DOI: 10.3842/umzh.v75i9.7241

UDC 512.5

Nripendu Bera[1] (Department of Mathematics, Jadavpur University, Kolkata, India),
Basudeb Dhara (Department of Mathematics, Belda College, Belda, Paschim Medinipur, India)

JORDAN HOMODERIVATION BEHAVIOR OF GENERALIZED DERIVATIONS IN PRIME RINGS

ПОВЕДІНКА ЖОРДАНОВОЇ ГОМОПОХІДНОЇ ДЛЯ УЗАГАЛЬНЕНИХ ПОХІДНИХ НА ПРОСТИХ КІЛЬЦЯХ

Suppose that R is a prime ring with $\mathrm{char}(R) \neq 2$ and $f(\xi_1, \ldots, \xi_n)$ is a noncentral multilinear polynomial over $C(= Z(U))$, where U is the Utumi quotient ring of R. An additive mapping $h\colon R \to R$ is called homoderivation if $h(ab) = h(a)h(b) + h(a)b + ah(b)$ for all $a, b \in R$. We investigate the behavior of three generalized derivations F, G, and H of R satisfying the condition

$$F(\xi^2) = G(\xi)^2 + H(\xi)\xi + \xi H(\xi)$$

for all $\xi \in f(R) = \{f(\xi_1, \ldots, \xi_n) \mid \xi_1, \ldots, \xi_n \in R\}$.

Припустимо, що R — просте кільце з $\mathrm{char}(R) \neq 2$, а $f(\xi_1, \ldots, \xi_n)$ — нецентральний мультилінійний поліном над $C(= Z(U))$, де U — фактор-кільце Утумі R. Адитивне відображення $h\colon R \to R$ називається гомопохідною, якщо $h(ab) = h(a)h(b) + h(a)b + ah(b)$ для всіх $a, b \in R$. Досліджено поведінку трьох узагальнених похідних F, G та H на R, що задовольняють умову

$$F(\xi^2) = G(\xi)^2 + H(\xi)\xi + \xi H(\xi)$$

для всіх $\xi \in f(R) = \{f(\xi_1, \ldots, \xi_n) | \xi_1, \ldots, \xi_n \in R\}$.

1. Introduction. In this paper we consider that R is a prime ring of characteristic different from 2. Also U is the Utumi quotient ring of R, $C = Z(U)$ is the extended centroid of R and $f(\xi_1, \ldots, \xi_n)$ is a noncentral multilinear polynomial over C. By the word derivation of R, we mean an additive mapping $d\colon R \to R$ such that $d(ab) = d(a)b + ad(b)$ holds for all $a, b \in R$. By the word Jordan derivation of R, we mean an additive mapping $d\colon R \to R$ such that $d(a^2) = d(a)a + ad(a)$ holds for all $a \in R$. Any derivation is Jordan derivation, but the converse is not true in general. Herstein [15] proved that every Jordan derivation in a prime ring of $\mathrm{char}(R) \neq 2$ is a derivation.

An additive mapping $F\colon R \to R$ is called a generalized derivation, if there exists a derivation d on R such that $F(ab) = F(a)b + ad(b)$ holds for all $a, b \in R$. Basic examples of generalized derivations are derivations, generalized inner derivations (i.e., maps of type $x \to a_1x + xb_1$ for some $a_1, b_1 \in R$). In [18], Lee proved that any generalized derivation of R can be uniquely extended to a generalized derivation of U and its form will be $g(x) = ax + d(x)$ for some $a \in U$, where d is the associated derivation. The Lie commutator of x, y is $[x, y]$ and also defined by $[x, y] = xy - yx$ for all $x, y \in R$; also the symbol $x \circ y$ stands for the anticommutator $xy + yx$.

An additive mapping $F\colon R \to R$ is called a homomorphism or antihomomorphism on R if $F(ab) = F(a)F(b)$ or $F(ab) = F(b)F(a)$ holds for all $a, b \in R$, respectively. The additive mapping F is called a Jordan homomorphism, if $F(a^2) = F(a)^2$ holds for all $a \in R$.

It is natural to consider a map which will behave like a derivation as well as a homomorphism. In this point of view, El Sofy [13] introduced the concept of homoderivation maps on a ring R. An

[1] Corresponding author, e-mail: nripendub@gmail.com.

additive mapping h from R into itself is called homoderivation if $h(ab) = h(a)h(b) + h(a)b + ah(b)$ for all $a, b \in R$. An example of such mapping is $h(a) = f(a) - a$ for all $a \in R$, where f is an endomorphism on R. As above, we can define Jordan homoderivation maps. An additive mapping h from R into itself is called Jordan homoderivation if $h(a^2) = h(a)h(a) + h(a)a + ah(a)$ for all $a \in R$. It is very clear that every homoderivation is Jordan homoderivation, but in general the converse need not be true. There are few papers in literature which studied the homoderivation maps in prime rings and obtained commutativity of ring under certain conditions (see [2–4, 21]).

In the spirit of consideration of above maps, in the present paper we consider three generalized derivations F, G, H which satisfy the situation

$$F(\xi^2) = G(\xi)^2 + H(\xi)\xi + \xi H(\xi)$$

for all $\xi \in f(R)$.

There are many papers in literature which studied the homomorphism or antihomomorphism behavior of generalized derivations in prime rings (see [1, 5, 9–12, 23]).

In the present paper, we study the Jordan homoderivation behavior of three generalized derivations in prime rings.

More precisely, we prove the following theorem.

Theorem 1.1. *Let R be a prime ring with $\mathrm{char}(R) \neq 2$ and $f(\xi_1, \ldots, \xi_n)$ be a noncentral multilinear polynomial over $C(= Z(U))$, where U be the Utumi quotient ring of R. If F, G and H are three generalized derivations on R satisfying*

$$F(\xi^2) = G(\xi)^2 + H(\xi)\xi + \xi H(\xi)$$

for all $\xi \in f(R)$, then d, g and h are three inner derivations or g is inner, d, h are outer, d, h are linearly C-dependent modulo inner derivations of U. Moreover, the forms of the maps are as follows:

(1) *there exist a derivation d on R and $\lambda_1, \lambda_2, \lambda_3 \in C$ such that $F(x) = \lambda_1 x + d(x)$, $G(x) = \lambda_2 x$ and $H(x) = \lambda_3 x + d(x)$ for all $x \in R$ with $\lambda_1 = \lambda_2^2 + 2\lambda_3$;*

(2) *there exist a derivation d on R, $a_1 \in U$ and $\lambda_1, \lambda_2, \lambda_3 \in C$ such that $F(x) = \lambda_1 x + d(x)$, $G(x) = \lambda_2 x$ and $H(x) = \lambda_3 x + [a_1, x] + d(x)$ for all $x \in R$ with $f(R)^2 \in C$ and $\lambda_1 = \lambda_2^2 + 2\lambda_3$.*

If we consider $H = 0$ in our main theorem, then we get the following corollary.

