) + t_{mn}^{(11)}(u) \\ &= V_{mn}^{(11)}(\Delta u). \end{aligned}$$

The proof of the second identity is similar to that of the first one. So, we omit the details.

Lemma 3. *The following identities are satisfied*:

(i) $\dfrac{P_{m-1}}{p_m}\Delta_m t_{mn}^{(10)}(u) = V_{mn}^{(10)}(\Delta u)$,

(ii) $\dfrac{Q_{n-1}}{q_n}\Delta_n t_{mn}^{(01)}(u) = V_{mn}^{(01)}(\Delta u)$.

Proof. (i) Taking backward difference of the sequence $t_{mn}^{(10)}(u)$ with respect to m and applying the Abel transformation for double sequences (see [1]), we have

$$\begin{aligned} \Delta_m t_{mn}^{(10)}(u) &= \Delta_m\left(\frac{1}{P_m}\sum_{j=0}^{m} p_j u_{jn}\right) \\ &= \frac{1}{P_m}\sum_{j=0}^{m} p_j u_{jn} - \frac{1}{P_{m-1}}\sum_{j=0}^{m-1} p_j u_{jn} \\ &= \frac{1}{P_m P_{m-1}}\left(P_{m-1}\sum_{j=0}^{m} p_j u_{jn} - P_m\sum_{j=0}^{m} p_j u_{jn}\right) \\ &= \frac{1}{P_m P_{m-1}}\left(P_{m-1}\sum_{j=0}^{m-1} p_j u_{jn} + P_{m-1}p_m u_{mn} - P_{m-1}\sum_{j=0}^{m-1} p_j u_{jn} - p_m\sum_{j=0}^{m-1} p_j u_{jn}\right). \end{aligned}$$

From this, we obtain

$$\Delta_m t_{mn}^{(10)}(u) = \frac{p_m}{P_m P_{m-1}}\sum_{j=1}^{m} P_{j-1}\Delta_j u_{jn} = \frac{p_m}{P_{m-1}}V_{mn}^{(10)}(\Delta u).$$

The identity (ii) can be verified in a similar way.

We note that the following lemma is given by Fekete [11]. We give a different proof for it.

Lemma 4. *Let* (u_{mn}) *be a double sequence. For sufficiently large* m *and* n,

(i) *if* $\lambda > 1$, *then*

$$\begin{aligned} u_{mn} - t_{mn}^{(11)}(u) &= \frac{P_{[\lambda m]}Q_{[\lambda n]}}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)}\Big(t_{[\lambda m],[\lambda n]}^{(11)}(u) - t_{[\lambda m],n}^{(11)}(u) - t_{m,[\lambda n]}^{(11)}(u) + t_{mn}^{(11)}(u)\Big) \\ &\quad + \frac{P_{[\lambda m]}}{P_{[\lambda m]} - P_m}\Big(t_{[\lambda m],n}^{(11)}(u) - t_{mn}^{(11)}(u)\Big) + \frac{Q_{[\lambda n]}}{Q_{[\lambda n]} - Q_n}\Big(t_{m,[\lambda n]}^{(11)}(u) - t_{mn}^{(11)}(u)\Big) \\ &\quad - \frac{1}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]} p_j q_k (u_{jk} - u_{mn}), \end{aligned}$$

(ii) *if* $0 < \lambda < 1$, *then*

$$u_{mn} - t^{(11)}_{mn}(u) = \frac{P_{[\lambda m]}Q_{[\lambda n]}}{(P_m - P_{[\lambda m]})(Q_n - Q_{[\lambda n]})}\Big(t^{(11)}_{mn}(u) - t^{(11)}_{[\lambda m],n}(u) - t^{(11)}_{m,[\lambda n]}(u) + t^{(11)}_{[\lambda m],[\lambda n]}(u)\Big)$$

$$+ \frac{P_{[\lambda m]}}{P_m - P_{[\lambda m]}}\Big(t^{(11)}_{mn}(u) - t^{(11)}_{[\lambda m],n}(u)\Big) + \frac{Q_{[\lambda n]}}{Q_n - Q_{[\lambda n]}}\Big(t^{(11)}_{mn}(u) - t^{(11)}_{m,[\lambda n]}(u)\Big)$$

$$+ \frac{1}{(P_m - P_{[\lambda m]})(Q_n - Q_{[\lambda n]})}\sum_{j=[\lambda m]+1}^{m}\sum_{k=[\lambda n]+1}^{n} p_j q_k (u_{mn} - u_{jk}).$$

