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COHOMOLOGY AND FORMAL DEFORMATIONS OF n-HOM – LIE COLOR ALGEBRAS

КОГОМОЛОГІЇ ТА ФОРМАЛЬНІ ДЕФОРМАЦІЇ АЛГЕБР КОЛЬОРІВ n-ХОМА – ЛІ

The aim of this paper is to provide a cohomology of n-Hom – Lie color algebras, in particular, a cohomology governing one-parameter formal deformations. Then we also study formal deformations of the n-Hom – Lie color algebras and introduce the notion of Nijenhuis operator on a n-Hom – Lie color algebra, which may give rise to infinitesimally trivial $(n-1)$-order deformations. Furthermore, in connection with Nijenhuis operators, we introduce and discuss the notion of product structure on n-Hom – Lie color algebras.

Мета цієї статті — визначити когомологію алгебри кольорів n-Хома – Лі та, зокрема, когомологію, що керує формальними однопараметричними деформаціями. Крім того, вивчаються формальні деформації алгебри кольорів n-Хома – Лі та введено поняття оператора Нойєнгайса на алгебрі кольорів n-Хома – Лі, що може привести до інфінітезимально тривіальної деформації $(n-1)$-го порядку. Крім того, у зв'язку з операторами Нойєнгайса введено та обговорено поняття структури добутку на алгебрах кольорів n-Хома – Лі.

1. Introduction. The generalization of Lie algebra, which is now known as Lie color algebra was introduced first by Ree [22]. This class includes Lie superalgebras which are $\mathbb{Z}_2$-graded and play an important role in supersymmetries. More generally, Lie color algebras play an important role in theoretical physics, see, for example, [26, 27]. Montgomery proved in [21] that simple Lie color algebras can be obtained from associative graded algebras, while the Ado theorem and the PBW theorem of Lie color algebras were proven by Scheunert [25]. In the last two decades, Lie color algebras have been developed as an interesting topic in mathematics and physics (see [8, 11, 12, 16, 19, 31] for more details).

Ternary Lie algebras and more generally n-ary Lie algebras are natural generalization of binary Lie algebras, where one considers n-ary operations and a generalization of Jacobi condition. The most common generalization consists of expressing the adjoint map as a derivation. The corresponding algebras were called n-Lie algebras and were first introduced and studied by Filippov in [15] and then completed by Kasymov in [18]. These algebras, in the ternary cas, appeared in the mathematical algebraic foundations of Nambu mechanics developed by Takhtajan and Daletskii in [14, 28, 29], as a generalization of Hamiltonian mechanics involving more than one Hamiltonian. Besides Nambu mechanics, n-Lie algebras revealed to have many applications in physics like string theory. The

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second approach of generalizing Jacobi condition to n-ary case consists of considering a summation over S_{2n-1} instead of S_3.

Hom-type generalizations of n-ary algebras were considered first in [9], where n-Hom – Lie algebras and other n-ary Hom-algebras of Lie type and associative type were introduced. The usual identities are twisted by linear maps. As a particular case one recovers Hom – Lie algebras which were motivated by quantum deformations of algebras of vector fields like Witt and Virasoro algebras. Further properties, construction methods, examples, cohomology and central extensions of n-ary Hom-algebras have been considered in [5 – 7].

A (co)homology theory with adjoint representation for n-Lie algebras was introduced by Takhtajan in [14, 29] and by Gautheron in [17] from deformation theory viewpoint. The general cohomology theory for n-Lie algebras, Leibniz n-algebras were established in [13, 23] and n-Hom – Lie algebras and superalgebras in [1, 3, 4].

Inspired by these works, we aim to study a cohomology and deformations of graded n-Hom – Lie algebras. Moreover, we consider a notion of Nijenhuis operator in connection with the study of $(n-1)$-order deformation of graded n-Hom – Lie algebras. In particular, we discuss a notion of product structure.

This paper is organized as follows. In Section 2, we recall some basic definitions on n-Hom – Lie color algebras. Section 3 is devoted to various constructions of n-Hom – Lie color algebras and Hom – Leibniz color algebras. Furthermore, we introduce a notion of representation of a n-Hom – Lie color algebra and construct the corresponding semidirect product. In Section 4, we study cohomologies with respect to given representations. In Section 5, we discuss formal and infinitesimal deformations of a n-Hom – Lie color algebra. Finally, in Section 6, we introduce a notion of Nijenhuis operators, which is connected to infinitesimally trivial $(n-1)$-order deformations. Moreover, we define a product structure on n-Hom – Lie color algebras using Nijenhuis conditions.

2. Basics on n-Hom – Lie color algebras. This section contains preliminaries and definitions on graded spaces, algebras and n-Hom – Lie color algebras which correspond to the graded case of n-Hom – Lie algebras (see [2, 10, 30] for more details).

Throughout this paper $\mathbb{K}$ will denote a commutative field of characteristic zero and Γ will stand for an Abelian group. A vector space $\mathfrak{g}$ is said to be a Γ-graded if we are given a family $(\mathfrak{g}_\gamma)_{\gamma\in\Gamma}$ of vector subspaces of $\mathfrak{g}$ such that $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$. An element $x \in \mathfrak{g}_\gamma$ is said to be homogeneous of degree γ. The set of homogeneous elements is denoted by $\mathcal{H}(\mathfrak{g})$. If the base field is considered as a graded vector space, it is understood that the graduation of $\mathbb{K}$ is given by $\mathbb{K}_0 = \mathbb{K}$ and $\mathbb{K}_\gamma = \{0\}$, if $\gamma \in \Gamma \setminus \{0\}$. Now, let $\mathfrak{g}$ and $\mathfrak{h}$ be two Γ-graded vector spaces. A linear map $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ is said to be homogeneous of degree $\xi \in \Gamma$, if $f(x)$ is homogeneous of degree $\gamma + \xi$ whenever the element $x \in \mathfrak{g}_\gamma$. The set of all linear maps of degree ξ will be denoted by $\mathrm{Hom}(\mathfrak{g},\mathfrak{h})_\xi$. Then the vector space of all linear maps of $\mathfrak{g}$ into $\mathfrak{h}$ is Γ-graded and denoted by $\mathrm{Hom}(\mathfrak{g},\mathfrak{h}) = \bigoplus_{\xi\in\Gamma} \mathrm{Hom}(\mathfrak{g},\mathfrak{h})_\xi$.

We mean by algebra (resp., Γ-graded algebra) $(\mathfrak{g}, \cdot)$ a vector space (resp., Γ-graded vector space) with multiplication, which we denote by the concatenation, such that $\mathfrak{g}_\gamma \mathfrak{g}_{\gamma'} \subseteq \mathfrak{g}_{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. In a graded case, a map $f : \mathfrak{g} \longrightarrow \mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are Γ-graded algebras, is called a Γ-graded algebra homomorphism if it is a degree zero algebra homomorphism.

We mean by Hom-algebra (resp., Γ-graded Hom-algebra) a triple $(\mathfrak{g}, \cdot, \alpha)$ consisting of a vector space (resp., Γ-graded vector space), a multiplication and an endomorphism α (twist map) (resp., degree zero endomorphism α).

For more detail about graded algebraic structures, we refer to [25]. In the following, we recall the definition of bicharacter on an Abelian group Γ.

Definition 2.1. *Let Γ be an Abelian group. A map $\varepsilon\colon \Gamma\times\Gamma\to\mathbb{K}\setminus\{0\}$ is called a bicharacter on Γ if the following identities are satisfied*:

$$\varepsilon(\gamma_1,\gamma_2)\varepsilon(\gamma_2,\gamma_1)=1,$$

$$\varepsilon(\gamma_1,\gamma_2+\gamma_3)=\varepsilon(\gamma_1,\gamma_2)\varepsilon(\gamma_1,\gamma_3),$$

$$\varepsilon(\gamma_1+\gamma_2,\gamma_3)=\varepsilon(\gamma_1,\gamma_3)\varepsilon(\gamma_2,\gamma_3)\quad\forall\gamma_1,\gamma_2,\gamma_3\in\Gamma.$$

In particular, the definition above implies the relations

$$\varepsilon(\gamma,0)=\varepsilon(0,\gamma)=1,\quad \varepsilon(\gamma,\gamma)=\pm1\quad\text{for all}\quad\gamma\in\Gamma.$$

Let $\mathfrak{g}=\bigoplus_{\gamma\in\Gamma}\mathfrak{g}_\gamma$ be a Γ-graded vector space. If x and x' are two homogeneous elements in $\mathfrak{g}$ of degree γ and γ', respectively, and ε is a bicharacter, then we shorten the notation by writing $\varepsilon(x,x')$ instead of $\varepsilon(\gamma,\gamma')$. If $X=(x_1,\ldots,x_p)\in\otimes^p\mathfrak{g}$, we set

$$\varepsilon(x,X_i)=\varepsilon\left(x,\sum_{k=1}^{i-1}x_k\right)\quad\text{for}\quad i>1\qquad\text{and}\qquad\varepsilon(x,X_i)=1\quad\text{for}\quad i=1,$$

$$\varepsilon(x,X^i)=\varepsilon\left(x,\sum_{k=i+1}^{p}x_i\right)\quad\text{for}\quad i<p\qquad\text{and}\qquad\varepsilon(x,X^i)=1\quad\text{for}\quad i=p,$$

$$\varepsilon(x,X_i^j)=\varepsilon\left(x,\sum_{k=i}^{j}x_k\right).$$

Then we define the general linear Lie color algebra $gl(\mathfrak{g})=\bigoplus_{\gamma\in\Gamma}gl(\mathfrak{g})_\gamma$, where

$$gl(\mathfrak{g})_\gamma=\big\{f:\mathfrak{g}\to\mathfrak{g}/f(\mathfrak{g}_{\gamma'})\subset\mathfrak{g}_{\gamma+\gamma'}\ \text{and}\ \alpha\circ f=f\circ\alpha\ \text{for all}\ \gamma'\in\Gamma\big\}.$$

In the following, we recall the notion of n-Hom – Lie color algebra given by Bakayoko and Silvestrov in [10], which is a generalization of n-Hom – Lie superalgebra introduced in [1].

Definition 2.2. *A n-Hom – Lie color algebra is a graded vector space $\mathfrak{g}=\bigoplus_{\gamma\in\Gamma}\mathfrak{g}_\gamma$ with a multilinear map $[\cdot,\ldots,\cdot]\colon\mathfrak{g}\times\ldots\times\mathfrak{g}\longrightarrow\mathfrak{g}$, a bicharacter $\varepsilon\colon\Gamma\times\Gamma\to\mathbb{K}\setminus\{0\}$ and a linear map $\alpha\colon\mathfrak{g}\longrightarrow\mathfrak{g}$ of degree zero such that*

$$\big[x_1,\ldots,x_i,x_{i+1},\ldots,x_n\big]=-\varepsilon(x_i,x_{i+1})\big[x_1,\ldots,x_{i+1},x_i,\ldots,x_n\big],\tag{2.1}$$

$$\big[\alpha(x_1),\ldots,\alpha(x_{n-1}),[y_1,y_2,\ldots,y_n]\big]$$
$$=\sum_{i=1}^{n}\varepsilon(X,Y_i)\big[\alpha(y_1),\ldots,\alpha(y_{i-1}),[x_1,x_2,\ldots,y_i],\alpha(y_{i+1}),\ldots,\alpha(y_n)\big].\tag{2.2}$$

The identity (2.2) is called *ε-n-Hom – Jacobi identity* and Eq. (2.1) is equivalent to

$$[x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n]=-\varepsilon(x_i,X_{i+1}^{j-1})\varepsilon(X_{i+1}^{j-1},x_j)\varepsilon(x_i,x_j)[x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n].\tag{2.3}$$

Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ and $(\mathfrak{g}', [\cdot, \ldots, \cdot]', \varepsilon, \alpha')$ be two n-Hom – Lie color algebras. A linear map of degree zero $f : \mathfrak{g} \to \mathfrak{g}'$ is a n-Hom – Lie color algebra *morphism* if it satisfies

$$f \circ \alpha = \alpha' \circ f,$$

$$f([x_1, \ldots, x_n]) = [f(x_1), \ldots, f(x_n)]'.$$

Definition 2.3. *Let* $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ *be a* n*-Hom – Lie color algebra. It is called*
multiplicative n*-Hom – Lie color algebra if* $\alpha[x_1, \ldots, x_n] = [\alpha(x_1), \ldots, \alpha(x_n)]$,
regular n*-Hom – Lie color algebra if* α *is an automorphism,*
involutive n*-Hom – Lie color algebra if* $\alpha^2 = Id$.