***Corollary* 1.1.** *Let R be a prime ring of $\mathrm{char}(R) \neq 2$ with Utumi quotient ring U and extended centroid C and $f(\xi_1, \ldots, \xi_n)$ be a multilinear polynomial over C, which is a noncentral valued on R. If F and G are two nonzero generalized derivations of R such that*

$$F(\xi^2) = G(\xi)^2$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$, then one of the following holds:

(1) *there exists $\lambda \in C$ such that $F(x) = \lambda^2 x$, $G(x) = \lambda x$ for all $x \in R$;*

(2) *there exist $\lambda \in C$ and $a \in U$ such that $F(x) = \lambda^2 x + [a, x]$, $G(x) = \lambda x$ for all $x \in R$ with $f(\xi_1, \ldots, \xi_n)^2$ is central valued on R.*

***Example* 1.1.** Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ be the set of all integers. Then it is clear that R is not prime ring, because $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0)$. Define the maps $F, G, d, g\colon R \to R$ by $F\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $d\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$ and $G\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}$, $g\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ on R. Then F and G are generalized derivations of R associated derivations d and g, respectively. Now we consider a multilinear polynomial $f(X, Y) = XY$, which is not central valued on R. We see that $F\big(f(X, Y)^2\big) = G\big(f(X, Y)\big)^2$ for all $X, Y \in R$ but $G(X) \neq \lambda X$, where $\lambda \in C$. This example show that the primeness hypothesis is not superfluous in our above corollary.

2. The matrix ring case and results for inner generalized derivations.

Lemma 2.1 [8, Lemma 1]. *Let C be an infinite field and $m \geq 2$. If $\kappa_1, \ldots, \kappa_t$ are not scalar matrices in $M_m(C)$, then there exists some invertible matrix $P \in M_m(C)$ such that all the entries of $P\kappa_1 P^{-1}, \ldots, P\kappa_t P^{-1}$ are nonzero.*

Lemma 2.2. *Suppose that $R = M_m(C)$, $m \geq 2$, be the ring of all $m \times m$ matrices over the field C and $f(R)$ be the set of all evaluations of the polynomial $f(\xi_1, \ldots, \xi_n)$ in R. If $A_1, A_2, A_3, a_3, a_4 \in R$ such that*

$$A_1 u^2 + u^2 A_2 + a_3 u a_3 u + u A_3 u + u a_4 u a_4 + a_3 u^2 a_4 = 0$$

for all $u \in f(R)$, then a_3 and a_4 are scalar matrices.

Proof. 1. *C is infinite field.* To prove this case of the lemma, we assume on contrary that $a_3 \notin Z(R)$ and $a_4 \notin Z(R)$. We prove that this case leads to a contradiction. Since $a_3 \notin Z(R)$ and $a_4 \notin Z(R)$, by Lemma 2.1 there exists a C-automorphism θ of $M_m(C)$ for which $a_3' = \theta(a_3)$ and $a_4' = \theta(a_4)$ have all nonzero entries. Clearly, $A_1' = \theta(A_1)$, $A_2' = \theta(A_2)$, $A_3' = \theta(A_3)$, a_3' and a_4' must satisfying the condition

$$A_1' f(\xi)^2 + f(\xi)^2 A_2' + a_3' f(\xi) a_3' f(\xi) + f(\xi) A_3' f(\xi) + f(\xi) a_4' f(\xi) a_4' + a_3' f(\xi)^2 a_4' = 0. \tag{1}$$

By e_{ij}, we consider a matrix unit with 1 in (i, j)-entry and zero elsewhere. Then it is obvious that $e_{ij}^2 = 0$. Since $f(\xi_1, \ldots, \xi_n)$ is not central valued, by [19] there exist some matrices $r_1, \ldots, r_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(r_1, \ldots, r_n) = \gamma e_{ij}$ with $i \neq j$.

Therefore, substituting $f(\xi) = \gamma e_{ij}$ in (1), we get

$$a_3' e_{ij} a_3' e_{ij} + e_{ij} A_3' e_{ij} + e_{ij} a_4' e_{ij} a_4' = 0.$$

Multiplying by e_{ij} in left-hand side of the above relation, we have

$$e_{ij} a_3' e_{ij} a_3' e_{ij} = 0,$$

which gives a contradiction, since $a_3' = \theta(a_3)$ have all nonzero entries. Thus, we conclude that a_3 is scalar matrix. Again we multiplying by e_{ij} in right-hand side of the above relation, we obtain

$$e_{ij} a_4' e_{ij} a_4' e_{ij} = 0,$$

which gives a contradiction, since $a_4' = \theta(a_4)$ have all nonzero entries. Thus, we conclude that a_4 is scalar matrix. Therefore, when C is infinite field, then a_3 and a_4 are scalar matrices.

2. C *is finite field.* Suppose that K be an infinite field which is an extension of C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(\xi_1, \ldots, \xi_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Now the generalized polynomial identity is

$$\begin{aligned}\Psi(\xi_1, \ldots, \xi_n) = A_1 f(\xi_1, \ldots, \xi_n)^2 + f(\xi_1, \ldots, \xi_n)^2 A_2 \\ + a_3 f(\xi_1, \ldots, \xi_n) a_3 f(\xi_1, \ldots, \xi_n) + f(\xi_1, \ldots, \xi_n) A_3 f(\xi_1, \ldots, \xi_n) \\ + f(\xi_1, \ldots, \xi_n) a_4 f(\xi_1, \ldots, \xi_n) a_4 + a_3 f(\xi_1, \ldots, \xi_n)^2 a_4 = 0.\end{aligned}$$

This is not only a generalized polynomial identity for R, but also a multihomogeneous of multidegree $(2, \ldots, 2)$ in the indeterminates $\xi_1, \ldots, \xi_n$.