Proof. (i) By the definition of the de la Vallée Poussin means of (u_{mn}) for $\lambda > 1$, we have

$$\tau^{>}_{mn}(u) = \frac{1}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]} p_j q_k u_{jk}$$

$$= \frac{1}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)}\left[\left(\sum_{j=0}^{[\lambda m]} - \sum_{j=0}^{m}\right)\left(\sum_{k=0}^{[\lambda n]} - \sum_{k=0}^{n}\right)\right] p_j q_k u_{jk}$$

$$= \frac{1}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)}\left(\sum_{j=0}^{[\lambda m]}\sum_{k=0}^{[\lambda n]} p_j q_k u_{jk} - \sum_{j=0}^{[\lambda m]}\sum_{k=0}^{n} p_j q_k u_{jk}\right.$$

$$\left. - \sum_{j=0}^{m}\sum_{k=0}^{[\lambda n]} p_j q_k u_{jk} + \sum_{j=0}^{m}\sum_{k=0}^{n} p_j q_k u_{jk}\right).$$

Therefore, we get

$$\tau^{>}_{mn}(u) = \frac{1}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)}$$

$$\times\Big(P_{[\lambda m]}Q_{[\lambda n]}t^{(11)}_{[\lambda m],[\lambda n]}(u) - P_{[\lambda m]}Q_n t^{(11)}_{[\lambda m],n}(u) - P_m Q_{[\lambda n]} t^{(11)}_{m,[\lambda n]}(u) + P_m Q_n t^{(11)}_{mn}(u)\Big)$$

and

$$\tau^{>}_{mn}(u) = \frac{P_{[\lambda m]}Q_{[\lambda n]}}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)} t^{(11)}_{[\lambda m],[\lambda n]}(u)$$

$$- \left[\frac{P_{[\lambda m]}Q_{[\lambda n]}}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)} t^{(11)}_{[\lambda m],n}(u) - \frac{P_{[\lambda m]}}{P_{[\lambda m]} - P_m} t^{(11)}_{[\lambda m],n}(u)\right]$$

$$- \left[\frac{P_{[\lambda m]}Q_{[\lambda n]}}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)} t^{(11)}_{m,[\lambda n]}(u) - \frac{Q_{[\lambda n]}}{Q_{[\lambda n]} - Q_n} t^{(11)}_{m,[\lambda n]}(u)\right]$$

$$+ \left[\frac{P_{[\lambda m]}Q_{[\lambda n]}}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)} t^{(11)}_{mn}(u) - \frac{P_{[\lambda m]}}{P_{[\lambda m]} - P_m} t^{(11)}_{mn}(u)\right.$$

$$-\frac{Q_{[\lambda n]}}{Q_{[\lambda n]}-Q_n}t^{(11)}_{mn}(u)+t^{(11)}_{mn}(u)\Bigg].$$

The difference $\tau^{>}_{mn}(u)-t^{(11)}_{mn}(u)$ can be written as

$$\tau^{>}_{mn}(u)-t^{(11)}_{mn}(u)=\frac{P_{[\lambda m]}Q_{[\lambda n]}}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\Big(t^{(11)}_{[\lambda m],[\lambda n]}(u)-t^{(11)}_{[\lambda m],n}(u)-t^{(11)}_{m,[\lambda n]}(u)+t^{(11)}_{m,n}(u)\Big)$$
$$+\frac{P_{[\lambda m]}}{P_{[\lambda m]}-P_m}\Big(t^{(11)}_{[\lambda m],n}(u)-t^{(11)}_{mn}(u)\Big)+\frac{Q_{[\lambda n]}}{Q_{[\lambda n]}-Q_n}\Big(t^{(11)}_{m,[\lambda n]}(u)-t^{(11)}_{mn}(u)\Big). \tag{11}$$

It follows from the identity

$$u_{mn}=\tau^{>}_{mn}(u)-\frac{1}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}p_jq_k(u_{jk}-u_{mn})$$

that

$$u_{mn}-t^{(11)}_{mn}(u)=(\tau^{>}_{mn}(u)-t^{(11)}_{mn}(u))-\frac{1}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}p_jq_k(u_{jk}-u_{mn}).$$

The proof is completed by using the identity (11).

The formula (ii) can be verified in a similar way.

***Remark* 2.** In analogy to Lemma 4, we have the following identities:

(i) for $\lambda>1$,

$$u_{mn}-t^{(10)}_{mn}(u)=\frac{P_{[\lambda m]}}{P_{[\lambda m]}-P_m}\Big(t^{(10)}_{[\lambda m],n}(u)-t^{(10)}_{mn}(u)\Big)-\frac{1}{P_{[\lambda m]}-P_m}\sum_{j=m+1}^{[\lambda m]}p_j(u_{jn}-u_{mn}),$$

(ii) for $0<\lambda<1$,

$$u_{mn}-t^{(10)}_{mn}(u)=\frac{P_{[\lambda m]}}{P_m-P_{[\lambda m]}}\Big(t^{(10)}_{mn}(u)-t^{(10)}_{[\lambda m],n}(u)\Big)+\frac{1}{P_m-P_{[\lambda m]}}\sum_{j=[\lambda m]+1}^{m}p_j(u_{mn}-u_{jn}).$$

We can show the identities as in the proof of the corresponding lemma for single sequences in [6]. We do not give details. Moreover, we note that we can represent the difference $u_{mn}-t^{(01)}_{mn}(u)$ in a similar way above.