***Remarks* 2.1.** 1. When $\Gamma = \{0\}$, the trivial group, we recover ordinary n-Hom – Lie algebras (see [9] for more details).

2. When $\Gamma = \mathbb{Z}_2$, $\varepsilon(x, y) = (-1)^{|x||y|}$, we obtain n-Hom – Lie superalgebras defined in [1].
3. If $n = 2$ (resp., $n = 3$) we recover Hom – Lie color algebras (resp., 3-Hom – Lie color algebras).
4. When $\alpha = Id$, we get n-Lie color algebras.

Definition 2.4. *Let* $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ *be a* n*-Hom – Lie color algebra. Then*:

1. *A* Γ*-graded subspace* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is a color subalgebra of* $\mathfrak{g}$, *if, for all* $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$,

$$\alpha(\mathfrak{h}_{\gamma_1}) \subseteq \mathfrak{h}_{\gamma_1} \qquad \textit{and} \qquad [\mathfrak{h}_{\gamma_1}, \mathfrak{h}_{\gamma_2}, \ldots, \mathfrak{h}_{\gamma_n}] \subseteq \mathfrak{h}_{\gamma_1+\ldots+\gamma_n}.$$

2. *A color ideal* $\mathfrak{I}$ *of* $\mathfrak{g}$ *is a color Hom-subalgebra of* $\mathfrak{g}$ *such that*, *for all* $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$,

$$\alpha(\mathfrak{I}_{\gamma_1}) \subseteq \mathfrak{I}_{\gamma_1} \qquad \textit{and} \qquad [\mathfrak{I}_{\gamma_1}, \mathfrak{g}_{\gamma_2}, \ldots, \mathfrak{g}_{\gamma_n}] \subseteq \mathfrak{I}_{\gamma_1+\ldots+\gamma_n}.$$

3. *A center of* $\mathfrak{g}$ *is the set* $Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y_1, \ldots, y_{n-1}] = 0 \ \forall y_1, \ldots, y_n \in \mathfrak{g}\}$. *It is easy to show that* $Z(\mathfrak{g})$ *is a color ideal of* $\mathfrak{g}$.

3. Constructions and representations of n-Hom – Lie color algebras. In this section, we show some constructions of n-Hom – Lie color algebras and Hom – Leibniz color algebras associated to n-Hom – Lie color algebras. Moreover, we introduce a notion of representation of n-Hom – Lie color algebras and construct the corresponding semidirect product.

3.1. Yau twist of n-Hom – Lie color algebras. In the following theorem, starting from a n-Hom – Lie color algebra and a n-Hom – Lie color algebra endomorphism, we construct a new n-Hom – Lie color algebra. We say that it is obtained by Yau twist.

Theorem 3.1. *Let* $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ *be a* n*-Hom – Lie color algebra and* $\beta : \mathfrak{g} \longrightarrow \mathfrak{g}$ *be a n-Hom – Lie color algebra endomorphism of degre zero. Then* $(\mathfrak{g}, [\cdot, \ldots, \cdot]_\beta, \varepsilon, \beta\circ\alpha)$, *where* $[\cdot, \ldots, \cdot]_\beta = \beta \circ [\cdot, \ldots, \cdot]$, *is a n-Hom – Lie color algebra.*

Moreover, *suppose that* $(\mathfrak{g}', [\cdot, \ldots, \cdot]', \varepsilon, \alpha')$ *is a* n*-Hom – Lie color algebra and* $\beta' : \mathfrak{g}' \longrightarrow \mathfrak{g}'$ *is a* n*-Hom – Lie color algebra endomorphism. If* $f : \mathfrak{g} \longrightarrow \mathfrak{g}'$ *is a* n*-Hom – Lie color algebra morphism that satisfies* $f \circ \beta = \beta' \circ f$, *then*

$$f : (\mathfrak{g}, [\cdot, \ldots, \cdot]_\beta, \varepsilon, \beta \circ \alpha) \longrightarrow (\mathfrak{g}', [\cdot, \ldots, \cdot]'_{\beta'}, \varepsilon, \beta' \circ \alpha')$$

is a morphism of n*-Hom – Lie color algebras.*

Proof. Obviously $[\cdot, \ldots, \cdot]_\beta$ is a ε-skew-symmetric. Furthermore, $(\mathfrak{g}, [\cdot, \ldots, \cdot]_\beta, \varepsilon, \beta\circ\alpha)$ satisfies the ε-n-Hom – Jacobi identity (2.2). Indeed,

$$[\beta \circ \alpha(x_1), \ldots, \beta \circ \alpha(x_{n-1}), [y_1, \ldots, y_n]_\beta]_\beta = \beta^2\big([\alpha(x_1), \ldots, \alpha(x_{n-1}), [y_1, \ldots, y_n]]\big)$$

$$= \sum_{i=1}^{n} \varepsilon(X, Y_i)\beta^2\Big(\big[\alpha(y_1), \ldots, \alpha(y_{i-1}), [x_1, x_2, \ldots, y_i]\alpha(y_{i+1}), \ldots, \alpha(y_n)\big]\Big)$$

$$= \sum_{i=1}^{n} \varepsilon(X, Y_i) \big[\beta \circ \alpha(y_1), \ldots, \beta \circ \alpha(y_{i-1}), [x_1, x_2, \ldots, y_i]_\beta, \beta \circ \alpha(y_{i+1}), \ldots, \beta \circ \alpha(y_n)\big]_\beta.$$

The second assertion follows from

$$f\big([x_1, \ldots, x_n]_\beta\big) = [f \circ \beta(x_1), \ldots, f \circ \beta(x_n)]'$$
$$= [\beta' \circ f(x_1), \ldots, \beta' \circ f(x_n)]' = [f(x_1), \ldots, f(x_n)]'_{\beta'}.$$

Theorem 3.1 is proved.

In particular, we have the following construction of n-Hom – Lie color algebra using n-Lie color algebras and n-Lie color algebra morphisms.

***Corollary* 3.1.** *Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon)$ be a n-Lie color algebra and $\alpha\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ be a n-Lie color algebra endomorphism. Then $(\mathfrak{g}, [\cdot, \ldots, \cdot]_\alpha, \varepsilon, \alpha)$, where $[\cdot, \ldots, \cdot]_\alpha = \alpha \circ [\cdot, \ldots, \cdot]$, is a n-Hom – Lie color algebra.*

***Example* 3.1 [10].** Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\varepsilon((i_1, i_2), (j_1, j_2)) = (-1)^{i_1 j_2 - i_2 j_1}$. Let L be a Γ-graded vector space

$$\mathfrak{g} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(1,1)}$$

with $\mathfrak{g}_{(0,0)} = \langle e_1, e_2 \rangle,\ \mathfrak{g}_{(0,1)} = \langle e_3 \rangle,\ \mathfrak{g}_{(1,0)} = \langle e_4 \rangle,\ \mathfrak{g}_{(1,1)} = \langle e_5 \rangle$.

The bracket $[\cdot, \cdot, \cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined with respect to basis $\{e_i \mid i = 1, \ldots, 5\}$ by

$$[e_2, e_3, e_4, e_5] = e_1, \qquad [e_1, e_3, e_4, e_5] = e_2, \qquad [e_1, e_2, e_4, e_5] = e_3,$$
$$[e_1, e_2, e_3, e_4] = 0, \qquad [e_1, e_2, e_3, e_5] = 0$$

makes $\mathfrak{g}$ into a five dimensional 4-Lie color algebra.

Now, we define a morphism $\alpha\colon \mathfrak{g} \to \mathfrak{g}$ by

$$\alpha(e_1) = e_2, \qquad \alpha(e_2) = e_1, \qquad \alpha(e_i) = e_i, \quad i = 3, 4, 5.$$

Then $\mathfrak{g}_\alpha = (\mathfrak{g}, [\cdot, \cdot, \cdot, \cdot]_\alpha, \varepsilon, \alpha)$ is a 4-Hom – Lie color algebra, where the new brackets are given as

$$[e_2, e_3, e_4, e_5]_\alpha = e_2, \qquad [e_1, e_3, e_4, e_5]_\alpha = e_1, \qquad [e_1, e_2, e_4, e_5]_\alpha = e_3,$$
$$[e_1, e_2, e_3, e_4]_\alpha = 0, \qquad [e_1, e_2, e_3, e_5]_\alpha = 0.$$

Let $\mathcal{G} = (\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom – Lie color algebra and $p \geq 0$. Define the pth derived $\mathcal{G}$ by

$$\mathcal{G}^p = (\mathfrak{g}, [\cdot, \ldots, \cdot]^p = \alpha^{2p-1} \circ [\cdot, \ldots, \cdot], \varepsilon, \alpha^{2p}).$$

Note that $\mathcal{G}^0 = \mathcal{G},\ \mathcal{G}^1 = (\mathfrak{g}, [\cdot, \ldots, \cdot]^1 = \alpha \circ [\cdot, \ldots, \cdot], \varepsilon, \alpha^2)$.

***Corollary* 3.2.** *With the above notations, the pth derived $\mathcal{G}$, $\mathcal{G}^p$, is also a n-Hom – Lie color algebra for each $p \geq 0$.*

Lemma 3.1. *Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a n-Hom – Lie color algebra and $\mathfrak{h}$ be a Γ-graded vector space. If there exists a bijective linear map of degree zero $f : \mathfrak{h} \to \mathfrak{g}$, then $(\mathfrak{h}, [\cdot, \ldots, \cdot]', \varepsilon, f^{-1} \circ \alpha \circ f)$ is a n-Hom – Lie color algebra, where the n-ary bracket $[\cdot, \ldots, \cdot]'$ is defined by*

$$[x_1, \ldots, x_n]' = f^{-1} \circ [f(x_1), \ldots, f(x_n)] \quad \forall x_i \in \mathfrak{h}.$$

Moreover, f is an algebra isomorphism.

Proof. Straightforward.

Proposition 3.1. *Let* $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$ *be a regular* n*-Hom–Lie color algebra. Then* $(V,[\cdot,\ldots,\cdot]_{\alpha^{-1}}=\alpha^{-1}\circ[\cdot,\ldots,\cdot],\varepsilon)$ *is a* n*-Lie color algebra.*

***Corollary* 3.3.** *Let* $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$ *be a* n*-Hom–Lie color algebra with* $n\geq 3$. *Let* $a_1,\ldots,a_p\in\mathfrak{g}_0$ *such that* $\alpha(a_i)=a_i$ *for all* $i\in\{1,\ldots,p\}$. *Then* $(\mathfrak{g},\{\cdot,\ldots,\cdot\},\varepsilon,\alpha)$ *is a* $(n-p)$*-Hom–Lie color algebra, where*

$$\{x_1,\ldots,x_{n-p}\}=[a_1,\ldots,a_p,x_1,\ldots,x_{n-p}]\quad\forall x_1,\ldots,x_{n-p}\in\mathfrak{g}.$$

3.2. From n***-Hom–Lie color algebras to Hom–Leibniz color algebras.*** In the following, we recall the definition of Hom–Leibniz color algebra introduced in [10]. We construct a Hom–Leibniz color algebra starting from a given n-Hom–Lie color algebra.