Hence, the complete linearization of $\Psi(\xi_1, \ldots, \xi_n)$ yields a multilinear generalized polynomial $\Theta(\xi_1, \ldots, \xi_n, t_1, \ldots, t_n)$ in $2n$ indeterminates, moreover,

$$\Theta(\xi_1, \ldots, \xi_n, t_1, \ldots, t_n) = 2^n \Psi(\xi_1, \ldots, \xi_n).$$

Clearly, the multilinear polynomial $\Theta(\xi_1, \ldots, \xi_n, t_1, \ldots, t_n) = 0$ is a generalized polynomial identity for R and $\overline{R}$ too. Since char $(C) \neq 2$, we obtain $\Psi(\xi_1, \ldots, \xi_n) = 0$ for all $\xi_1, \ldots, \xi_n \in \overline{R}$ and then conclusion follows as above when C was infinite.

Lemma 2.2 is proved.

Lemma 2.3. *Suppose that R is a prime ring of* $\mathrm{char}(R) \neq 2$ *and $f(x_1, \ldots, x_n)$ is a noncentral multilinear polynomial over C. If $A_1, A_2, A_3, a_3, a_4 \in R$ such that R satisfies $\Psi(\xi_1, \ldots, \xi_n)$, then $a_3 \in C$ and $a_4 \in C$.*

Proof. We will show this case by contradiction. Suppose that both a_3 and a_4 are not central. By hypothesis, we have

$$\begin{aligned}\Psi(\xi_1, \ldots, \xi_n) = A_1 f(\xi_1, \ldots, \xi_n)^2 + f(\xi_1, \ldots, \xi_n)^2 A_2 \\ + a_3 f(\xi_1, \ldots, \xi_n) a_3 f(\xi_1, \ldots, \xi_n) + f(\xi_1, \ldots, \xi_n) A_3 f(\xi_1, \ldots, \xi_n) \\ + f(\xi_1, \ldots, \xi_n) a_4 f(\xi_1, \ldots, \xi_n) a_4 + a_3 f(\xi_1, \ldots, \xi_n)^2 a_4 = 0\end{aligned}$$

for all $\xi_1, \ldots, \xi_n \in R$.

Since R and U satisfy the same generalized polynomial identities (see [7]), so, U satisfies $\Psi(\xi_1, \ldots, \xi_n) = 0$. Suppose that R does not satisfy any nontrivial GPI. Let $T = U *_C C\{\xi_1, \xi_2, \ldots, \xi_n\}$, the free product of U and $C\{\xi_1, \ldots, \xi_n\}$, the free C-algebra in noncommuting indeterminates $\xi_1, \xi_2, \ldots, \xi_n$. Then $\Psi(\xi_1, \ldots, \xi_n)$ is zero element in T. This gives that $\{A_1, a_3, 1\}$ is linearly C-dependent, hence, there exist $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 A_1 + \alpha_2 a_3 + \alpha_3 .1 = 0$. If $\alpha_1 = 0$, then $\alpha_2 \neq 0$ and so $a_3 = -\alpha_2^{-1} \alpha_3 \in C$, a contradiction. Therefore, either $\alpha_1 \neq 0$. Then $A_1 = \alpha a_3 + \beta$, where $\alpha = -\alpha_1^{-1} \alpha_2, \beta = -\alpha_1^{-1} \alpha_3$. Hence,

$$\begin{aligned}(\alpha a_3 + \beta) f(\xi_1, \ldots, \xi_n)^2 + f(\xi_1, \ldots, \xi_n)^2 A_2 \\ + a_3 f(\xi_1, \ldots, \xi_n) a_3 f(\xi_1, \ldots, \xi_n) + f(\xi_1, \ldots, \xi_n) A_3 f(\xi_1, \ldots, \xi_n) \\ + f(\xi_1, \ldots, \xi_n) a_4 f(\xi_1, \ldots, \xi_n) a_4 + a_3 f(\xi_1, \ldots, \xi_n)^2 a_4 = 0.\end{aligned}$$

Since $\{a_3, 1\}$ is linearly C-independent, we have

$$\alpha a_3 f(\xi_1,\ldots,\xi_n)^2 + a_3 f(\xi_1,\ldots,\xi_n) a_3 f(\xi_1,\ldots,\xi_n) + a_3 f(\xi_1,\ldots,\xi_n)^2 a_4 = 0,$$

that is,

$$a_3 f(\xi_1,\ldots,\xi_n)\{\alpha f(\xi_1,\ldots,\xi_n) + a_3 f(\xi_1,\ldots,\xi_n) + f(\xi_1,\ldots,\xi_n) a_4\} = 0.$$

Since $a_3 \notin C$, the term $a_3 f(\xi_1,\ldots,\xi_n) a_3 f(\xi_1,\ldots,\xi_n)$ can not be canceled and, hence, $a_3 f(\xi_1,\ldots,$ $\xi_n) a_3 f(\xi_1,\ldots,\xi_n) = 0$ in T which implies $a_3 = 0$, a contradiction.

Next suppose that U satisfies the nontrivial GPI $\Psi(\xi_1,\ldots,\xi_n) = 0$. Then, by the well-known theorem of Martindale in [20], U is a primitive ring with nonzero socle H and with C as its associated division ring. By Jacobson's theorem [16, p. 75], U is isomorphic to a dense ring of linear transformations of a vector space V over C.

If V is finite dimensional over C, that is, $\dim_C V = m$, then, by density of U, we have $U \cong M_m(C)$. As U is noncommutative, $m \geq 2$. By Lemma 2.2, $a_3 \in C$ and $a_4 \in C$, a contradiction.