Lemma 5. *Let (P_m) and (Q_n) be regularly varying of index δ.*

(i) *If (u_{mn}) is $(\overline{N},p,*;1,0)$ summable to s and the condition $\dfrac{P_{m-1}}{p_m}\Delta_m u_{mn}\geq -C$ is satisfied for some $C>0$, then (u_{mn}) is P-convergent to s.*

(ii) *If (u_{mn}) is $(\overline{N},*,q;0,1)$ summable to s and the condition $\dfrac{Q_{n-1}}{q_n}\Delta_n u_{mn}\geq -C$ is satisfied for some $C>0$, then (u_{mn}) is P-convergent to s.*

Proof. The proof of Lemma 5 can be done by the similar technique as in the proof of Theorem 3. So, we omit it.

Lemma 6. *Let (P_m) and (Q_n) be regularly varying of index δ.*

(i) *If (u_{mn}) is $(\overline{N}, p, *; 1, 0)$ summable to s and (u_{mn}) is slowly oscillating in sense $(1, 0)$, then (u_{mn}) is P-convergent to s.*

(ii) *If (u_{mn}) is $(\overline{N}, *, q; 0, 1)$ summable to s and (u_{mn}) is slowly oscillating in sense $(0, 1)$, then (u_{mn}) is P-convergent to s.*

Proof. The proof of Lemma 6 can be done by the similar technique as in the proof of Theorem 5. So, we omit it.

4. A one-sided Tauberian theorem for double sequences. In this section, we extend Hardy–Landau type Tauberian theorem given for single sequences to double sequences. Moreover, we establish a one-sided Tauberian theorem under some conditions on the sequence $\big(V_{mn}^{(11)}(\Delta u)\big)$ for the $(\overline{N}, p, q; 1, 1)$ summability method.

Theorem 3. *Let (P_m) and (Q_n) be regularly varying sequences of index δ, and suppose that (u_{mn}) is bounded. If (u_{mn}) is $(\overline{N}, p, q; 1, 1)$ summable to s and the conditions*

$$\frac{P_{m-1}}{p_m}\Delta_m V_{mn}^{(11)}(\Delta u) \geq -C, \qquad \frac{Q_{n-1}}{q_n}\Delta_n V_{mn}^{(11)}(\Delta u) \geq -C, \tag{12}$$

$$\frac{P_{m-1}}{p_m}\Delta_m V_{mn}^{(10)}(\Delta u) \geq -C, \qquad \frac{Q_{n-1}}{q_n}\Delta_n V_{mn}^{(01)}(\Delta u) \geq -C \tag{13}$$

are satisfied for some $C > 0$, then (u_{mn}) is P-convergent to s.

Proof. Since (u_{mn}) is bounded and $(\overline{N}, p, q; 1, 1)$ summable to s, $\big(t_{mn}^{(11)}(u)\big)$ is P-convergent to s. We know that $(\overline{N}, p, q; 1, 1)$, $(\overline{N}, p, *; 1, 0)$ and $(\overline{N}, *, q; 0, 1)$ summability methods are regular, so $\big(t_{mn}^{(11)}(u)\big)$ is $(\overline{N}, p, q; 1, 1)$ summable to s, $(t_{mn}^{(10)}(u))$ is $(\overline{N}, p, q; 1, 1)$ summable to s and $\big(t_{mn}^{(01)}(u)\big)$ is $(\overline{N}, p, q; 1, 1)$ summable to s. It follows from Lemma 1 that

$$\big(V_{mn}^{(11)}(\Delta u)\big) \text{ is } (\overline{N}, p, q; 1, 1) \quad \text{summable to} \quad 0. \tag{14}$$

If we replace u_{mn} by $V_{mn}^{(11)}(\Delta u)$ in Lemma 4(i), we have

$$\begin{aligned} &V_{mn}^{(11)}(\Delta u) - t_{mn}^{(11)}\big(V^{(11)}(\Delta u)\big) \\ &\quad= \frac{P_{[\lambda m]}Q_{[\lambda n]}}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)}\Big(t_{[\lambda m],[\lambda n]}^{(11)}\big(V^{(11)}(\Delta u)\big) - t_{[\lambda m],n}^{(11)}\big(V^{(11)}(\Delta u)\big) \\ &\qquad - t_{m,[\lambda n]}^{(11)}\big(V^{(11)}(\Delta u)\big) + t_{mn}^{(11)}\big(V^{(11)}(\Delta u)\big)\Big) \\ &\qquad + \frac{P_{[\lambda m]}}{P_{[\lambda m]} - P_m}\Big(t_{[\lambda m],n}^{(11)}\big(V^{(11)}(\Delta u)\big) - t_{mn}^{(11)}\big(V^{(11)}(\Delta u)\big)\Big) \\ &\qquad + \frac{Q_{[\lambda n]}}{Q_{[\lambda n]} - Q_n}\Big(t_{m,[\lambda n]}^{(11)}\big(V^{(11)}(\Delta u)\big) - t_{mn}^{(11)}\big(V^{(11)}(\Delta u)\big)\Big) \\ &\qquad - \frac{1}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]} p_j q_k\big(V_{jk}^{(11)}(\Delta u) - V_{mn}^{(11)}(\Delta u)\big). \end{aligned} \tag{15}$$