Definition 3.1. *A Hom–Leibniz color algebra is a quadruple* $(\mathcal{L},[\cdot,\cdot],\varepsilon,\alpha)$ *consisting of a* Γ*-graded vector space* $\mathcal{L}$, *a bicharacter* ε, *a bilinear map of degree zero* $[\cdot,\cdot]\colon\mathcal{L}\times\mathcal{L}\to\mathcal{L}$ *and a homomorphism* $\alpha\colon\mathcal{L}\to\mathcal{L}$ *such that, for any homogeneous elements* $x,y,z\in\mathcal{L}$,

$$[\alpha(x),[y,z]]-\varepsilon(x,y)[\alpha(y),[x,z]]=[[x,y],\alpha(z)]\quad(\varepsilon\text{-Hom–Leibniz identity}).\tag{3.1}$$

In particular, if α *is a morphism of Hom–Leibniz color algebra* $(i.e., \alpha\circ[\cdot,\cdot]=[\cdot,\cdot]\circ\alpha^{\otimes 2})$, *we call* $(\mathcal{L},[\cdot,\cdot],\varepsilon,\alpha)$ *a multiplicative Hom–Leibniz color algebra.*

Let $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$ be a n-Hom–Lie color algebra. We define a Γ-graded space $\mathcal{L}=\mathcal{L}(\mathfrak{g}):=\bigwedge^{n-1}\mathfrak{g}$, which is called fundamental set, and, for a fundamental object $X=x_1\wedge\ldots\wedge x_{n-1}\in\mathcal{L}$, an adjoint map ad_X as a linear map on $\mathfrak{g}$ by

$$ad_X\cdot y=[x_1,\ldots,x_{n-1},y].\tag{3.2}$$

We define a linear map $\tilde{\alpha}\colon\mathcal{L}\longrightarrow\mathcal{L}$ by

$$\tilde{\alpha}(X)=\alpha(x_1)\wedge\ldots\wedge\alpha(x_{n-1}).$$

Then the color ε-n-Hom–Jacobi identity (2.2) may be written in terms of adjoint maps as

$$ad_{\tilde{\alpha}(X)}[y_1,\ldots,y_n]=\sum_{i=1}^{n}\varepsilon(X,Y_i)[\alpha(y_1),\ldots,\alpha(y_{i-1}),ad_X y_i,\ldots,\alpha(y_n)].$$

Now, we define a bilinear map of degree zero $[\cdot,\cdot]_\alpha\colon\mathcal{L}\times\mathcal{L}\longrightarrow\mathcal{L}$ by

$$[X,Y]_\alpha=\sum_{i=1}^{n-1}\varepsilon(X,Y_i)\big(\alpha(y_1),\ldots,\alpha(y_{i-1}),ad_X y_i,\ldots,\alpha(y_{n-1})\big).\tag{3.3}$$

Proposition 3.2. *With the above notations, the map* ad *satisfies, for all* $X,Y\in\mathcal{H}(\mathcal{L}(\mathfrak{g}))$ *and* $z\in\mathcal{H}(\mathfrak{g})$, *the equality*

$$ad_{[X,Y]_\alpha}\alpha(z)=ad_{\tilde{\alpha}(X)}(ad_Y z)-\varepsilon(X,Y)ad_{\tilde{\alpha}(Y)}(ad_X z).\tag{3.4}$$

Moreover, the quadruple $(\mathcal{L},[\cdot,\cdot]_\alpha,\varepsilon,\tilde{\alpha})$ *is a Hom–Leibniz color algebra.*

Proof. It is easy to show that the identity (2.2) is equivalent to (3.4). Let $X, Y, Z \in \mathcal{H}(\mathcal{L})$, the ε-Hom – Leibniz identity (3.1) can be written using the bracket $[\cdot,\cdot]_\alpha$ and the twist $\tilde{\alpha}$ as

$$[\tilde{\alpha}(X), [Y, Z]_\alpha]_\alpha - \varepsilon(X, Y)[\tilde{\alpha}(Y), [X, Z]_\alpha]_\alpha = [[X, Y]_\alpha, \tilde{\alpha}(Z)]_\alpha. \tag{3.5}$$

Then we have

$$\begin{aligned}
&[\tilde{\alpha}(X), [Y, Z]_\alpha]_\alpha \\
&\quad= \sum_{i=1}^{n-1}\sum_{j<i}^{n-1} \varepsilon(X, Z_j)\varepsilon(Y, Z_i)(\alpha^2(z_1), \ldots, \alpha(ad_X z_j), \ldots, \alpha(ad_Y z_i), \ldots, \alpha^2(z_{n-1})) \\
&\qquad+ \sum_{i=1}^{n-1}\sum_{j>i}^{n-1} \varepsilon(X, Y + Z_j)\varepsilon(Y, Z_i)(\alpha^2(z_1), \ldots, \alpha(ad_X z_i), \ldots, \alpha(ad_Y z_j), \ldots, \alpha^2(z_{n-1})) \\
&\qquad+ \sum_{i=1}^{n-1} \varepsilon(X + Y, Z_i)(\alpha^2(z_1), \ldots, (ad_{\tilde{\alpha}(X)} ad_y z_i), \ldots, \alpha^2(z_{n-1})).
\end{aligned}$$

Using the ε-skew-symmetry in X and Y, we obtain

$$\begin{aligned}
&[\tilde{\alpha}(Y), [X, Z]_\alpha]_\alpha \\
&\quad= \sum_{i=1}^{n-1}\sum_{j<i}^{n-1} \varepsilon(Y, Z_j)\varepsilon(X, Z_i)(\alpha^2(z_1), \ldots, \alpha(ad_Y z_j), \ldots, \alpha(ad_X z_i), \ldots, \alpha^2(z_{n-1})) \\
&\qquad+ \sum_{i=1}^{n-1}\sum_{j>i}^{n-1} \varepsilon(Y, X)\varepsilon(Y, Z_j)\varepsilon(X, Z_i) \\
&\qquad\times (\alpha^2(z_1), \ldots, \alpha(ad_Y z_i), \ldots, \alpha(ad_X z_j), \ldots, \alpha^2(z_{n-1})) \\
&\qquad+ \sum_{i=1}^{n-1} \varepsilon(X + Y, Z_i)(\alpha^2(z_1), \ldots, (ad_{\tilde{\alpha}(Y)} ad_X z_i), \ldots, \alpha^2(z_{n-1})).
\end{aligned}$$

So, the right-hand side of Eq. (3.5) is equal to

$$\sum_{i=1}^{n-1} \varepsilon(X + Y, Z_i)\Big(\alpha^2(z_1), \ldots, \big((ad_{\tilde{\alpha}(X)} ad_y z_i) - \varepsilon(X, Y)(ad_{\tilde{\alpha}(Y)} ad_X z_i)\big), \ldots, \alpha^2(z_{n-1})\Big).$$

The left-hand side of Eq. (3.5) is equal to

$$\begin{aligned}
&[[X, Y]_\alpha, \tilde{\alpha}(Z)]_\alpha \\
&\quad= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \varepsilon(X, Y_j)\varepsilon(X, Z_i)\varepsilon(Y, Z_i) \\
&\qquad\times \Big(\alpha^2(z_1), \ldots, [\alpha(y_1), \ldots, ad_X z_j, \ldots, \alpha(y_{n-1}), \alpha(z_i)], \ldots, \alpha^2(z_{n-1})\Big)
\end{aligned}$$

$$= \sum_{i=1}^{n-1} \varepsilon(X+Y, Z_i)\Big(\alpha^2(z_1), \ldots, \alpha^2(z_{i-1}), ad_{[X,Y]_\alpha}\alpha(z_i), \ldots, \alpha^2(z_{n-1})\Big)$$

by Eq. (3.4), the ε-Hom – Leibniz identity holds.

Proposition 3.2 is proved.

3.3. Representations of n-Hom – Lie color algebras. In this subsection, we introduce a notion of representation of n-Hom – Lie color algebras, generalizing representations of n-Hom – Lie superalgebras (see in [1]) to the Γ-graded case. In the sequel, we consider only multiplicative n-Hom – Lie color algebras.

Definition 3.2. *A representation of a n-Hom – Lie color algebra $\mathcal{G} = (\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ on a Γ-graded space M is a ε-skew-symmetric linear map of degree zero $\rho\colon \bigwedge^{n-1}\mathfrak{g} \longrightarrow \mathrm{End}(M)$, and μ an endomorphism on M satisfying, for $X = (x_1, \ldots, x_{n-1})$, $Y = (y_1, \ldots, y_{n-1}) \in \mathcal{H}(\bigwedge^{n-1}\mathfrak{g})$ and $x_n \in \mathcal{H}(\mathfrak{g})$,*

$$\rho(\tilde{\alpha}(X)) \circ \mu = \mu \circ \rho(X), \tag{3.6}$$

$$\begin{aligned} &\rho(\alpha(x_1), \ldots, \alpha(x_{n-1})) \circ \rho(y_1, \ldots, y_{n-1}) \\ &\qquad - \varepsilon(X, Y)\rho(\alpha(y_1), \ldots, \alpha(y_{n-1})) \circ \rho(x_1, \ldots, x_{n-1}) \\ &\quad = \sum_{i=1}^{n-1} \varepsilon(X, Y_i)\rho(\alpha(y_1), \ldots, \alpha(y_{i-1}), ad_X(y_i), \ldots, \alpha(y_{n-1})) \circ \mu, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\rho\big([x_1, x_2, \ldots, x_n], \alpha(y_1), \alpha(y_2), \ldots, \alpha(y_{n-2})\big) \circ \mu \\ &\quad = \sum_{i=1}^{n}(-1)^{n-i}\varepsilon(x_i, X^i)\rho(\alpha(x_1), \ldots, \hat{x_i}, \ldots, \alpha(x_n))\rho(x_i, y_1, y_2, \ldots, y_{n-2}). \end{aligned} \tag{3.8}$$

We denote this representation by a triple (M, ρ, μ).

***Remarks* 3.1.** 1. Condition (3.7) can be written in terms of fundamental object on $\mathcal{L}$ defined in Subsection 3.2 and the bracket of Hom – Leibniz color algebra defined in (3.3) as

$$\rho(\tilde{\alpha}(X)) \circ \rho(Y) - \varepsilon(X, Y)\rho(\tilde{\alpha}(Y)) \circ \rho(X) = \rho([X, Y]_\alpha) \circ \mu.$$

2. Two representations (M, ρ, μ) and (M', ρ', μ') of a n-Hom – Lie color algebra $\mathcal{G}$ are equivalent if there exists $f : M \to M'$, an isomorphism of Γ-graded vector space of degree zero, such that $f(\rho(X)m) = \rho'(X)f(m)$ and $f \circ \mu = \mu' \circ f$ for all $X \in \mathcal{H}(\mathcal{L})$, $m \in M$ and $m' \in M'$.

3. If $\alpha = id_{\mathfrak{g}}$ and $\mu = id_M$, we recover representations of n-Lie color algebras.

***Example* 3.2.** Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom – Lie color algebra. The map ad defined in Eq. (3.2) is a representation on $\mathfrak{g}$ where the endomorphism μ is the twist map α. The identity (3.7) is equivalent to n-Hom – Jacobi identity Eq. (2.2). It is called the adjoint representation.

Proposition 3.3. *Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a multiplicative n-Hom – Lie color algebra. Then (M, ρ, μ) is a representation of $\mathfrak{g}$ if and only if $(\mathfrak{g} \oplus M, [\cdot, \ldots, \cdot]_\rho, \varepsilon, \tilde{\alpha})$ is a multiplicative n-Hom – Lie color algebra with the bracket operation $[\cdot, \ldots, \cdot]_\rho\colon \wedge^n(\mathfrak{g} \oplus M) \longrightarrow \mathfrak{g} \oplus M$ defined by*

$$[x_1 + m_1, \ldots, x_n + m_n]_\rho = [x_1, \ldots, x_n] + \sum_{i=1}^{n}(-1)^{n-i}\varepsilon(x_i, X^i)\rho(x_1, \ldots, \hat{x_i}, \ldots, x_n)m_i$$

and the linear map of degree zero $\alpha_{\mathfrak{g}\oplus M} : \mathfrak{g}\oplus M \longrightarrow \mathfrak{g}\oplus M$ *given by*

$$\alpha_{\mathfrak{g}\oplus M}(x+m) = \alpha(x) + \mu(m)$$

for all $x_i \in \mathcal{H}(\mathfrak{g})$ *and* $m_i \in M$, $i \in \{1,\ldots,n\}$. *It is called the semidirect product of the* n*-Hom–Lie color algebra* $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$ *by the representation* (M,ρ,μ) *denoted by* $\mathfrak{g}\ltimes_{\rho}^{\alpha,\mu} M$. *Note that* $\mathfrak{g}\oplus M$ *is a* Γ*-graded space, where* $(\mathfrak{g}\oplus M)_\gamma = \mathfrak{g}_\gamma \oplus M_\gamma$, *implying that* $x+m \in \mathcal{H}(\mathfrak{g}\oplus M)$, *so* $\overline{x_i+m_i} = \overline{x_i} + \overline{m_i}$.