If V is infinite dimensional over C, then, by [22, Lemma 2], the set $f(U)$ is dense on U. Thus, U satisfies

$$A_1\xi^2 + \xi^2 A_2 + a_3\xi a_3\xi + \xi A_3\xi + \xi a_4\xi a_4 + a_3\xi^2 a_4.$$

Since $a_3 \notin C$ and $a_4 \notin C$, they can not commute any nonzero ideal of U, i.e., $[a_3, H] \neq (0)$ and $[a_4, H] \neq (0)$. Therefore, there exists $h_1, h_2 \in H$ such that $[a_3, h_1] \neq 0$ and $[a_4, h_2] \neq 0$. By [14], there exists idempotent $e \in H$ such that $a_3h_1, h_1a_3, a_4h_2, h_2a_4, h_1, h_2 \in eUe$. Since U satisfies generalized identity

$$e\big\{A_1(e\xi e)^2 + (e\xi e)^2 A_2 + a_3(e\xi e)a_3(e\xi e) + (e\xi e)A_3(e\xi e) + (e\xi e)a_4(e\xi e)a_4 + a_3(e\xi e)^2 a_4\big\}e,$$

the subring eUe satisfies

$$eA_1e\xi^2 + \xi^2 eA_2e + ea_3e\xi ea_3e\xi + \xi eA_3e\xi + \xi ea_4e\xi ea_4e + ea_3e\xi^2 ea_4e.$$

Since $eUe \cong M_t(C)$ with $t = \dim_C Ve$, by above argument a_3 or a_4 are central elements of eUe. But then we have contradiction as $a_3h_1 = (ea_3e)h_1 = h_1ea_3e = h_1a_3$ and $a_4h_2 = (ea_4e)h_2 = h_2(ea_4e) = h_2a_4$. Therefore, we get that a_3 and a_4 are central.

Lemma 2.3 is proved.

Lemma 2.4. *Let R be a prime ring of* $\mathrm{char}(R) \neq 2$, *C the extended centroid of R and $f(x_1,\ldots,x_n)$ a noncentral multilinear polynomial over C. If $a_1, a_2, a_3 \in R$ such that*

$$a_1 f(\xi)^2 + f(\xi)a_2 f(\xi) + f(\xi)^2 a_3 = 0$$

for all $\xi = (\xi_1,\ldots,\xi_n) \in R^n$, then a_2 is central.

Proof. By using the similar argument above as in Lemma 2.3, we get that a_2 is central.

Lemma 2.5 [10, Lemma 2.9]. *Let R be a noncommutative prime ring and $p(\xi_1,\ldots,\xi_n)$ be any polynomial over C. If there exist $a_1, a_2, a_3, a_4 \in U$ such that*

$$a_1 p(\xi) + p(\xi)a_2 + a_3 p(\xi) a_4 = 0$$

for all $\xi = (\xi_1,\ldots,\xi_n) \in R^n$, then one of the following holds:

(1) $a_2, a_4 \in C$ *and* $a_1 + a_2 + a_3a_4 = 0$;

(2) $a_1, a_3 \in C$ *and* $a_1 + a_2 + a_3a_4 = 0$;

(3) $a_1 + a_2 + a_3a_4 = 0$ *and* $p(\xi)$ *is central valued on* R.

Proposition 2.1. *Suppose that* R *is a prime ring of* $\mathrm{char}(R) \neq 2$ *and* F, G, H *are three inner generalized derivations on* R. *If*

$$F(\xi^2) = G(\xi)^2 + H(\xi)\xi + \xi H(\xi)$$

for all $\xi \in f(R)$, *then one of the following holds*:

(1) *there exist a derivation* d *on* R *and* $\lambda_1, \lambda_2, \lambda_3 \in C$ *such that* $F(x) = \lambda_1 x + d(x)$, $G(x) = \lambda_2 x$ *and* $H(x) = \lambda_3 x + d(x)$ *for all* $x \in R$ *with* $\lambda_1 = \lambda_2^2 + 2\lambda_3$;

(2) *there exist* $a_1, a_2 \in U$ *and* $\lambda_1, \lambda_2, \lambda_3 \in C$ *such that* $F(x) = \lambda_1 x + [a_1, x]$, $G(x) = \lambda_2 x$ *and* $H(x) = \lambda_3 x + [a_2, x]$ *for all* $x \in R$ *with* $f(R)^2 \in C$ *and* $\lambda_1 = \lambda_2^2 + 2\lambda_3$.

Proof. By our hypothesis, generalized derivations F, G and H all are inner. Then there exist $a_1, a_2, a_3, a_4, a_5, a_6 \in U$ such that $F(x) = a_1x + xa_2$, $G(x) = a_3x + xa_4$ and $H(x) = a_5x + xa_6$ for all $x \in R$. Now $F(\xi^2) = G(\xi)^2 + H(\xi)\xi + \xi H(\xi)$ for all $\xi \in f(R)$ gives

$$(a_5 - a_1)f(\xi)^2 + f(\xi)^2(a_6 - a_2) + a_3f(\xi)a_3f(\xi)$$
$$+ f(\xi)(a_5 + a_4a_3 + a_6)f(\xi) + f(\xi)a_4f(\xi)a_4 + a_3f(\xi)^2a_4 = 0 \qquad (2)$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. By Lemma 2.3, we get that a_3 and a_4 are central. Then (2) reduces to

$$(a_3^2 + a_4^2 + a_3a_4 + a_5 - a_1)f(\xi)^2 + f(\xi)^2(a_6 - a_2) + f(\xi)(a_5 + a_3a_4 + a_6)f(\xi) = 0 \qquad (3)$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. Now applying the Lemma 2.5, we obtain that $a_5 + a_3a_4 + a_6 \in C$, that is, $a_5 + a_6 \in C$ and (3) reduces to

$$\big(a_3^2 + a_4^2 + a_3a_4 + a_5 - a_1\big)f(\xi)^2 + f(\xi)^2(2a_6 - a_2 + a_5 + a_3a_4) = 0$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. By application of Lemma 2.5, we have

(i) $a_3^2 + a_4^2 + a_3a_4 + a_5 - a_1 \in c$, that is, $a_5 - a_1 \in c$, $2a_6 - a_2 + a_5 + a_3a_4 \in C$, that is, $a_6 - a_2 \in C$ and $a_3^2 + a_4^2 + a_3a_4 + a_5 - a_1 + 2a_6 - a_2 + a_5 + a_3a_4 = 0$, that is, $(a_3 + a_4)^2 + 2(a_5 + a_6) = a_1 + a_2$. Since $a_5 - a_1 \in c$ and $a_6 - a_2 \in C$, so $a_1 + a_2 \in C$.