Since (P_m) and (Q_n) are regularly varying of index δ, it is plain that for all $\lambda > 1$ and sufficiently large $m, n \geq 0$,

$$\frac{\lambda^\delta}{2(\lambda^\delta-1)} \le \frac{P_{[\lambda m]}}{P_{[\lambda m]}-P_m} \le \frac{3\lambda^\alpha}{2(\lambda^\delta-1)}, \tag{16}$$

$$\frac{\lambda^\delta}{2(\lambda^\delta-1)} \le \frac{Q_{[\lambda n]}}{Q_{[\lambda n]}-Q_n} \le \frac{3\lambda^\delta}{2(\lambda^\delta-1)}. \tag{17}$$

By (14), (16) and (17), for all $\lambda > 1$, we obtain

$$\lim_{m,n\to\infty} \frac{P_{[\lambda m]}Q_{[\lambda n]}}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\Big(t^{(11)}_{[\lambda m],[\lambda n]}\big(V^{(11)}(\Delta u)\big) - t^{(11)}_{[\lambda m],n}\big(V^{(11)}(\Delta u)\big)$$

$$-t^{(11)}_{m,[\lambda n]}\big(V^{(11)}(\Delta u)\big) + t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big) = 0, \tag{18}$$

$$\lim_{m,n\to\infty} \frac{P_{[\lambda m]}}{P_{[\lambda m]}-P_m}\Big(t^{(11)}_{[\lambda m],n}\big(V^{(11)}(\Delta u)\big) - t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big) = 0, \tag{19}$$

and

$$\lim_{m,n\to\infty} \frac{Q_{[\lambda n]}}{Q_{[\lambda n]}-Q_n}\Big(t^{(11)}_{m,[\lambda n]}\big(V^{(11)}(\Delta u)\big) - t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big) = 0. \tag{20}$$

Taking $\limsup$ of both sides of the identity (15) as $m, n \to \infty$, we get

$$\limsup_{m,n\to\infty}\Big(V^{(11)}_{mn}(\Delta u) - t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)$$

$$\le \limsup_{m,n\to\infty} \frac{P_{[\lambda m]}Q_{[\lambda n]}}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\Big(t^{(11)}_{[\lambda m],[\lambda n]}\big(V^{(11)}(\Delta u)\big) - t^{(11)}_{[\lambda m],n}\big(V^{(11)}(\Delta u)\big)$$

$$-t^{(11)}_{m,[\lambda n]}\big(V^{(11)}(\Delta u)\big) + t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)$$

$$+\limsup_{m,n\to\infty} \frac{P_{[\lambda m]}}{P_{[\lambda m]}-P_m}\Big(t^{(11)}_{[\lambda m],n}\big(V^{(11)}(\Delta u)\big) - t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)$$

$$+\limsup_{m,n\to\infty} \frac{Q_{[\lambda n]}}{Q_{[\lambda n]}-Q_n}\Big(t^{(11)}_{m,[\lambda n]}\big(V^{(11)}(\Delta u)\big) - t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)$$

$$+\limsup_{m,n\to\infty}\left(-\frac{1}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]} p_j q_k (V^{(11)}_{jk}(\Delta u) - V^{(11)}_{mn}(\Delta u))\right).$$

Taking (18), (19), and (20) into account, we have

$$\limsup_{m,n\to\infty}\Big(V^{(11)}_{mn}(\Delta u) - t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)$$

$$\le \limsup_{m,n\to\infty}\left(-\frac{1}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]} p_j q_k \big(V^{(11)}_{jk}(\Delta u) - V^{(11)}_{mn}(\Delta u)\big)\right).$$

Hence, we conclude by the condition (12) that

$$\limsup_{m,n\to\infty}\Big(V^{(11)}_{mn}(\Delta u)-t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)$$

$$\leq \limsup_{m,n\to\infty}\Bigg(-\frac{1}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}p_jq_k$$

$$\times\Bigg(\sum_{r=m+1}^{j}\Delta_r V^{(11)}_{rk}(\Delta u)+\sum_{s=n+1}^{k}\Delta_s V^{(11)}_{ms}(\Delta u)\Bigg)\Bigg)$$

$$\leq C\limsup_{m,n\to\infty}\frac{1}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}p_jq_k\Bigg(\sum_{r=m+1}^{j}\frac{p_r}{P_{r-1}}+\sum_{s=n+1}^{k}\frac{q_s}{Q_{s-1}}\Bigg)$$