Proof. It is easy to show that $[\cdot,\ldots,\cdot]_\rho$ is ε-skew-symmetric using the ε-skew-symmetry of $[\cdot,\ldots,\cdot]$ and ρ. Let $x_1+v_1,\ldots,x_{n-1}+v_{n-1}$ and $y_1+w_1,\ldots,y_n+w_n \in \mathcal{H}(\mathfrak{g}\oplus M)$, the identity (2.2) is given in terms of $[\cdot,\ldots,\cdot]_\rho$ and $\alpha_{\mathfrak{g}\oplus M}$ by

$$\begin{aligned}[]
[\alpha_{\mathfrak{g}\oplus M}(x_1+v_1),\ldots,\alpha_{\mathfrak{g}\oplus M}(x_{n-1}+v_{n-1}),[y_1+w_1,\ldots,y_n+w_n]_\rho]_\rho \\
= \sum_{i=1}^{n}\varepsilon(X,Y_i)\big[\alpha_{\mathfrak{g}\oplus M}(y_1+w_1),\ldots,\alpha_{\mathfrak{g}\oplus M}(y_{i-1}+w_{i-1}), \\
[x_1+v_1,\ldots,x_{n-1}+v_{n-1},y_i+w_i]_\rho,\ldots,\alpha_{\mathfrak{g}\oplus M}(y_n+w_n)\big]_\rho. \qquad (3.9)
\end{aligned}$$

The left-hand side of (3.9) is equal to

$$\begin{aligned}
&[\alpha(x_1),\ldots,\alpha(x_{n-1}),[y_1,\ldots,y_n]] \\
&\quad+\sum_{i=1}^{n-1}(-1)^{n-i}\varepsilon(x_i,X^i+Y)\rho(\alpha(x_1),\ldots,\hat{x_i},\ldots,\alpha(x_{n-1}),[y_1,\ldots,y_n])\mu(v_i) \\
&\quad+\sum_{i=1}^{n}(-1)^{n-i}\varepsilon(y_i,Y^i)\rho(\alpha(x_1),\ldots,\alpha(x_n))\rho(y_1,\ldots,\hat{y_i},\ldots,y_n)w_i.
\end{aligned}$$

The right-hand side of (3.9), for a fixed $i\in\{1,\ldots,n\}$, is equal to

$$\begin{aligned}
&\varepsilon(X,Y_i)[\alpha(y_1),\ldots,\alpha(y_{i-1}),[x_1,\ldots,y_i],\ldots,\alpha(y_n)] \\
&\quad+\sum_{j<i}(-1)^{n-j}\varepsilon(X,Y_i)\varepsilon(y_j,X+Y^j) \\
&\quad\times\rho\Big(\alpha(y_1),\ldots,\hat{y_j},\ldots,\alpha(y_{i-1}),[x_1,\ldots,x_{n-1},y_i],\ldots,\alpha(y_n)\Big)\mu(w_j) \\
&\quad+\sum_{j>i}(-1)^{n-j}\varepsilon(X,Y_i)\varepsilon(y_j,Y^j) \\
&\quad\times\rho\big(\alpha(y_1),\ldots,\alpha(y_{i-1}),[x_1,\ldots,x_{n-1},y_i],\ldots,\hat{y_j},\ldots,\alpha(y_n)\big)\mu(w_j) \\
&\quad+\sum_{j=1}^{n-1}(-1)^{i+j}\varepsilon(X,Y_i)\varepsilon(X+y_i,Y^i)\varepsilon(x_j,X^j+y_i) \\
&\quad\times\rho\big(\alpha(y_1),\ldots,\hat{y_i},\ldots,\alpha(y_n)\big)\rho(x_1,\ldots,\hat{x_j},\ldots,x_{n-1},y_i)v_j
\end{aligned}$$

$$+\,(-1)^{n-i}\varepsilon(X,Y_i)\varepsilon(X+y_i,Y^i)\rho(\alpha(y_1),\ldots,\hat{y_i},\ldots,\alpha(y_n))\rho(x_1,\ldots,x_{n-1})w_i.$$

Then (M,ρ,μ) is a representation of the multiplicative n-Hom–Lie color algebra $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$ if and only if $(\mathfrak{g}\oplus M,[\cdot,\ldots,\cdot]_\rho,\varepsilon,\alpha_{\mathfrak{g}\oplus M})$ is a multiplicative n-Hom–Lie color algebra.

Proposition 3.3 is proved.

Proposition 3.4. *Let* $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon)$ *be a* n*-Lie color algebra,* (M,ρ) *be a representation of* $\mathfrak{g}$, $\alpha\colon\mathfrak{g}\to\mathfrak{g}$ *be an algebra morphism and* $\mu\colon M\to M$ *be a linear map of degree zero such that* $\rho(\tilde{\alpha}(X))\circ\mu=\mu\circ\rho(X)$ *for all* $X\in\mathcal{L}$*. Then* $(M,\tilde{\rho}=\mu\circ\rho,\mu)$ *is a representation of the multiplicative* n*-Hom–Lie color algebra* $(\mathfrak{g},[\cdot,\ldots,\cdot]_\alpha,\varepsilon,\alpha)$ *given in Corollary* 3.1.

Proof. It is easy to verify that, for $X=(x_1,\ldots,x_{n-1})$, $Y=(y_1,\ldots,y_{n-1})\in\mathcal{H}(\mathcal{L})$ and $x_n,y_n\in\mathcal{H}(\mathfrak{g})$,

$$\tilde{\rho}(\tilde{\alpha}(X))\circ\mu=\mu\circ\tilde{\rho}(X).$$

Since (M,ρ) is a representation of $\mathfrak{g}$, then we have

$$\begin{aligned}
&\tilde{\rho}([x_1,\ldots,x_n]_\alpha,\alpha(y_1),\ldots,\alpha(y_{n-2}))\circ\mu\\
&\quad=\mu\circ\tilde{\rho}([x_1,\ldots,x_n],y_1,\ldots,y_{n-2})\\
&\quad=\sum_{i=1}^{n}(-1)^{n-1}\varepsilon(x_i,X^i)\mu^2\circ\rho(x_1,\ldots,\hat{x_i},\ldots,x_n)\rho(x_i,y_1,\ldots,y_n)\\
&\quad=\sum_{i=1}^{n}(-1)^{n-1}\varepsilon(x_i,X^i)\mu\circ\rho(\alpha(x_1),\ldots,\hat{x_i},\ldots,\alpha(x_n))\mu\circ\rho(x_i,y_1,\ldots,y_n)\\
&\quad=\sum_{i=1}^{n}(-1)^{n-1}\varepsilon(x_i,X^i)\tilde{\rho}(\alpha(x_1),\ldots,\hat{x_i},\ldots,\alpha(x_n))\tilde{\rho}(x_i,y_1,\ldots,y_n).
\end{aligned}$$

Thus, condition (3.8) holds. Similarly, Eq. (3.7) is valid for $\tilde{\rho}$. Then $(M,\tilde{\rho},\mu)$ is a representation of the multiplicative n-Hom–Lie color algebra $(\mathfrak{g},[\cdot,\ldots,\cdot]_\alpha,\varepsilon,\alpha)$.

Proposition 3.4 is proved.

4. Cohomology of n-Hom–Lie color algebras. In this section, we study a cohomology of a multiplicative n-Hom–Lie color algebra with respect to a given representation. Let $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$ be a n-Hom–Lie color algebra and (M,ρ,μ) is a $\mathcal{L}(\mathfrak{g})$-module. A p-cochain is a ε-skew-symmetric multilinear map $\varphi\colon\underbrace{\mathcal{L}(\mathfrak{g})\otimes\ldots\otimes\mathcal{L}(\mathfrak{g})}_{p-1}\wedge\mathfrak{g}\longrightarrow M$ such that

$$\mu\circ\varphi(X_1,\ldots,X_p,z)=\varphi(\tilde{\alpha}(X_1),\ldots,\tilde{\alpha}(X_p),\alpha(z)).$$

The space of all p-cochains is Γ-graded and is denoted by $\mathcal{C}^p_{\alpha,\mu}(\mathfrak{g},M)$.

Thus, we can define the coboundary operator of the cohomology of a n-Hom–Lie color algebra $\mathfrak{g}$ with coefficients in M by using the structure of its induced Hom–Leibniz color algebra as follows.

Definition 4.1. *We call*, *for* $p\geq 1$, *a* p*-coboundary operator of a* n*-Hom–Lie color algebra* $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$, *a linear map* $\delta^p:\mathcal{C}^p_{\alpha,\mu}(\mathfrak{g},M)\to\mathcal{C}^{p+1}_{\alpha,\mu}(\mathfrak{g},M)$ *defined by*

$$\begin{aligned}
&\delta^p\varphi(X_1,\ldots,X_p,z)\\
&\quad=\sum_{1\leq i<j}^{p}(-1)^i\varepsilon(X_i,X_{i+1}+\ldots+X_{j-1})
\end{aligned}$$

$$\times\,\varphi\big(\tilde{\alpha}(X_1),\ldots,\widehat{\tilde{\alpha}(X_i)},\ldots,\tilde{\alpha}(X_{j-1}),[X_i,X_j]_\alpha,\ldots,\tilde{\alpha}(X_p),\alpha(z)\big)$$
$$+\sum_{i=1}^{p}(-1)^i\varepsilon(X_i,X_{i+1}+\ldots+X_p)\varphi\big(\tilde{\alpha}(X_1),\ldots,\widehat{\tilde{\alpha}(X_i)},\ldots,\tilde{\alpha}(X_p),ad(X_i)(z)\big)$$
$$+\sum_{i=1}^{p}(-1)^{i+1}\varepsilon(\varphi+X_1+\ldots+X_{i-1},X_i)\rho(\tilde{\alpha}^{p-1}(X_i))\Big(\varphi\big(X_1,\ldots,\widehat{X_i},\ldots,X_p,z\big)\Big)$$
$$+\,(-1)^{p+1}\big(\varphi(X_1,\ldots,X_{p-1})X_p\big)\bullet_\alpha\alpha^p(z),$$

where

$$\big(\varphi(X_1,\ldots,X_{p-1})X_p\big)\bullet_\alpha\alpha^p(z)$$
$$=\sum_{i=1}^{n-1}(-1)^{n-i}\varepsilon\big(\varphi+X_1+\ldots+X_{p-1},x_p^1+\ldots+\hat{x}_p^i+\ldots+x_p^{n-1}+z\big)$$
$$\times\,\varepsilon\big(x_p^i,x_p^{i+1}+\ldots+x_p^{n-1}+z\big)$$
$$\times\,\rho\big(\alpha^{p-1}(x_p^1),\ldots,\alpha^{p-1}(x_p^{n-1}),\alpha^{p-1}(z)\big)\varphi(X_1,\ldots,X_{p-1},x_p^i)$$

for $X_i=(x_i^j)_{1\le j\le n-1}\in\mathcal{H}(\mathcal{L}(\mathfrak{g}))$, $1\le i\le p$, *and* $z\in\mathcal{H}(\mathfrak{g})$.