Therefore, in this case we get $F(x) = a_1x + xa_2 = a_1x + x(a_1 + a_2) - xa_1 = [a_1, x] + (a_1 + a_2)x$, where $a_1 + a_2 \in C$ and $G(x) = a_3x + xa_4 = (a_3 + a_4)x$, $a_3 + a_4 \in C$. Also

$$H(x) = a_5x + xa_6 = (a_5 - a_1)x + a_1x + x(a_6 - a_2) + xa_2$$
$$= (a_5 + a_6 - a_1 - a_2)x + (a_1x + xa_2)$$
$$= (a_5 + a_6 - a_1 - a_2)x + [a_1, x] + (a_1 + a_2)x = (a_5 + a_6)x + [a_1, x],$$

where $a_5 + a_6 \in C$, $(a_3 + a_4)^2 + 2(a_5 + a_6) = a_1 + a_2$, this is, conclusion (1).

(ii) $f(x_1, \ldots, x_n)^2$ is central valued and $a_3^2 + a_4^2 + a_3a_4 + a_5 - a_1 + 2a_6 - a_2 + a_5 + a_3a_4 = 0$, that is, $(a_3 + a_4)^2 + 2(a_5 + a_6) = a_1 + a_2$ implying $a_1 + a_2 \in C$.

Therefore, in this case we get $F(x) = a_1x + xa_2 = a_1x + x(a_1 + a_2) - xa_1 = [a_1, x] + (a_1 + a_2)x$, where $a_1 + a_2 \in C$ and $G(x) = a_3x + xa_4 = (a_3 + a_4)x$, $a_3 + a_4 \in C$. Also

$$H(x) = a_5x + xa_6 = a_5x + x(a_5 + a_6) - xa_5$$

$$= [a_5, x] + (a_5 + a_6)x = [a_5 - a_1, x] + [a_1, x] + (a_5 + a_6)x,$$

where $a_5 + a_6 \in C$, also $(a_3 + a_4)^2 + 2(a_5 + a_6) = a_1 + a_2$ with $f(x_1, \ldots, x_n)^2$ is central valued on R, this is, conclusion (2).

Proposition 2.1 is proved.

3. Proof of Theorem 1.1. Here R is a prime ring and U the Utumi quotient ring of R and $C = Z(U)$ (see [6] for more details). It is well-known that any derivation of R can be uniquely extended to a derivation of U. Now we consider $f(\xi_1, \ldots, \xi_n)$ be a noncentral multilinear polynomial over the field C and d be a derivation on R.

We shall use the notation

$$f(\xi_1, \ldots, \xi_n) = \xi_1\xi_2 \ldots \xi_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_\sigma \xi_{\sigma(1)}\xi_{\sigma(2)} \cdots \xi_{\sigma(n)}$$

for some $\alpha_\sigma \in C$, and S_n denotes the symmetric group of degree n. Then we have

$$d(f(\xi_1, \ldots, \xi_n)) = f^d(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, d(\xi_i), \ldots, \xi_n),$$

where $f^d(\xi_1, \ldots, \xi_n)$ be the polynomials obtained from $f(\xi_1, \ldots, \xi_n)$ replacing each coefficients α_σ with $d(\alpha_\sigma)$.

By [18, Theorem 3] every generalized derivation g of R can be uniquely extended to a generalized derivation of U and its form will be $g(x) = ax + d(x)$ for all $x \in U$, where $a \in U$ and d is a derivation of U. Thus, we can assume that $F(x) = ax + d(x)$, $G(x) = bx + g(x)$ and $H(x) = cx + h(x)$ for all $x \in R$ with some fixed $a, b, c \in U$ and d, g, h are derivations on U. Thus, by [7, 19], our hypothesis yields

$$\begin{aligned} &af(\xi)^2 + d\big(f(\xi)\big)f(\xi) + f(\xi)d\big(f(\xi)\big) \\ &\quad = bf(\xi)bf(\xi) + cf(\xi)^2 + f(\xi)cf(\xi) + bf(\xi)g(f(\xi)) \\ &\qquad + g(f(\xi))bf(\xi) + g\big(f(\xi)\big)^2 + h\big(f(\xi)\big)f(\xi) + f(\xi)h\big(f(\xi)\big) \end{aligned} \tag{4}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$. If d, g and h are three inner derivations, then F, G and H are three inner generalized derivations and in this case conclusions follow by Proposition 2.1. Therefore, to prove Theorem 1.1, we need to consider the following cases:

1) d, g are inner and h is outer;
2) g, h are inner and d is outer;
3) d, h are inner and g is outer;
4) h is inner and d, g are outer;
5) d is inner and g, h are outer;
6) g is inner and d, h are outer;
7) d, g, h all are outer.

We divide these 7 cases into the following cases:

Case 1: d, g are inner and h is outer.

Let $d(x) = [p, x]$ and $g(x) = [q, x]$ for all $x \in R$ and for some $p, q \in U$. By (4), we get

$$af(\xi)^2 + [p, f(\xi)]f(\xi) + f(\xi)[p, f(\xi)]$$
$$= bf(\xi)bf(\xi) + cf(\xi)^2 + f(\xi)cf(\xi) + bf(\xi)[q, f(\xi)]$$
$$+ [q, f(\xi)]bf(\xi) + \big([q, f(\xi)]\big)^2 + h\big(f(\xi)\big)f(\xi) + f(\xi)h\big(f(\xi)\big) \tag{5}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$. Since h is outer derivation, by Kharchenko's theorem [17], we may replace $h(f(\xi_1, \ldots, \xi_n))$ by $f^h(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, z_i, \ldots, \xi_n)$ in (5) and then U satisfies blended component

$$\sum_i f(\xi_1, \ldots, z_i, \ldots, \xi_n)f(\xi_1, \ldots, \xi_n) + f(\xi_1, \ldots, \xi_n)\sum_i f(\xi_1, \ldots, z_i, \ldots, \xi_n) = 0.$$

In particular, for $z_1 = \xi_1$ and $z_2 = \ldots = z_n = 0$, we have $2f(\xi_1, \ldots, \xi_n)^2 = 0$. Since $\mathrm{Char}(R) \neq 2$, so $f(\xi_1, \ldots, \xi_n)^2 = 0$, which implies $f(\xi_1, \ldots, \xi_n) = 0$, a contradiction.

Case 2: g, h are inner and d is outer.