for some $C>0$. Therefore, we obtain

$$\limsup_{m,n\to\infty}\Big(V^{(11)}_{mn}(\Delta u)-t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)$$

$$\leq C\limsup_{m,n\to\infty}\frac{1}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}p_jq_k\left(\frac{P_j-P_m}{P_m}+\frac{Q_k-Q_n}{Q_n}\right)$$

$$\leq C\limsup_{m,n\to\infty}\left(\frac{P_{[\lambda m]}-P_m}{P_m}+\frac{Q_{[\lambda n]}-Q_n}{Q_n}\right)\frac{1}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}p_jq_k$$

$$= C\limsup_{m,n\to\infty}\left(\frac{P_{[\lambda m]}-P_m}{P_m}+\frac{Q_{[\lambda n]}-Q_n}{Q_n}\right).$$

Since (P_m) and (Q_n) are regularly varying sequences of index δ, we get

$$\limsup_{m,n\to\infty}\Big(V^{(11)}_{mn}(\Delta u)-t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)\leq 2C(\lambda^{\delta}-1).$$

Taking the limit of both sides as $\lambda\to 1^+$, we have

$$\limsup_{m,n\to\infty}\Big(V^{(11)}_{mn}(\Delta u)-t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)\leq 0. \tag{21}$$

In a similar way, using Lemma 4(ii), we obtain

$$\liminf_{m,n\to\infty}\Big(V^{(11)}_{mn}(\Delta u)-t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)\geq 0. \tag{22}$$

By the inequalities (21) and (22), we have

$$V^{(11)}_{mn}(\Delta u)=o(1). \tag{23}$$

Since (u_{mn}) is $(\overline{N},p,q;1,1)$ summable to s, then $\big(t^{(01)}_{mn}(u)\big)$ is $(\overline{N},p,*;1,0)$ summable to s. Moreover, $\big(t^{(10)}_{mn}(u)\big)$ is $(\overline{N},*,q;0,1)$ summable to s. As a result, we get that $\big(t^{(01)}_{mn}(V^{(10)}(\Delta u))\big)$ is

$(\overline{N}, p, *; 1, 0)$ summable to 0 by the identity (9) and $\left(t^{(10)}_{mn}\left(V^{(01)}(\Delta u)\right)\right)$ is $(\overline{N}, *, q; 0, 1)$ summable to 0 by the identity (10).

Using the identity (9), we obtain

$$\frac{P_{m-1}}{p_m}\Delta_m V^{(01)}_{mn}(\Delta u) - \frac{P_{m-1}}{p_m}\Delta_m t^{(10)}_{mn}\left(V^{(01)}(\Delta u)\right) = \frac{P_{m-1}}{p_m}\Delta_m V^{(11)}_{mn}(\Delta u).$$

By (23) and (12), Lemmas 2 and 3(i), it follows that

$$\frac{P_{m-1}}{p_m}\Delta_m V^{(01)}_{mn}(\Delta u) \geq -C$$

for some $C > 0$. Moreover, $\dfrac{P_{m-1}}{p_m}\Delta_m t^{(01)}_{mn}\left(V^{(01)}(\Delta u)\right) \geq -C$ for some $C > 0$. Since the sequence $\left(t^{(01)}_{mn}\left(V^{(01)}(\Delta u)\right)\right)$ is $(\overline{N}, p, *; 1, 0)$ summable to 0, then we get that $\left(t^{(01)}_{mn}\left(V^{(01)}(\Delta u)\right)\right)$ is P-convergent to 0 from Lemma 5(i). Therefore, we obtain that $\left(V^{(01)}_{mn}(\Delta u)\right)$ is $(\overline{N}, *, q; 0, 1)$ summable to 0. By condition (13) and Lemma 5(ii), we have

$$V^{(01)}_{mn}(\Delta u) = o(1). \tag{24}$$

Similarly, from (10), (23), Lemmas 2 and 3(ii), we obtain

$$\frac{Q_{n-1}}{q_n}\Delta_n V^{(10)}_{mn}(\Delta u) \geq -C$$

for some $C > 0$. Moreover, $\dfrac{Q_{n-1}}{q_n}\Delta_n t^{(10)}_{mn}(V^{(10)}(\Delta u)) \geq -C$, for some $C > 0$. Since the sequence $\left(t^{(10)}_{mn}(V^{(10)}(\Delta u))\right)$ is $(\overline{N}, p, *; 1, 0)$ summable to 0, then we have that $\left(t^{(10)}_{mn}(V^{(10)}(\Delta u))\right)$ is P-convergent to 0 from Lemma 5(ii). Hence, we obtain that $\left(V^{(10)}_{mn}(\Delta u)\right)$ is $(\overline{N}, p, *; 1, 0)$ summable to 0. By condition (13) and Lemma 5(i), we get

$$V^{(10)}_{mn}(\Delta u) = o(1). \tag{25}$$

The proof is completed by using Lemma 1 by (23), (24), and (25).