Proposition 4.1. *Let* $\varphi\in\mathcal{C}^p_{\alpha,\mu}(\mathfrak{g},M)$ *be a p-cochain, then*

$$\delta^{p+1}\circ\delta^p(\varphi)=0.$$

Proof. Let φ be a p-cochain, $X_i=(x_i^j)_{1\le j\le n-1}\in\mathcal{H}(\mathcal{L}(\mathfrak{g}))$, $1\le i\le p+2$, and $z\in\mathcal{H}(\mathfrak{g})$. We set

$$\delta^p=\delta_1^p+\delta_2^p+\delta_3^p+\delta_4^p \qquad \text{and} \qquad \delta^{p+1}\circ\delta^p=\sum_{i,j=1}^{4}\Upsilon_{ij},$$

where $\Upsilon_{ij}=\delta_i^{p+1}\circ\delta_j^p$ and

$$\delta_1^p\varphi(X_1,\ldots,X_p,z)=\sum_{1\le i<j}^{p}(-1)^i\varepsilon(X_{i+1}+\ldots+X_{j-1},X_i)$$
$$\times\,\varphi\big(\tilde{\alpha}(X_1),\ldots,\widehat{\tilde{\alpha}(X_i)},\ldots,[X_i,X_j]_\alpha,\ldots,\tilde{\alpha}(X_p),\alpha(z)\big),$$
$$\delta_2^p\varphi(X_1,\ldots,X_p,z)=\sum_{i=1}^{p}(-1)^i\varepsilon(X_{i+1}+\ldots+X_p,X_i)$$
$$\times\,\varphi\big(\tilde{\alpha}(X_1),\ldots,\widehat{\tilde{\alpha}(X_i)},\ldots,\tilde{\alpha}(X_p),ad(X_i)(z)\big),$$

$$\delta_3^p\varphi(X_1,\ldots,X_p,z)=\sum_{i=1}^{p}(-1)^{i+1}\varepsilon(\varphi+X_1+\ldots+X_{i-1},X_i)$$
$$\times\rho(\tilde\alpha^{p-1}(X_i))\Big(\varphi\big(X_1,\ldots,\widehat{X_i},\ldots,X_p,z\big)\Big),$$
$$\delta_4^p\varphi(X_1,\ldots,X_p,z)=(-1)^{p+1}\big(\varphi(X_1,\ldots,X_{p-1})X_p\big)\bullet_\alpha\alpha^p(z).$$

To simplify the notations we replace $ad(X)(z)$ by $X\cdot z$. Let first prove that $\Upsilon_{11}+\Upsilon_{12}+\Upsilon_{21}+\Upsilon_{22}=0$, $(X_i)_{1\le i\le p}\in\mathcal{L}(\mathfrak{g})$ and $z\in\mathfrak{g}$.

Let us compute first $\Upsilon_{11}(\varphi)(X_1,\ldots,X_p,z)$. We have

$$\Upsilon_{11}(\varphi)(X_1,\ldots,X_p,z)$$
$$=\sum_{1\le i<k<j}^{p}(-1)^{i+k}\varepsilon(X_{i+1}+\ldots+X_{j-1},X_i)\varepsilon(X_{k+1}+\ldots+X_{j-1},X_k)$$
$$\times\varphi\big(\tilde\alpha^2(X_1),\ldots,\widehat{X_i},\ldots,\widehat{\tilde\alpha(X_k)},\ldots,[\tilde\alpha(X_k),[X_i,X_j]_\alpha]_\alpha,\ldots,\tilde\alpha^2(X_p),\alpha^2(z)\big)$$
$$+\sum_{1\le i<k<j}^{p}(-1)^{i+k-1}\varepsilon(X_{i+1}+\ldots+\widehat{X_k}+\ldots+X_{j-1},X_i)\varepsilon(X_{k+1}+\ldots+X_{j-1},X_k)$$
$$\times\varphi\big(\tilde\alpha^2(X_1),\ldots,\widehat{\tilde\alpha(X_i)},\ldots,\widehat{X_k},\ldots,[\tilde\alpha(X_i),[X_k,X_j]_\alpha]_\alpha,\ldots,\tilde\alpha^2(X_p),\alpha^2(z)\big)$$
$$+\sum_{1\le i<k<j}^{p}(-1)^{i+k-1}\varepsilon(X_{i+1}+\ldots+\widehat{X_k}+\ldots+X_{j-1},X_i)\varepsilon(X_{k+1}+\ldots+X_{j-1},X_k)$$
$$\times\varphi\big(\tilde\alpha^2(X_1),\ldots,\widehat{X_i},\ldots,\widehat{[X_i,X_k]}_\alpha,\ldots,[[X_i,X_k]_\alpha,\tilde\alpha(X_j)]_\alpha,\ldots,\tilde\alpha^2(X_p),\alpha^2(z)\big).$$

Whence applying the Hom – Leibniz identity (3.1) to $X_i,\ X_j,\ X_k\in\mathcal{H}(\mathcal{L}(\mathfrak{g}))$, we find $\Upsilon_{11}=0$.

On the other hand, we obtain

$$(\Upsilon_{21}(\varphi)+\Upsilon_{12}(\varphi))(X_1,\ldots,X_p,z)$$
$$=\sum_{1\le i<j}^{p}(-1)^{i+j-1}\varepsilon(X_{i+1}+\ldots+\widehat{X_j}+\ldots+X_p,X_i)\varepsilon(X_{j+1}+\ldots+X_p,X_j)$$
$$\times\varphi\big(\tilde\alpha^2(X_1),\ldots,\widehat{X}_i,\ldots,\widehat{[X_i,X_j]}_\alpha,\ldots,\tilde\alpha^2(X_p),[X_i,X_j]_\alpha\cdot\alpha(z)\big)$$

and

$$\Upsilon_{22}(\varphi)(X_1,\ldots,X_p,z)$$
$$=\sum_{1\le i<j}^{p}(-1)^{i+j}\varepsilon(X_{i+1}+\ldots+X_p,X_i)\varepsilon(X_{j+1}+\ldots+X_p,X_j)$$
$$\times\varphi\big(\tilde\alpha^2(X_1),\ldots,\widehat{X}_i,\ldots,\widehat{\tilde\alpha(X_j)},\ldots,\tilde\alpha^2(X_p),\big(\tilde\alpha(X_j)(X_i\cdot z)\big)\big)$$

$$+\sum_{1\le i<j}^{p}(-1)^{i+j-1}\varepsilon(X_{j+1}+\ldots+X_{p+1},X_j)\varepsilon(X_{j+1}+\ldots+\widehat{X_j}+\ldots+X_p,X_i)$$
$$\times\,\varphi\big(\tilde\alpha^2(X_1),\ldots,\widehat{\tilde\alpha(X_i)},\ldots,\widehat{X}_j,\ldots,\tilde\alpha^2(X_p),\big(\tilde\alpha(X_i)(X_j\cdot z)\big)\big).$$

Then, applying (2.2) to $X_i,\ X_j\in\mathcal{H}(\mathcal{L}(\mathfrak{g}))$ and $z\in\mathcal{H}(\mathfrak{g})$, we get $\Upsilon_{12}+\Upsilon_{21}+\Upsilon_{22}=0$.

On the other hand, we have

$$\Upsilon_{31}\varphi(X_1,\ldots,X_{p+1},z)$$
$$=\sum_{1\le i<j<k}^{p+1}\Big\{(-1)^{k+i+1}\varepsilon(\varphi+X_1+\ldots+X_{k-1},X_k)\varepsilon(X_{i+1}+\ldots+X_{j-1},X_i)$$
$$\times\,\rho\big(\tilde\alpha^p(X_k)\big)\varphi(\tilde\alpha(X_1),\ldots,\widehat{X}_i,\ldots,[X_i,X_j]_\alpha,\ldots,\widehat{X}_k,\ldots,\alpha(z))$$
$$+\,(-1)^{j+i+1}\varepsilon(\varphi+X_1+\ldots+X_{j-1},X_j)\varepsilon(X_{i+1}+\ldots+\widehat{X_j}+\ldots+X_{k-1},X_i)$$
$$\times\,\rho\big(\tilde\alpha^{p-1}(X_j)\big)\varphi(\tilde\alpha(X_1),\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,[X_i,X_k]_\alpha,\ldots,\alpha(z))$$
$$+\,(-1)^{j+i}\varepsilon(\varphi+X_1+\ldots+X_{i-1},X_i)\varepsilon(X_{j+1}+\ldots+X_{k-1},X_j)$$
$$\times\,\rho\big(\tilde\alpha^{p-1}(X_i)\big)\varphi(\tilde\alpha(X_1),\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,[X_j,X_k]_\alpha,\ldots,\alpha(z))\Big\},$$

$$\Upsilon_{13}\varphi(X_1,\ldots,X_{p+1},z)=-\Upsilon_{31}\varphi(X_1,\ldots,X_{p+1},z)$$
$$+\sum_{1\le i<j}^{p+1}(-1)^{i+j+1}\varepsilon(\varphi+X_1+\ldots+X_{i-1},X_i)$$
$$\times\,\varepsilon(\varphi+X_1+\ldots+\widehat{X_i}+\ldots+X_{j-1},X_j)$$
$$\times\,\rho\big(\tilde\alpha^{p-1}([X_i,X_j]_\alpha)\big)\mu\big(\varphi(X_1,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,z)\big)$$

and

$$\Upsilon_{33}\varphi(X_1,\ldots,X_{p+1},z)$$
$$=\sum_{1\le i<j}^{p+1}\Big\{(-1)^{i+j+1}\varepsilon(\varphi+X_1+\ldots+X_{i-1},X_i)\varepsilon(\varphi+X_1+\ldots+X_{j-1},X_j)\rho\big(\tilde\alpha^{p-1}(X_i)\big)$$
$$\times\,\Big(\rho\big(\tilde\alpha^p(X_j)\big)\big(\varphi(X_1,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,z)\big)\Big)$$
$$+\,(-1)^{i+j}\varepsilon(\varphi+X_1+\ldots+X_{i-1},X_i)\varepsilon(\varphi+X_1+\ldots+\widehat{X_i}+\ldots+X_{j-1},X_j)$$
$$\times\,\rho\big(\tilde\alpha^{p-1}(X_j)\big)\Big(\rho\big(\tilde\alpha^{p-1}(X_i)\big)\big(\varphi(X_1,\ldots,\widehat{X}_j,\ldots,\widehat{X}_i,\ldots,z)\big)\Big)\Big\}.$$

Then, applying (3.7) to $\tilde\alpha^p(X_i)\in\mathcal{L}(\mathfrak{g}),\ \tilde\alpha^p(X_j)\in\mathcal{L}(\mathfrak{g})$ and $\varphi(X_1,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,z)\in M$, we have

$$\Upsilon_{13} + \Upsilon_{33} + \Upsilon_{31} = 0.$$

By the same calculation, we can prove that

$$\Upsilon_{23} + \Upsilon_{32} = 0,$$

$$\Upsilon_{14} + \Upsilon_{41} + \Upsilon_{24} + \Upsilon_{42} + \Upsilon_{34} + \Upsilon_{43} + \Upsilon_{44} = 0.$$

Proposition 4.1 is proved.

Definition 4.2. *We define the graded space of p-cocycles by $\mathcal{Z}^p_{\alpha,\mu}(\mathfrak{g}, M) = \{\varphi \in \mathcal{C}^p_{\alpha,\mu}(\mathfrak{g}, M) : \delta^p\varphi = 0\}$ and p-coboundaries by $\mathcal{B}^p_{\alpha,\mu}(\mathfrak{g}, M) = \{\psi = \delta^{p-1}\varphi : \varphi \in \mathcal{C}^{p-1}_{\alpha,\mu}(\mathfrak{g}, M)\}$.*

Lemma 4.1. $\mathcal{B}^p_{\alpha,\mu}(\mathfrak{g}, M) \subset \mathcal{Z}^p_{\alpha,\mu}(\mathfrak{g}, M)$.