Let $g(x) = [p, x]$ and $h(x) = [q, x]$ for all $x \in R$ and for some $p, q \in U$. By (4), we get

$$af(\xi)^2 + d\big(f(\xi)\big)f(\xi) + f(\xi)d\big(f(\xi)\big)$$
$$= bf(\xi)bf(\xi) + cf(\xi)^2 + f(\xi)cf(\xi) + bf(\xi)[p, f(\xi)]$$
$$+ [p, f(\xi)]bf(\xi) + \big([p, f(\xi)]\big)^2 + [q, f(\xi)]f(\xi) + f(\xi)[q, f(\xi)] \tag{6}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$. Since d is outer derivation, by Kharchenko's theorem [17], we may replace $d(f(\xi_1, \ldots, \xi_n))$ by $f^d(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, x_i, \ldots, \xi_n)$ in (6) and then U satisfies blended component

$$\sum_i f(\xi_1, \ldots, x_i, \ldots, \xi_n)f(\xi_1, \ldots, \xi_n) + f(\xi_1, \ldots, \xi_n)\sum_i f(\xi_1, \ldots, x_i, \ldots, \xi_n) = 0.$$

In particular, for $x_1 = \xi_1$ and $x_2 = \ldots = x_n = 0$, we have $2f(\xi_1, \ldots, \xi_n)^2 = 0$, which implies $f(\xi_1, \ldots, \xi_n) = 0$, a contradiction.

Case 3: d, h are inner and g is outer.

Let $d(x) = [p, x]$ and $h(x) = [q, x]$ for all $x \in R$ and for some $p, q \in U$. By (4), we obtain

$$af(\xi)^2 + [p, f(\xi)]f(\xi) + f(\xi)[p, f(\xi)]$$
$$= bf(\xi)bf(\xi) + cf(\xi)^2 + f(\xi)cf(\xi) + bf(\xi)g(f(\xi))$$
$$+ g(f(\xi))bf(\xi) + g(f(\xi))^2 + [q, f(\xi)]f(\xi) + f(\xi)[q, f(\xi)] \tag{7}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$. Since g is outer derivation, by Kharchenko's theorem [17], we may replace $g(f(\xi_1, \ldots, \xi_n))$ by $f^g(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)$ in (7), and then U satisfies blended component

$$bf(\xi_1, \ldots, \xi_n)\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)$$

$$+\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)bf(\xi_1,\ldots,\xi_n)+f^g(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)$$

$$+\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f^g(\xi_1,\ldots,\xi_n)+\left(\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\right)^2=0. \tag{8}$$

Putting $y_i=-y_i$, we get

$$-bf(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)$$

$$-\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)bf(\xi_1,\ldots,\xi_n)-f^g(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)$$

$$-\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f^g(\xi_1,\ldots,\xi_n)+\left(\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\right)^2=0. \tag{9}$$

Now adding (8) and (9), we get $2\Big(\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\Big)^2=0$. In particular, for $y_1=\xi_1$ and $y_2=\ldots=y_n=0$, we have $2f(\xi_1,\ldots,\xi_n)^2=0$. Since $\mathrm{Char}R\neq 2$, so $f(\xi_1,\ldots,\xi_n)^2=0$, which implies $f(\xi_1,\ldots,\xi_n)=0$, a contradiction.

Case 4: h is inner and $d,\ g$ are outer.

Let $h(x)=[p,x]$ for all $x\in R$ and for some $p\in U$. By (4), we get

$$af(\xi)^2+d\big(f(\xi)\big)f(\xi)+f(\xi)d\big(f(\xi)\big)$$
$$=bf(\xi)bf(\xi)+cf(\xi)^2+f(\xi)cf(\xi)+bf(\xi)g(f(\xi))$$
$$+g(f(\xi))bf(\xi)+g\big(f(\xi)\big)^2+[p,f(\xi)]f(\xi)+f(\xi)[p,f(\xi)] \tag{10}$$

for all $\xi=(\xi_1,\ldots,\xi_n)\in U^n$.

Subcase 4.1: *$d,\ g$ are linearly C-independent modulo inner derivations of U.*

Since d and g are outer derivations, by Kharchenko's theorem [17], we may replace $d(f(\xi_1,\ldots,\xi_n))$ by

$$f^d(\xi_1,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,x_i,\ldots,\xi_n)$$

and $g(f(\xi_1,\ldots,\xi_n))$ by

$$f^g(\xi_1,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n).$$

Then U satisfies blended component

$$\sum_i f(\xi_1,\ldots,x_i,\ldots,\xi_n)f(\xi_1,\ldots,\xi_n)+f(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,x_i,\ldots,\xi_n)=0.$$

In particular, for $x_1=\xi_1$ and $x_2=\ldots=x_n=0$, we have $2f(\xi_1,\ldots,\xi_n)^2=0$, which implies $f(\xi_1,\ldots,\xi_n)=0$, a contradiction.

Subcase 4.2: *d, g are linearly C-dependent modulo inner derivations of U.*

In this subcase we get $d(x) = \alpha g(x) + [q, x]$ for all $x \in U$, for some $q \in U$ and $0 \neq \alpha \in C$. Then from (10) we obtain

$$af(\xi)^2 + \big(\alpha g(f(\xi)) + [q, f(\xi)]\big)f(\xi) + f(\xi)\big(\alpha g(f(\xi)) + [q, f(\xi)]\big)$$
$$= bf(\xi)bf(\xi) + cf(\xi)^2 + f(\xi)cf(\xi) + bf(\xi)g(f(\xi))$$
$$+ g(f(\xi))bf(\xi) + g\big(f(\xi)\big)^2 + [p, f(\xi)]f(\xi) + f(\xi)[p, f(\xi)]$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$. Since g is outer derivations, by Kharchenko's theorem [17], we may replace $g(f(\xi_1, \ldots, \xi_n))$ by

$$f^g(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n).$$

Then U satisfies blended component

$$\alpha \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)f(\xi_1, \ldots, \xi_n) + \alpha f(\xi_1, \ldots, \xi_n) \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)$$
$$= bf(\xi_1, \ldots, \xi_n) \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)bf(\xi_1, \ldots, \xi_n)$$
$$+ f^g(\xi_1, \ldots, \xi_n) \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)f^g(\xi_1, \ldots, \xi_n)$$
$$+ \left(\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)\right)^2. \tag{11}$$

Putting $y_i = -y_i$, we have

$$-\alpha \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)f(\xi_1, \ldots, \xi_n) - \alpha f(\xi_1, \ldots, \xi_n) \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)$$
$$= -bf(\xi_1, \ldots, \xi_n) \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n) - \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)bf(\xi_1, \ldots, \xi_n)$$
$$- f^g(\xi_1, \ldots, \xi_n) \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n) - \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)f^g(\xi_1, \ldots, \xi_n)$$
$$+ \left(\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)\right)^2. \tag{12}$$

Now adding (11) and (12), we get $2\Big(\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)\Big)^2 = 0$. In particular, for $y_1 = \xi_1$ and $y_2 = \ldots = y_n = 0$, we have $2f(\xi_1, \ldots, \xi_n)^2 = 0$. Since $\mathrm{Char} R \neq 2$, so $f(\xi_1, \ldots, \xi_n)^2 = 0$, which implies $f(\xi_1, \ldots, \xi_n) = 0$, a contradiction.