The following corollary is a version of the Hardy–Landau theorem for the $(\overline{N}, p, q; 1, 1)$ summability method given by Chen and Hsu [5].

***Corollary* 1.** *Let (P_m) and (Q_n) be regularly varying of index δ, and suppose that (u_{mn}) is bounded. If (u_{mn}) is $(\overline{N}, p, q; 1, 1)$ summable to s, the conditions*

$$\frac{P_{m-1}}{p_m}\Delta_m u_{mn} \geq -C,$$

$$\frac{Q_{n-1}}{q_n}\Delta_n u_{mn} \geq -C$$

are satisfied for some $C > 0$, then (u_{mn}) is P-convergent to s.

Proof. Since (u_{mn}) is $(\overline{N}, p, q; 1, 1)$ summable to s, $\left(t^{(11)}_{mn}(u)\right)$ is P-convergent to s. If we apply a similar calculation for $V^{(11)}_{mn}(\Delta u)$ as in the proof of Theorem 3 to the sequence (u_{mn}), one can easily obtain that (u_{mn}) is P-convergent to s.

If we take $p_m = q_n = 1$ for all $m,\ n$ in Corollary 1, we present the following classical Tauberian result for the $(C, 1, 1)$ summability.

***Corollary* 2** [17]**.** *Suppose that* (u_{mn}) *is bounded. If* (u_{mn}) *is* $(C,1,1)$ *summable to* s, *the conditions*

$$m\Delta_m u_{mn} \geq -C,$$

$$n\Delta_n u_{mn} \geq -C$$

are satisfied for some $C > 0$, *then* (u_{mn}) *is* P*-convergent to* s.

Finally, the following theorem can be given in this section.

Theorem 4. *Let* (P_m) *and* (Q_n) *be regularly varying of index* δ, *and suppose that* (u_{mn}) *is bounded the conditions* (12) *is satisfied. If* (u_{mn}) *is* $(\overline{N},p,*;1,0)$ *summable to* s *and* $(\overline{N},*,q;0,1)$ *summable to* s, *then* (u_{mn}) *is* P*-convergent to* s.

Proof. The conditions that (u_{mn}) is $(\overline{N},p,*;1,0)$ summable to s, and $(\overline{N},*,q;0,1)$ summable to s imply that (u_{mn}) is $(\overline{N},p,q;1,1)$ summable to s. We obtain that $\big(V^{(11)}_{mn}(\Delta u)\big)$ is P-convergent to 0 by condition (12) and Lemma 4 as in the proof of Theorem 3. Since the sequences $\big(t^{(10)}_{mn}(u)\big)$ and $\big(t^{(01)}_{mn}(u)\big)$ are P-convergent to s, then the proof is completed by Lemma 1.

5. A Schmidt's type theorem for double sequences. In this section, we obtain some new Tauberian theorems under some conditions related to the oscillatory behavior of a double sequence $\big(V^{(11)}_{mn}(\Delta u)\big)$ or (u_{mn}).

Theorem 5. *Let* (P_m) *and* (Q_n) *be regularly varying of positive index* δ, *and suppose that* (u_{mn}) *is bounded. If* (u_{mn}) *is* $(\overline{N},p,q;1,1)$ *summable to* s *and* $\big(V^{(11)}_{mn}(\Delta u)\big)$ *is slowly oscillating in senses* $(1,0)$ *and* $(0,1)$, *then* (u_{mn}) *is* P*-convergent to* s.

Proof. Since (u_{mn}) is bounded and $(\overline{N},p,q;1,1)$ summable to s, it can be verified in exactly the same way as in Theorem 3 that $\big(V^{(11)}_{mn}(\Delta u)\big)$ is $(\overline{N},p,q;1,1)$ summable to 0.

If we replace u_{mn} by $V^{(11)}_{mn}(\Delta u)$ in Lemma 4(i), we obtain

$$\begin{aligned}
&V^{(11)}_{mn}(\Delta u) - t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\\
&\quad= \frac{P_{[\lambda m]}Q_{[\lambda n]}}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\Big(t^{(11)}_{[\lambda m],[\lambda n]}\big(V^{(11)}(\Delta u)\big) - t^{(11)}_{[\lambda m],n}\big(V^{(11)}(\Delta u)\big)\\
&\qquad -t^{(11)}_{m,[\lambda n]}\big(V^{(11)}(\Delta u)\big) + t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)\\
&\qquad + \frac{P_{[\lambda m]}}{P_{[\lambda m]}-P_m}\Big(t^{(11)}_{[\lambda m],n}\big(V^{(11)}(\Delta u)\big) - t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)\\
&\qquad + \frac{Q_{[\lambda n]}}{Q_{[\lambda n]}-Q_n}\Big(t^{(11)}_{m,[\lambda n]}\big(V^{(11)}(\Delta u)\big) - t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big)\\
&\qquad - \frac{1}{(P_{[\lambda m]}-P_m)(Q_{[\lambda n]}-Q_n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]} p_j q_k\Big(V^{(11)}_{jk}(\Delta u) - V^{(11)}_{mn}(\Delta u)\Big)
\end{aligned}$$