Definition 4.3. *We call the p th-cohomology group the quotient*

$$\mathcal{H}^p_{\alpha,\mu}(\mathfrak{g}, M) = \mathcal{Z}^p_{\alpha,\mu}(\mathfrak{g}, M)/\mathcal{B}^p_{\alpha,\mu}(\mathfrak{g}, M).$$

***Remarks* 4.1.** Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a n-Hom – Lie color algebra. Then:

1. A 1-cochain ϕ is called 1-cocycle if and only if

$$\phi \circ ad_X(x_n) = \sum_{i=1}^{n} (-1)^{n-i} \varepsilon(\phi, \hat{X}) \varepsilon(x_i, X^i) \rho(x_1, \ldots, \hat{x_i}, \ldots, x_n) \phi(x_i). \tag{4.1}$$

In particular, if $M = \mathfrak{g}$, Eq. (4.1) can be rewritten as

$$\sum_{i=1}^{n} \varepsilon(\phi, X_i)[x_1, \ldots, \phi(x_i), \ldots, x_n] - \phi([x_1, x_2, \ldots, x_n]) = 0.$$

2. A 2-cochain ψ is called 2-cocycle if and only if, for all homogeneous elements $X = (x_1, \ldots, x_{n-1})$, $Y = (y_1, \ldots, y_{n-1}) \in \mathcal{L}(\mathfrak{g})$ and $y_n \in \mathfrak{g}$,

$$\begin{aligned} &\rho(\tilde{\alpha}(X))\psi(y_1, \ldots, y_n) + \psi(\alpha(x_1), \ldots, \alpha(x_{n-1}), ad_Y y_n) \\ &\quad = \sum_{i=1}^{n} \varepsilon(X, Y_i) \psi(\alpha(y_1), \ldots, \alpha(y_{i-1}), [x_1, \ldots, x_{n-1}, y_i], \ldots, \alpha(y_n)) \\ &\qquad + \sum_{i=1}^{n} (-1)^{n-i} \varepsilon(\psi + X, \hat{Y}_n) \varepsilon(y_i, Y_n^i) \rho(\alpha(y_1), \ldots, \hat{y_i}, \ldots, \alpha(y_n)) \psi(x_1, \ldots, x_{n-1}, y_i). \end{aligned} \tag{4.2}$$

5. Deformations of n-Hom – Lie color algebras. In this section, we study formal deformations and discuss equivalent deformations of n-Hom – Lie color algebras.

Definition 5.1. *Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a n-Hom – Lie color algebra and $\omega_i : \mathfrak{g}^{\otimes n} \to \mathfrak{g}$ be ε-skew-symmetric linear maps of degree zero. Consider a λ-parametrized family of n-linear operations $(\lambda \in \mathbb{K})$:*

$$[x_1, \ldots, x_n]_\lambda = [x_1, \ldots, x_n] + \sum_{i=1}^{\infty} \lambda^i \omega_i(x_1, \ldots, x_n).$$

The tuple $\mathfrak{g}_\lambda := (\mathfrak{g}, [\cdot, \ldots, \cdot]_\lambda, \varepsilon, \alpha)$ is a one-parameter formal deformation of $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ generated by ω_i if it defines a n-Hom – Lie color algebra.

***Remarks* 5.1.** 1. If $\lambda^2 = 0$ $(k = 1)$, the deformation is called infinitesimal.

2. If $\lambda^n = 0$, the deformation is said to be of order $n - 1$.

Let ω and ω' be two 2-cochains on a n-Hom – Lie color algebra $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ with coefficients in the adjoint representation. Define the bracket $[\,,\,] : \mathcal{C}^2_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g}) \times \mathcal{C}^2_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g}) \to \mathcal{C}^3_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g})$ for $X, Y \in \mathcal{H}(\mathcal{L})$ and $z \in \mathcal{H}(\mathfrak{g})$ by

$$\begin{aligned}[\omega, \omega'](X, Y, z) = \omega(\tilde{\alpha}(X), \omega'(Y, z)) - \varepsilon(X, Y)\omega(\tilde{\alpha}(Y), \omega'(X, z)) \\ - \omega(\omega'(X, \bullet) \circ \tilde{\alpha}(Y), \alpha(z)) + \omega'(\tilde{\alpha}(X), \omega(Y, z)) \\ - \varepsilon(X, Y)\omega'(\tilde{\alpha}(Y), \omega(X, z)) - \omega'(\omega(X, \bullet) \circ \tilde{\alpha}(Y), \alpha(z)),\end{aligned}$$

where

$$\omega(X, \bullet) \circ \tilde{\alpha}(Y) = \sum_{k=1}^{n-1} \varepsilon(X, Y_i)\alpha(y_1) \wedge \ldots \wedge \omega(X, y_k) \wedge \ldots \wedge \alpha(y_{n-1})$$

for all $X, Y \in \mathcal{H}(\mathcal{L}(\mathfrak{g}))$ and $z \in \mathcal{H}(\mathfrak{g})$.

Theorem 5.1. *With the above notations, the* 2*-cochains* $\omega_i,\ i \geq 1$, *generate a one-parameter formal deformation* $\mathfrak{g}_\lambda$ *of order* k *of a* n*-Hom – Lie color algebra* $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ *if and only if the following conditions hold*:

$$\delta^2\omega_1 = 0, \tag{5.1}$$

$$\delta^2\omega_l + \frac{1}{2} \sum_{i=1}^{l-1} [\omega_i, \omega_{l-i}] = 0, \quad 2 \leq l \leq k, \tag{5.2}$$

$$\frac{1}{2} \sum_{i=l-k}^{k} [\omega_i, \omega_{l-i}] = 0, \quad n \leq l \leq 2k. \tag{5.3}$$

Proof. Let $\omega_i,\ i \geq 1$, be 2-cochains generating a one-parameter formal deformation $\mathfrak{g}_\lambda := (\mathfrak{g}, [\cdot, \ldots, \cdot]_\lambda, \varepsilon, \alpha)$ of order k of a n-Hom – Lie color algebra $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$. Then $\mathfrak{g}_\lambda$ is also a n-Hom – Lie color algebra. According to Proposition 3.2, the ε-n-Hom – Jacobi identity (2.2) on $\mathfrak{g}_\lambda$ is equivalent to

$$ad^\lambda_{[X,Y]^\lambda_\alpha}\alpha(z) = ad^\lambda_{\tilde{\alpha}(X)}(ad^\lambda_Y z) - \varepsilon(X, Y)ad^\lambda_{\tilde{\alpha}(Y)}(ad^\lambda_X z), \tag{5.4}$$

where

$$[X, Y]^\lambda_\alpha = [X, Y]_\alpha + \sum_{i=1}^{k} \lambda^i \omega_i(X, \bullet) \circ \tilde{\alpha}(Y)$$

and the adjoint map on $\mathfrak{g}_\lambda$ is given by

$$ad^\lambda_X z = ad_X z + \sum_{i=1}^{k} \lambda^i \omega_i(X, z).$$

The left-hand side of (5.4) is equal to

$$ad^{\lambda}_{[X,Y]^{\lambda}_{\alpha}}\alpha(z) = ad_{[X,Y]_{\alpha}}\alpha(z) + \sum_{i=1}^{k}\lambda^{i}\big(\omega_i([X,Y]_{\alpha},\alpha(z))$$

$$+ ad_{\omega_i(X,\bullet)\circ\tilde{\alpha}(Y)}\alpha(z)\big) + \sum_{i,j=1}^{k}\lambda^{i+j} ad_{\omega_i(\omega_j(X,\bullet)\circ\tilde{\alpha}(Y))}\alpha(z).$$

The right-hand side of (5.4) is equal to

$$ad^{\lambda}_{\tilde{\alpha}(X)}(ad^{\lambda}_{Y}z) = ad_{\tilde{\alpha}(X)}(ad_Y z) + \sum_{i=1}^{k}\lambda^{i}\big(\omega_i(\tilde{\alpha}(X), ad_Y z)$$

$$+ ad_{\tilde{\alpha}(X)}\omega_i(Y,z)\big) + \sum_{i,j=1}^{k}\lambda^{i+j}\omega_i(\tilde{\alpha}(X),\omega_j(Y,z))$$

and

$$ad^{\lambda}_{\tilde{\alpha}(Y)}(ad^{\lambda}_{X}z) = ad_{\tilde{\alpha}(Y)}(ad_X z) + \sum_{i=1}^{k}\lambda^{i}\big(\omega_i(\tilde{\alpha}(Y), ad_X z)$$

$$+ ad_{\tilde{\alpha}(Y)}\omega_i(X,z)\big) + \sum_{i,j=1}^{k}\lambda^{i+j}\omega_i(\tilde{\alpha}(Y),\omega_j(X,z)).$$

Comparing the coefficients of λ^l, we obtain conditions (5.1), (5.2) and (5.3), respectively.

Theorem 5.1 is proved.

***Remarks* 5.2.** 1. Equation (5.1) means that ω_1 is always a 2-cocycle on $\mathfrak{g}$.

2. If $\mathfrak{g}_\lambda$ is a deformation of order k, then, by Eq. (5.3), we deduce that $(\mathfrak{g},\omega_k,\varepsilon,\alpha)$ is a n-Hom – Lie color algebra.

3. In particular, consider an infinitesimal deformation of $\mathfrak{g}_\lambda$ generated by $\omega\colon \wedge^n \mathfrak{g}\to\mathfrak{g}$ defined as

$$[\cdot,\ldots,\cdot]_\lambda = [\cdot,\ldots,\cdot] + \lambda\omega(\cdot,\ldots,\cdot).$$

The linear map ω generates an infinitesimal deformation of the multiplicative n-Hom – Lie color algebra $\mathfrak{g}$ if and only if:

(a) $(\mathfrak{g},\omega,\varepsilon,\alpha)$ is a multiplicative n-Hom – Lie color algebra,

(b) ω is a 2-cocycle of $\mathfrak{g}$ with coefficients in the adjoint representation, that is, ω satisfies condition (4.2) for $\rho = ad$.

Definition 5.2. *Two formal deformations $\mathfrak{g}_\lambda$ and $\mathfrak{g}_{\lambda'}$ of a n-Hom – Lie color algebra $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$ are said to be equivalent if there exists a formal isomorphism $\phi_\lambda\colon \mathfrak{g}_\lambda\to\mathfrak{g}_{\lambda'}$, where $\phi_\lambda = \sum_{i\geq 0}\phi_i\lambda^i$ and $\phi_i\colon \mathfrak{g}\to\mathfrak{g}$ are linear maps of degree zero such that $\phi_0 = id_{\mathfrak{g}}$, $\phi_i\circ\alpha = \alpha\circ\phi_i$ and*

$$\phi_\lambda\circ[x_1,\ldots,x_n]_\lambda = [\phi_\lambda(x_1),\ldots,\phi_\lambda(x_n)]_{\lambda'}. \tag{5.5}$$

It is denoted by $\mathfrak{g}_\lambda\sim\mathfrak{g}_{\lambda'}$. A formal deformation $\mathfrak{g}_\lambda$ is said to be trivial if $\mathfrak{g}_\lambda\sim\mathfrak{g}_0$.

Theorem 5.2. *Let $\mathfrak{g}_\lambda$ and $\mathfrak{g}_{\lambda'}$ be two equivalent deformations of a n-Hom – Lie color algebra $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$ generated by ω and ω', respectively. Then ω and ω' belong to the same cohomology class in the cohomology group $\mathcal{H}^2_{\alpha,\mu}(\mathfrak{g},\mathfrak{g})$.*

Proof. It is enough to prove that $\omega - \omega' \in B^2(\mathfrak{g}, \mathfrak{g})$.

We have two equivalent deformations $\mathfrak{g}_\lambda$ and $\mathfrak{g}_{\lambda'}$, then identification of coefficients of λ in (5.5) leads to

$$\omega(x_1, \ldots, x_n) + \phi_1[x_1, \ldots, x_n] = \omega'(x_1, \ldots, x_n) + [\phi(x_1), \ldots, x_n] + \ldots + [x_1, \ldots, \phi(x_n)].$$

Thus,

$$\begin{aligned}\omega(x_1, \ldots, x_n) &- \omega'(x_1, \ldots, x_n) \\ &= -\phi[x_1, \ldots, x_n] + [\phi(x_1), \ldots, x_n] + \ldots + [x_1, \ldots, \phi(x_n)] \\ &= -\phi[x_1, \ldots, x_n] + \sum_{i=1}^{n} (-1)^{n-i} \varepsilon(x_i, X^i)[x_1, \ldots, \hat{x_i}, \ldots, x_n, \phi(x_i)] \\ &= \delta^1 \phi(X, x_n).\end{aligned}$$

Therefore, $\omega - \omega' \in B^2(\mathfrak{g}, \mathfrak{g})$.