Case 5: d is inner and g, h are outer.

Let $d(x) = [p, x]$ for all $x \in R$ and for some $p \in U$. By (4), we obtain

$$
\begin{aligned}
&af(\xi)^2 + [p, f(\xi)]f(\xi) + f(\xi)[p, f(\xi)] \\
&\quad = bf(\xi)bf(\xi) + cf(\xi)^2 + f(\xi)cf(\xi) + bf(\xi)g(f(\xi)) \\
&\qquad + g(f(\xi))bf(\xi) + g(f(\xi))^2 + h\big(f(\xi)\big)f(\xi) + f(\xi)h\big(f(\xi)\big)
\end{aligned} \tag{13}
$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$.

Subcase 5.1: *g, h are linearly C-independent modulo inner derivations of U.*

Since g and h are outer derivations, by Kharchenko's theorem [17], we may replace $g(f(\xi_1, \ldots, \xi_n))$ by

$$f^g(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)$$

and $h(f(\xi_1, \ldots, \xi_n))$ by

$$f^h(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, z_i, \ldots, \xi_n),$$

and U satisfies a blended component

$$\sum_i f(\xi_1, \ldots, z_i, \ldots, \xi_n)f(\xi_1, \ldots, \xi_n) + f(\xi_1, \ldots, \xi_n)\sum_i f(\xi_1, \ldots, z_i, \ldots, \xi_n) = 0.$$

In particular, for $z_1 = \xi_1$ and $z_2 = \ldots = z_n = 0$, we have $2f(\xi_1, \ldots, \xi_n)^2 = 0$, which implies $f(\xi_1, \ldots, \xi_n) = 0$, a contradiction.

Subcase 5.2: *g, h are linearly C-dependent modulo inner derivations of U.*

Here we get $h(x) = \alpha g(x) + [q, x]$ for all $x \in U$, for some $q \in U$ and $0 \neq \alpha \in C$. Then from (13) we obtain

$$
\begin{aligned}
&af(\xi)^2 + [p, f(\xi)]f(\xi) + f(\xi)[p, f(\xi)] \\
&\quad = bf(\xi)bf(\xi) + cf(\xi)^2 + f(\xi)cf(\xi) + bf(\xi)g(f(\xi)) + g(f(\xi))bf(\xi) \\
&\qquad + g(f(\xi))^2 + \big(\alpha g(f(\xi)) + [q, f(\xi)]\big)f(\xi) + f(\xi)\big(\alpha g(f(\xi)) + [q, f(\xi)]\big)
\end{aligned}
$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$. Since g is outer derivations, by Kharchenko's theorem [17], we may replace $g(f(\xi_1, \ldots, \xi_n))$ by

$$f^g(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n).$$

Then U satisfies blended component

$$
\begin{aligned}
&bf(\xi_1, \ldots, \xi_n)\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)bf(\xi_1, \ldots, \xi_n) \\
&\quad + f^g(\xi_1, \ldots, \xi_n)\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)f^g(\xi_1, \ldots, \xi_n) \\
&\quad + \left(\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)\right)^2 + \alpha\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)f(\xi_1, \ldots, \xi_n) \\
&\quad + \alpha f(\xi_1, \ldots, \xi_n)\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n) = 0.
\end{aligned} \tag{14}
$$

Putting $y_i = -y_i$, we have

$$-bf(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n) - \sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)bf(\xi_1,\ldots,\xi_n)$$
$$- f^g(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n) - \sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f^g(\xi_1,\ldots,\xi_n)$$
$$+ \left(\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\right)^2 - \alpha\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f(\xi_1,\ldots,\xi_n)$$
$$- \alpha f(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n) = 0. \tag{15}$$

Now adding (14) and (15), we get $2\Big(\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\Big)^2 = 0$. In particular, for $y_1 = \xi_1$ and $y_2 = \ldots = y_n = 0$, we obtain $2f(\xi_1,\ldots,\xi_n)^2 = 0$. Since $\mathrm{Char}R \neq 2$, so $f(\xi_1,\ldots,\xi_n)^2 = 0$, which implies $f(\xi_1,\ldots,\xi_n) = 0$, a contradiction.

Case 6: g is inner and d, h are outer.

Let $g(x) = [p,x]$ for all $x \in R$ and for some $p \in U$. By (4), U satisfies

$$af(\xi)^2 + d\big(f(\xi)\big)f(\xi) + f(\xi)d\big(f(\xi)\big)$$
$$= bf(\xi)bf(\xi) + cf(\xi)^2 + f(\xi)cf(\xi) + bf(\xi)[p,f(\xi)]$$
$$+ [p,f(\xi)]bf(\xi) + \big([p,f(\xi)]\big)^2 + h\big(f(\xi)\big)f(\xi) + f(\xi)h\big(f(\xi)\big). \tag{16}$$

Subcase 6.1: *d, h are linearly C-independent modulo inner derivations of U.*

Since d and h are outer derivations, by Kharchenko's theorem [17], we may replace $d(f(\xi_1,\ldots,\xi_n))$ by

$$f^d(\xi_1,\ldots,\xi_n) + \sum_i f(\xi_1,\ldots,x_i,\ldots,\xi_n)$$

and $h(f(\xi_1,\ldots,\xi_n))$ by

$$f^h(\xi_1,\ldots,\xi_n) + \sum_i f(\xi_1,\ldots,z_i,\ldots,\xi_n),$$

and U satisfies a blended component

$$\sum_i f(\xi_1,\ldots,x_i,\ldots,\xi_n)f(\xi_1,\ldots,\xi_n) + f(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,x_i,\ldots,\xi_n) = 0.$$

In particular, for $x_1 = \xi_1$ and $x_2 = \ldots = x_n = 0$, we have $2f(\xi_1,\ldots,\xi_n)^2 = 0$, which implies $f(\xi_1,\ldots,\xi_n) = 0$, a contradiction.