for $\lambda > 1$. From this, we get

$$\Big|V^{(11)}_{mn}(\Delta u) - t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big|$$

$$= \left| \frac{P_{[\lambda m]} Q_{[\lambda n]}}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)} \left(t^{(11)}_{[\lambda m],[\lambda n]} \big(V^{(11)}(\Delta u)\big) - t^{(11)}_{[\lambda m],n} \big(V^{(11)}(\Delta u)\big) \right.\right.$$

$$\left.\left. - t^{(11)}_{m,[\lambda n]} \big(V^{(11)}(\Delta u)\big) + t^{(11)}_{mn} \big(V^{(11)}(\Delta u)\big) \right) \right|$$

$$+ \left| \frac{P_{[\lambda m]}}{P_{[\lambda m]} - P_m} \left(t^{(11)}_{[\lambda m],n} \big(V^{(11)}(\Delta u)\big) - t^{(11)}_{mn} \big(V^{(11)}(\Delta u)\big) \right) \right|$$

$$+ \left| \frac{Q_{[\lambda n]}}{Q_{[\lambda n]} - Q_n} \left(t^{(11)}_{m,[\lambda n]} \big(V^{(11)}(\Delta u)\big) - t^{(11)}_{mn} \big(V^{(11)}(\Delta u)\big) \right) \right|$$

$$+ \left| - \frac{1}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} p_j q_k (V^{(11)}_{jk}(\Delta u) - V^{(11)}_{mn}(\Delta u)) \right|. \qquad (26)$$

Since the sequence $\left(t^{(11)}_{mn} \big(V^{(11)}(\Delta u)\big) \right)$ is P-convergent, (P_m) and (Q_n) are regularly varying of positive indices, then the terms on the right-hand side of the last inequality vanish.

From the last term on the right-hand side of the inequality (26), we have

$$\left| - \frac{1}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} p_j q_k (V^{(11)}_{jk}(\Delta u) - V^{(11)}_{mn}(\Delta u)) \right|$$

$$\leq \frac{1}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} p_j q_k$$

$$\times \left(\left| \sum_{r=m+1}^{j} \Delta_r V^{(11)}_{rk}(\Delta u) \right| + \left| \sum_{s=n+1}^{k} \Delta_s V^{(11)}_{ms}(\Delta u) \right| \right)$$

and then

$$\left| - \frac{1}{(P_{[\lambda m]} - P_m)(Q_{[\lambda n]} - Q_n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} p_j q_k (V^{(11)}_{jk}(\Delta u) - V^{(11)}_{mn}(\Delta u)) \right|$$

$$\leq \max_{m+1 \leq j \leq [\lambda m]} \left| \sum_{r=m+1}^{j} \Delta_r V^{(11)}_{rk}(\Delta u) \right| + \max_{n+1 \leq k \leq [\lambda n]} \left| \sum_{s=n+1}^{k} \Delta_s V^{(11)}_{ms}(\Delta u) \right|.$$

Taking $\limsup$ of both sides of the inequality (26) as $m, n \to \infty$, then we obtain

$$\limsup_{m,n \to \infty} \left| V^{(11)}_{mn}(\Delta u) - t^{(11)}_{mn} \big(V^{(11)}(\Delta u)\big) \right|$$

$$\leq \limsup_{m,n \to \infty} \max_{m+1 \leq j \leq [\lambda m]} \left| \sum_{r=m+1}^{j} \Delta_r V^{(11)}_{rk}(\Delta u) \right| + \limsup_{m,n \to \infty} \max_{n+1 \leq k \leq [\lambda n]} \left| \sum_{s=n+1}^{k} \Delta_s V^{(11)}_{ms}(\Delta u) \right|.$$

Therefore, taking the limit of both sides as $\lambda \to 1^+$, we get

$$\limsup_{m,n\to\infty}\Big|V^{(11)}_{mn}(\Delta u)-t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big|$$

$$\le \lim_{\lambda\to1^+}\limsup_{m,n\to\infty}\max_{m+1\le j\le[\lambda m]}\left|\sum_{r=m+1}^{j}\Delta_r V^{(11)}_{rk}(\Delta u)\right|$$

$$+\lim_{\lambda\to1^+}\limsup_{m,n\to\infty}\max_{n+1\le k\le[\lambda n]}\left|\sum_{s=n+1}^{k}\Delta_s V^{(11)}_{ms}(\Delta u)\right|.$$