Theorem 5.2 is proved.

6. Nijenhuis operators on n-Hom – Lie color algebras. Motivated by the infinitesimally trivial deformation introduced in this section, we define the notion of Nijenhuis operator for a multiplicative n-Hom – Lie color algebras which is a generalization of Nijenhuis operator on n-Lie algebras given in [20]. Then we define the notion of a product structure on a n-Hom – Lie color algebra using Nijenhuis operators (see [24] for non-graded case).

6.1. Definitions and constructions. Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a n-Hom – Lie color algebra and $\mathfrak{g}_\lambda := (\mathfrak{g}, [\cdot, \ldots, \cdot]_\lambda, \varepsilon, \alpha)$ be a deformation of $\mathfrak{g}$ of order $(n-1)$.

Definition 6.1. *The deformation $\mathfrak{g}_\lambda$ is said to be infinitesimally trivial if there exists a linear map of degree zero $\mathcal{N} : \mathfrak{g} \to \mathfrak{g}$ such that $\mathcal{T}_\lambda = id + \lambda\mathcal{N} : \mathfrak{g}_\lambda \to \mathfrak{g}$ is an algebra morphism, that is, for all $x_1, \ldots, x_n \in \mathcal{H}(\mathfrak{g})$, we have*

$$\mathcal{T}_\lambda \circ \alpha = \alpha \circ \mathcal{T}_\lambda, \tag{6.1}$$

$$\mathcal{T}_\lambda[x_1, \ldots, x_n]_\lambda = [\mathcal{T}_\lambda(x_1), \ldots, \mathcal{T}_\lambda(x_n)]. \tag{6.2}$$

The condition (6.1) is equivalent to

$$\mathcal{N} \circ \alpha = \alpha \circ \mathcal{N}.$$

The left-hand side of Eq. (6.2) equals to

$$\begin{aligned}&[x_1, \ldots, x_n] + \lambda\big(\omega_1(x_1, \ldots, x_n) + \mathcal{N}[x_1, \ldots, x_n]\big) \\ &\quad + \sum_{j=1}^{n-2} \lambda^{j+1}\big(\mathcal{N}\omega_j(x_1, \ldots, x_n) + \omega_{j+1}(x_1, \ldots, x_n)\big) + \lambda^n \mathcal{N}\omega_{n-1}(x_1, \ldots, x_n).\end{aligned}$$

The right-hand side of Eq. (6.2) equals to

$$[x_1, \ldots, x_n] + \lambda \sum_{i_1=1}^{n} [x_1, \ldots, \mathcal{N}x_{i_1}, \ldots, x_n] + \lambda^2 \sum_{i_1 < i_2}^{n} [x_1, \ldots, \mathcal{N}x_{i_1}, \ldots, \mathcal{N}x_{i_2}, \ldots, x_n] + \ldots$$

$$+ \lambda^{n-1} \sum_{i_1<i_2<\ldots<i_{n-1}} [x_1, \ldots, \mathcal{N}x_{i_1}, \ldots, \mathcal{N}x_{i_2}, \ldots, \mathcal{N}x_{i_{n-1}}, x_n]$$

$$+ \lambda^n [\mathcal{N}x_1, \ldots, \mathcal{N}x_2, \ldots, \mathcal{N}x_n].$$

Therefore, by identification of coefficients, we have

$$\omega_1(x_1, \ldots, x_n) + \mathcal{N}[x_1, \ldots, x_n] = \sum_{i_1=1}^{n} [x_1, \ldots, \mathcal{N}x_{i_1}, \ldots, x_n],$$

$$\mathcal{N}\omega_{n-1}(x_1, \ldots, x_n) = [\mathcal{N}x_1, \ldots, \mathcal{N}x_2, \ldots, \mathcal{N}x_n],$$

$$\mathcal{N}\omega_l(x_1, \ldots, x_n) + \omega_{l-1}(x_1, \ldots, x_n) = \sum_{i_1<i_2<\ldots<i_l} [x_1, \ldots, \mathcal{N}x_{i_1}, \ldots, \mathcal{N}x_{i_2}, \ldots, \mathcal{N}x_{i_l}, \ldots, x_n]$$

for all $2 \leq l \leq n-1$.

Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a n-Hom – Lie color algebra, and $\mathcal{N} : \mathfrak{g} \to \mathfrak{g}$ be a linear map of degree zero. Define a n-ary bracket $[\cdot, \ldots, \cdot]^1_{\mathcal{N}} : \wedge^n \mathfrak{g} \to \mathfrak{g}$ by

$$[x_1, \ldots, x_n]^1_{\mathcal{N}} = \sum_{i=1}^{n} [x_1, \ldots, \mathcal{N}x_i, \ldots, x_n] - \mathcal{N}[x_1, x_2, \ldots, x_n].$$

By induction, we define n-ary brackets $[\cdot, \ldots, \cdot]^j_{\mathcal{N}} : \wedge^n \mathfrak{g} \to \mathfrak{g}$, $2 \leq j \leq n-1$,

$$[x_1, \ldots, x_n]^j_{\mathcal{N}} = \sum_{i_1<i_2<\ldots<i_j} [\ldots, \mathcal{N}x_{i_1}, \ldots, \mathcal{N}x_{i_2}, \ldots, \mathcal{N}x_{i_j}, \ldots] - \mathcal{N}[x_1, \ldots, x_n]^{j-1}_{\mathcal{N}}.$$

In particular, we have

$$[x_1, \ldots, x_n]^{n-1}_{\mathcal{N}} = \sum_{i_1<i_2<\ldots<i_{n-1}} [\mathcal{N}x_{i_1}, \ldots, \mathcal{N}x_{i_{n-1}}, x_n] - \mathcal{N}[x_1, \ldots, x_n]^{n-2}_{\mathcal{N}}.$$

These observations motivate the following definition.

Definition 6.2. *Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a n-Hom – Lie color algebra. A linear map of degree zero $\mathcal{N} : \mathfrak{g} \to \mathfrak{g}$ is called a Nijenhuis operator if it satisfies $\mathcal{N} \circ \alpha = \alpha \circ \mathcal{N}$ and*

$$[\mathcal{N}x_1, \ldots, \mathcal{N}x_2, \ldots, \mathcal{N}x_n] = \mathcal{N}[x_1, \ldots, x_n]^{n-1}_{\mathcal{N}}.$$

The above condition can be written as

$$\sum_{j=0}^{n} (-1)^{n-j} \mathcal{N}^{n-j} \left(\sum_{i_1<i_2<\ldots<i_j} [\ldots, \mathcal{N}x_{i_1}, \ldots, \mathcal{N}x_{i_2}, \ldots, \mathcal{N}x_{i_j}, \ldots] \right) = 0.$$

We have seen that any trivial deformation produces a Nijenhuis operator. Conversely, any Nijenhuis operator gives a trivial deformation as the following theorem shows.

Theorem 6.1. *Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a n-Hom – Lie color algebra and $\mathcal{N}$ be a Nijenhuis operator on $(\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$. Then the bracket*

$$[x_1, \ldots, x_1]_\lambda = [x_1, \ldots, x_1] + \sum_{i=1}^{n-1} \lambda^i [x_1, \ldots, x_n]^i_{\mathcal{N}}$$

defines a deformation of $\mathfrak{g}$ which is infinitesimally trivial.

Proof. Follows from the above characterization of identity (6.2), Theorem 5.1 and Lemma 3.1.

Proposition 6.1. *Let $\mathcal{N}$ be a bijective Nijenhuis operator on a n-Hom – Lie color algebra $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$, $a\in\mathfrak{g}_0$ such that $\alpha(a)=a$ and $\mathcal{N}(a)\in Z(\mathfrak{g})$. Then $\mathcal{N}$ is a Nijenhuis operator on the $(n-1)$-Hom – Lie color algebra $(\mathfrak{g},\{\cdot,\ldots,\cdot\},\varepsilon,\alpha)$.*

Recall that, if $(\mathfrak{g},[\cdot,\cdot],\varepsilon,\alpha)$ is a Hom – Lie color algebra, the Nijenhuis operator condition writes as

$$[\mathcal{N}x,\mathcal{N}y]=\mathcal{N}[\mathcal{N}x,y]+\mathcal{N}[x,\mathcal{N}y]-\mathcal{N}^2[x,y].$$

***Corollary* 6.1.** *Let $\mathcal{N}$ be a Nijenhuis operator on a 3-Hom – Lie color algebra $(\mathfrak{g},[\cdot,\cdot,\cdot],\varepsilon,\alpha)$. If $\mathcal{N}$ is a bijection, then it is a Nijenhuis operator on the Hom – Lie color algebra $(\mathfrak{g},\{\cdot,\cdot\},\varepsilon,\alpha)$ such that $\mathcal{N}(a)\in Z(\mathfrak{g})$.*

6.2. Product structures on n-Hom – Lie color algebras. In this subsection, we study a notion of product structure on a n-Hom – Lie color algebra and show that it leads to a special decomposition of the original n-Hom – Lie color algebra. Moreover, we introduce a notion of strict product structure on a n-Hom – Lie color algebra and provide example.

Definition 6.3. *Let $\mathcal{G}=(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$ be a n-Hom – Lie color algebra.*

An almost product structure on $\mathcal{G}$ is a linear map of degree zero $\mathcal{P}\colon\mathfrak{g}\to\mathfrak{g}$, $\mathcal{P}\neq\pm Id_{\mathfrak{g}}$, satisfying $\mathcal{P}^2=Id_{\mathfrak{g}}$.

An almost product structure is called product stucture on $\mathcal{G}$ if it is a Nijenhuis operator.

***Remark* 6.1.** One can understand a product structure on $\mathcal{G}$ as a linear map of degree zero $\mathcal{P}\colon\mathfrak{g}\to\mathfrak{g}$ satisfying

$$\mathcal{P}^2=Id,\qquad \mathcal{P}\alpha=\alpha\mathcal{P},$$

$$[\mathcal{P}x_1,\mathcal{P}x_2,\ldots,\mathcal{P}x_n]=\sum_{j=0}^{n-1}(-1)^{n-j-1}\mathcal{P}^{\mu_{n-j}}\left(\sum_{i_1<i_2<\ldots<i_j}^{n}[\ldots,\mathcal{P}x_{i_1},\ldots,\mathcal{P}x_{i_j},\ldots]\right) \tag{6.3}$$

for $x_1,\ldots,x_n\in\mathfrak{g}$ and $\mu_k=\begin{cases}1, & \text{if } k \text{ is odd},\\ 0, & \text{if } k \text{ is even}.\end{cases}$

Theorem 6.2. *Let $\mathcal{G}=(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$ be a n-Hom – Lie color algebra. Then $\mathcal{G}$ has a product structure if and only if $\mathfrak{g}=\bigoplus_{\gamma\in\Gamma}\mathfrak{g}_\gamma$ admits a decomposition $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-$ with $\mathfrak{g}_\gamma=(\mathfrak{g}_\gamma)_+\oplus(\mathfrak{g}_\gamma)_-$ and where the eigenspaces $\mathfrak{g}_+=\bigoplus_{\gamma\in\Gamma}(\mathfrak{g}_\gamma)_+$ and $\mathfrak{g}_-=\bigoplus_{\gamma\in\Gamma}(\mathfrak{g}_\gamma)_-$ of $\mathfrak{g}$ associated to the eigenvalues 1 and -1 respectively are subalgebras.*

Proof. Let $\mathcal{P}$ be a product structure on $(\mathfrak{g},[\cdot,\ldots,\cdot],\varepsilon,\alpha)$. According to Eq. (6.3), for all element $x_i\in\mathfrak{g}_+$, we have

$$\begin{aligned}[x_1,\ldots,x_n]&=[\mathcal{P}x_1,\ldots,\mathcal{P}x_n]\\&=\sum_{j=0}^{n-1}(-1)^{n-j-1}\mathcal{P}^{\mu_{n-j}}\left(\sum_{i_1<i_2<\ldots<i_j}[\ldots,\mathcal{P}x_{i_1},\ldots,\mathcal{P}x_{i_2},\ldots,\mathcal{P}x_{i_j},\ldots]\right)\\&=\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n}{j}\mathcal{P}^{\mu_{n-j}}[x_1,\ldots,x_n].\end{aligned}$$

Then we obtain

$$[x_1,\ldots,x_n]=\sum_{2j+1<n}\binom{n}{2j+1}\mathcal{P}[x_1,\ldots,x_n]-\sum_{2j<n}\binom{n}{2j}[x_1,\ldots,x_n],\quad \text{if}\quad n\quad \text{is even,} \tag{6.4}$$

$$[x_1,\ldots,x_n]=\sum_{2j<n}\binom{n}{2j}\mathcal{P}[x_1,\ldots,x_n]-\sum_{2j+1<n}\binom{n}{2j+1}[x_1,\ldots,x_n],\quad \text{if}\quad n\quad \text{is odd.} \tag{6.5}$$

By the binomial theorem, we get

$$\sum_{2j<n}\binom{n}{2j}-\sum_{2j+1<n}\binom{n}{2j+1}=(-1)^{n+1}. \tag{6.6}$$

Apply the above condition to Eqs. (6.4) and (6.5), we have, for all $x_i\in\mathfrak{g}_+$, $\mathcal{P}[x_1,\ldots,x_n]=[x_1,\ldots,x_n]$. Let $x\in\mathfrak{g}_+$, then we obtain $\mathcal{P}\circ\alpha(x)=\alpha\circ\mathcal{P}(x)=\alpha(x)$, which implies that $\alpha(x)\subseteq\mathfrak{g}_+$. So, $\mathfrak{g}_+$ is subalgebra of $\mathfrak{g}$. Similarly, we show that $\mathfrak{g}_-$ is subalgebra of $\mathfrak{g}$.

Conversely, we define a linear map of degree zero $\mathcal{P}\colon\mathfrak{g}\to\mathfrak{g}$ such that, for all $x\in\mathfrak{g}_+$ and $y\in\mathfrak{g}_-$,

$$\mathcal{P}(x+y)=x-y. \tag{6.7}$$

We have $\mathcal{P}^2(x+y)=\mathcal{P}(x-y)=x+y$, then $\mathcal{P}^2=Id$. Since $x\in\mathfrak{g}_+$, then $\alpha(x)\in\mathfrak{g}_+$. Thus, $\mathcal{P}\circ\alpha(x)=\alpha(x)=\alpha\circ\mathcal{P}(x)$. Similarly, $\mathcal{P}\circ\alpha(y)=\alpha\circ\mathcal{P}(y)$.

If n is even, since $\mathfrak{g}_+$ is a subalgebra of $\mathfrak{g}$, then, for $x_i\in\mathfrak{g}_+$, we have

$$\begin{aligned}
&\sum_{j=0}^{n-1}(-1)^{n-j-1}\mathcal{P}^{\mu_{n-j}}\left(\sum_{i_1<i_2<\ldots<i_j}[\ldots,\mathcal{P}x_{i_1},\ldots,\mathcal{P}x_{i_2},\ldots,\mathcal{P}x_{i_j},\ldots]\right)\\
&\quad=\sum_{2j+1<n}\binom{n}{2j+1}\mathcal{P}[x_1,\ldots,x_n]-\sum_{2j<n}\binom{n}{2j}[x_1,\ldots,x_n]\\
&\quad=\sum_{2j+1<n}\binom{n}{2j+1}[x_1,\ldots,x_n]-\sum_{2j<n}\binom{n}{2j}[x_1,\ldots,x_n]\\
&\quad=\left(\sum_{2j+1<n}\binom{n}{2j+1}-\sum_{2j<n}\binom{n}{2j}\right)[x_1,\ldots,x_n]\\
&\quad\overset{(6.6)}{=}[\mathcal{P}x_1,\mathcal{P}x_2,\ldots,\mathcal{P}x_n].
\end{aligned}$$

Also if n is odd, we have

$$\begin{aligned}
&\sum_{j=0}^{n-1}(-1)^{n-j-1}\mathcal{P}^{\mu_{n-j}}\left(\sum_{i_1<i_2<\ldots<i_j}[\ldots,\mathcal{P}x_{i_1},\ldots,\mathcal{P}x_{i_2},\ldots,\mathcal{P}x_{i_j},\ldots]\right)\\
&\quad=\left(\sum_{2j<n}\binom{n}{2j}-\sum_{2j+1<n}\binom{n}{2j+1}\right)[x_1,\ldots,x_n]
\end{aligned}$$

$$\overset{(6.6)}{=} [\mathcal{P}x_1, \mathcal{P}x_2, \ldots, \mathcal{P}x_n].$$

One may check for all $x_i \in \mathfrak{g}_-$ similarly. Then $\mathcal{P}$ is a product structure on $\mathcal{G}$.

Theorem 6.2 is proved.

Let $\mathcal{G} = (\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a n-Hom – Lie color algebra and $\Theta \colon \mathfrak{g} \to \mathfrak{g}$ be a linear map of degree γ. Then Θ is said in the *centroid* of $\mathcal{G}$ if, for all homogeneous elements $x_i \in \mathfrak{g}$, $\Theta \circ \alpha = \alpha \circ \Theta$ and

$$\Theta[x_1, x_2, \ldots, x_n] = [\Theta x_1, x_2, \ldots, x_n]. \tag{6.8}$$

The above identity is equivalent to

$$\Theta[x_1, x_2, \ldots, x_n] = \varepsilon(\gamma, X_i)\Big[x_1, \ldots, \underbrace{\Theta x_i}_{i\text{th place}}, \ldots, x_n\Big]. \tag{6.9}$$

Definition 6.4. *An almost product structure $\mathcal{P}$ on $\mathcal{G}$ is called a strict product structure if it is an element of the centroid.*

Lemma 6.1. *Let $\mathcal{P}$ be a strict product structure on $\mathcal{G}$. Then $\mathcal{P}$ is a product structure on $\mathcal{G}$ such that $\Big[\mathfrak{g}_+, \ldots, \mathfrak{g}_+, \underbrace{\mathfrak{g}_+}_{i\text{th place}}, \mathfrak{g}_-, \ldots, \mathfrak{g}_-\Big] = 0$ for all $1 \le i \le n-1$.*

Proof. The identity (6.3) is equivalent to

$$\sum_{j=0}^{n} (-1)^{n-j} \mathcal{P}^{\mu_{n-j}} \left(\sum_{i_1 < i_2 < \ldots < i_j}^{n} [\ldots, \mathcal{P}x_{i_1}, \ldots, \mathcal{P}x_{i_2}, \ldots, \mathcal{P}x_{i_j}, \ldots] \right) = 0.$$

Then, if $\mathcal{P}$ is a strict product structure on $\mathcal{G}$ and $x_i \in \mathfrak{g}$, we have

$$\sum_{j=0}^{n} (-1)^{n-j} \mathcal{P}^{\mu_{n-j}} \left(\sum_{i_1 < i_2 < \ldots < i_j}^{n} [\ldots, \mathcal{P}x_{i_1}, \ldots, \mathcal{P}x_{i_2}, \ldots, \mathcal{P}x_{i_j}, \ldots] \right)$$

$$= \sum_{j=0}^{n} (-1)^{n-j} \left(\sum_{i_1 < i_2 < \ldots < i_j}^{n} \mathcal{P}^{\mu_{n-j}} \mathcal{P}^{\mu_j} [x_1, \ldots, x_n] \right)$$

$$= \left(\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \right) \mathcal{P}^{\mu_n} [x_1, \ldots, x_n] = 0.$$

Thus, $\mathcal{P}$ is a product structure.

Fix i such that $0 < i < n$, and let (k, l) such that $0 < k \le i < l \le n$ with $x_k \in \mathfrak{g}_+$ and $x_l \in \mathfrak{g}_-$. According to Eq. (6.8), we get

$$\mathcal{P}[x_1, \ldots, x_i, x_{i+1}, \ldots, x_n] = [\mathcal{P}x_1, \ldots, x_i, \ldots, x_i] = [x_1, \ldots, x_i, \ldots, x_i].$$

On the other hand, by Eq. (6.9) we have

$$\mathcal{P}[x_1, \ldots, x_i, x_{i+1}, \ldots, x_i] = [x_1, \ldots, x_i, \mathcal{P}x_{i+1}, \ldots, x_i]$$

$$= -[x_1, \ldots, x_i, x_{i+1}, \ldots, x_i].$$

Then we obtain

$$\Big[\mathfrak{g}_+, \ldots, \ \mathfrak{g}_+, \ \underbrace{\mathfrak{g}_+}_{i\text{th place}}, \ \mathfrak{g}_-, \ldots, \ \mathfrak{g}_-\Big] = 0.$$

Lemma 6.1 is proved.

Proposition 6.2. *Let $\mathcal{G} = (\mathfrak{g}, [\cdot, \ldots, \cdot], \varepsilon, \alpha)$ be a n-Hom – Lie color algebra. Then $\mathcal{G}$ has a strict product structure if and only if $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ admits a decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are graded subalgebras of $\mathfrak{g}$ such that $\Big[\mathfrak{g}_+, \ldots, \ \mathfrak{g}_+, \ \underbrace{\mathfrak{g}_+}_{i\text{th place}}, \ \mathfrak{g}_-, \ldots, \ \mathfrak{g}_-\Big] = 0, \ 1 \leq i \leq n-1$.*

Proof. The first implication is a direct computation from Lemma 6.1. Conversely, on the basis of Theorem 6.2, the map $\mathcal{P}$, defined in Eq. (6.7), is an almost product structure and, for all $x_k = x_k^+ + x_k^- \in \mathfrak{g}$, where $x_k^+ \in \mathfrak{g}_+$ and $x_k^- \in \mathfrak{g}_-$, we have

$$\begin{aligned}
\mathcal{P}[x_1, x_2, \ldots, x_n] &= \mathcal{P}[x_1^+ + x_1^-, x_2^+ + x_2^-, \ldots, x_n^+ + x_n^-] \\
&= \mathcal{P}[x_1^+, x_2^+, \ldots, x_n^+] + \mathcal{P}[x_1^-, x_2^-, \ldots, x_n^-] \\
&= [x_1^+, x_2^+, \ldots, x_n^+] - [x_1^-, x_2^-, \ldots, x_n^-] \\
&= [\mathcal{P}x_1^+, x_2^+, \ldots, x_n^+] + [\mathcal{P}x_1^-, x_2^-, \ldots, x_n^-] = [\mathcal{P}x_1, x_2, \ldots, x_n].
\end{aligned}$$

Then $\mathcal{P}$ is a strict product structure on $\mathcal{G}$.

Proposition 6.2 is proved.

Example **6.1.** Let $\mathfrak{g}_\alpha = (\mathfrak{g}, [\cdot, \cdot, \cdot, \cdot]_\alpha, \varepsilon, \alpha)$ be the 4-Hom – Lie color algebra defined in Example 3.1. Define a linear map of degree zero $\mathcal{P} : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathcal{P}(e_1) = e_1, \qquad \mathcal{P}(e_2) = e_2, \qquad \mathcal{P}(e_3) = -e_3, \qquad \mathcal{P}(e_4) = -e_4 \qquad \text{and} \qquad \mathcal{P}(e_5) = -e_5.$$

It easy to prove that $\mathcal{P}$ is a strict product structure, therefore it is a product structure. Using Theorem 6.2, we deduce that the graded subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are generated by $\langle e_1, e_2 \rangle$ and $\langle e_3, e_4, e_5 \rangle$, respectively. Thus,

$$\mathfrak{g}_+ = \mathfrak{g}_{(0,0)} \qquad \text{and} \qquad \mathfrak{g}_- = \mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(1,1)}.$$

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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