Since $\big(V^{(11)}_{mn}(\Delta u)\big)$ is slowly oscillating in sense $(1,0)$ and $(0,1)$, then

$$\limsup_{m,n\to\infty}\Big|V^{(11)}_{mn}(\Delta u)-t^{(11)}_{mn}\big(V^{(11)}(\Delta u)\big)\Big|\le 0.$$

Hence, we obtain

$$V^{(11)}_{mn}(\Delta u)=o(1). \tag{27}$$

On the other hand, if (u_{mn}) is $(\overline{N},p,q;1,1)$ summable to s, then $\big(t^{(01)}_{mn}(u)\big)$ is $(\overline{N},p,*;1,0)$ summable to s. Moreover, $(t^{(10)}_{mn}(u))$ is $(\overline{N},*,q;0,1)$ summable to s. Therefore, we get that $\big(V^{(01)}_{mn}(\Delta u)\big)$ is $(\overline{N},p,*;1,0)$ summable to 0 by the identity (10), and $\big(V^{(10)}_{mn}(\Delta u)\big)$ is $(\overline{N},*,q;0,1)$ summable to 0 by the identity (9).

Since the boundedness of the sequence (u_{mn}) implies the boundedness of the sequence $\big(V^{(11)}_{mn}(\Delta u)\big)$, it follows from Lemmas 2 and 3 that $\Big(t^{(10)}_{mn}(V^{(01)}(\Delta u))\Big)$ is slowly oscillating in sense $(1,0)$. Using the identity (9), we have

$$V^{(01)}_{mn}(\Delta u)-t^{(10)}_{mn}\big(V^{(01)}(\Delta u)\big)=V^{(11)}_{mn}(\Delta u).$$

Hence, we have the slow oscillation of the sequence $\big(V^{(01)}_{mn}(\Delta u)\big)$ in sense $(1,0)$. Therefore, we obtain

$$V^{(01)}_{mn}(\Delta u)=o(1) \tag{28}$$

by Lemma 6(i). Similarly, since the sequence $\big(V^{(11)}_{mn}(\Delta u)\big)$ is P-convergent and bounded, then $\big(V^{(10)}_{mn}(\Delta u)\big)$ is slowly oscillating in sense $(0,1)$ by Lemmas 2 and 3. Hence, we get

$$V^{(10)}_{mn}(\Delta u)=o(1) \tag{29}$$

by Lemma 6(ii).

Lemma 1 completes the proof by (27), (28), and (29).

The following corollary is a version of the generalized Littlewood theorem for the $(\overline{N},p,q;1,1)$ integrability method given by Chen and Hsu [5].

***Corollary* 3.** *Let (P_m) and (Q_n) be regularly varying of index δ, and suppose that (u_{mn}) is bounded. If (u_{mn}) is $(\overline{N},p,q;1,1)$ summable to s and (u_{mn}) is slowly oscillating in senses $(1,0)$ and $(0,1)$, then (u_{mn}) is P-convergent to s.*

Proof. Since (u_{mn}) is $(\overline{N},p,q;1,1)$ summable to s, it is clear that $\big(t^{(11)}_{mn}(u)\big)$ is P-convergent to s. If we apply similar calculation for $\big(V^{(11)}_{mn}(\Delta u)\big)$ as in the proof of Theorem 5 to the sequence (u_{mn}), we conclude that (u_{mn}) is P-convergent to s.

If we take $p_m=q_n=1$ for all m,n in Corollary 3, we present the following classical Tauberian result for the $(C,1,1)$ summability.

***Corollary* 4** [17]. *Suppose that (u_{mn}) is bounded. If (u_{mn}) is $(C,1,1)$ summable to s and (u_{mn}) is slowly oscillating in senses $(1,0)$ and $(0,1)$, then (u_{mn}) is P-convergent to s.*

Finally, the following theorem can be given in this section.

Theorem 6. *Let (P_m) and (Q_n) be regularly varying of index δ, and suppose that (u_{mn}) is bounded. If (u_{mn}) is $(\overline{N},p,*;1,0)$ summable to s, $(\overline{N},*,q;0,1)$ summable to s, and $\big(V^{(11)}_{mn}(\Delta u)\big)$ is slowly oscillating in senses $(1,0)$ and $(0,1)$, then (u_{mn}) is P-convergent to s.*

Proof. The conditions that (u_{mn}) is $(\overline{N},p,*;1,0)$ summable to s, and $(\overline{N},*,q;0,1)$ summable to s imply that (u_{mn}) is $(\overline{N},p,q;1,1)$ summable to s. We obtain that $\big(V^{(11)}_{mn}(\Delta u)\big)$ is P-convergent to 0 by the slow oscillation of $\big(V^{(11)}_{mn}(\Delta u)\big)$ in sense $(1,0)$ and $(0,1)$ and Lemma 4 as the proof of Theorem 5. Since the sequences $(t^{(10)}_{mn}(u))$ and $\big(t^{(01)}_{mn}(u)\big)$ are P-convergent to s, then the proof is completed by Lemma 1.

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