Subcase 6.2: *d, h are linearly C-dependent modulo inner derivations of U.*

Here we obtain $h(x) = \alpha d(x) + [q,x]$ for all $x \in U$, for some $q \in U$ and $0 \neq \alpha \in C$. Then from (16) we get

$$af(\xi)^2 + d\big(f(\xi)\big)f(\xi) + f(\xi)d\big(f(\xi)\big)$$

$$= bf(\xi)bf(\xi) + cf(\xi)^2 + f(\xi)cf(\xi) + bf(\xi)[p, f(\xi)] + [p, f(\xi)]bf(\xi)$$
$$+ \big([p, f(\xi)]\big)^2 + \big(\alpha d(f(\xi)) + [q, f(\xi)]\big)f(\xi) + f(\xi)\big(\alpha d(f(\xi)) + [q, f(\xi)]\big) \tag{17}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$. Since d is outer derivations, by Kharchenko's theorem [17], we may replace $d(f(\xi_1, \ldots, \xi_n))$ by

$$f^d(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, x_i, \ldots, \xi_n).$$

Then U satisfies blended component

$$\sum_i f(\xi_1, \ldots, x_i, \ldots, \xi_n)f(\xi_1, \ldots, \xi_n) + f(\xi_1, \ldots, \xi_n)\sum_i f(\xi_1, \ldots, x_i, \ldots, \xi_n)$$
$$= \alpha\left(\sum_i f(\xi_1, \ldots, x_i, \ldots, \xi_n)\right)f(\xi_1, \ldots, \xi_n)$$
$$+ f(\xi_1, \ldots, \xi_n)\alpha\sum_i f(\xi_1, \ldots, x_i, \ldots, \xi_n) = 0.$$

In particular, for $x_1 = \xi_1$ and $x_2 = \ldots = x_n = 0$, we have

$$(2\alpha - 2)f(\xi_1, \ldots, \xi_n)^2 = 0,$$

implying $\alpha = 1$. Then (17) reduce to

$$af(\xi)^2 = \big((b + p)f(\xi) - f(\xi)p\big)^2$$
$$+ \big(cf(\xi) + [q, f(\xi)]\big)f(\xi) + f(\xi)\big(cf(\xi) + [q, f(\xi)]\big). \tag{18}$$

Then by Lemma 2.3, we get that $b + p, p \in C$, that is, $b, p \in C$ also the equation (18) becomes

$$(a - b^2 - c - q)f(\xi)^2 = f(\xi)cf(\xi) - f(\xi)^2q. \tag{19}$$

By Lemma 2.4, we have that $c \in C$, so (19) reduce to

$$(a - b^2 - c - q)f(\xi)^2 + f(\xi)^2(q - c) = 0,$$

by application of Lemma 2.5, we obtain

(1) $a - b^2 - c - q, q - c \in C$ and $a - b^2 - c - q + q - c = 0$, that is, $a = b^2 + 2c$. Now $b, p, c \in C$ and $a - b^2 - c - q, q - c \in C$ implies that $a, b, c, p, q \in C$ with $a = b^2 + 2c$.

Therefore, in this section we get $F(x) = ax + d(x)$, where $a \in C$ and $G(x) = bx + [p, x] = bx$, where $b \in C$. Also $H(x) = cx + \alpha d(x) + [q, x] = cx + d(x)$, where $c \in C$ with $a = b^2 + 2c$. This is conclusion (1) of Theorem 1.1.

(2) $f(\xi)^2 \in C$ and $a - b^2 - c - q + q - c = 0$, that is, $a = b^2 + 2c$. Now $b, p, c \in C$ and $a = b^2 + 2c$ implies that $a, b, p, c \in C$ with $a = b^2 + 2c$.

Therefore, in this section we get $F(x) = ax + d(x)$, where $a \in C$ and $G(x) = bx + [p, x] = bx$, $b \in C$. Also $H(x) = cx + \alpha d(x) + [q, x] = cx + d(x) + [q, x]$, where $c \in C$ with $a = b^2 + 2c$. This is conclusion (2) of Theorem 1.1.

Case 7: d, g, h all are outer.

Subcase 7.1: *d, g, h are linearly C-independent modulo inner derivations of U.*

Since d, g and h are all outer derivations, by Kharchenko's theorem [17], we may replace $d(f(\xi_1,\ldots,\xi_n))$ by

$$f^d(\xi_1,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,x_i,\ldots,\xi_n),$$

$g(f(\xi_1,\ldots,\xi_n))$ by

$$f^g(\xi_1,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)$$

and $h(f(\xi_1,\ldots,\xi_n))$ by

$$f^h(\xi_1,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,z_i,\ldots,\xi_n).$$

Then U satisfies a blended component

$$\sum_i f(\xi_1,\ldots,x_i,\ldots,\xi_n)f(\xi_1,\ldots,\xi_n)+f(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,x_i,\ldots,\xi_n)=0.$$

In particular, for $x_1=\xi_1$ and $x_2=\ldots=x_n=0$, we have $2f(\xi_1,\ldots,\xi_n)^2=0$, which implies $f(\xi_1,\ldots,\xi_n)=0$, a contradiction.

Subcase 7.2: *d, g, h are linearly C-dependent modulo inner derivations of U.*

This implies there exist $\alpha,\beta,\gamma\in C$ and $q\in U$ such that

$$\alpha d(x)+\beta g(x)+\gamma h(x)=[q,x] \tag{20}$$

for all $x\in R$. If we consider $\beta=\gamma=0$, then inevitably $\alpha\neq 0$, which implies the contradiction that d is inner. So to move forward we have to consider $(\beta,\gamma)\neq(0,0)$. Without loss of generality we assume that $\gamma\neq 0$. By (20) we obtain

$$h(x)=\alpha' d(x)+\beta' g(x)+[q',x]$$

for all $x\in R$, where $\alpha'=-\gamma^{-1}\alpha$, $\beta'=-\gamma^{-1}\beta$, $q'=\gamma^{-1}q$.

By (4), we get

$$\begin{aligned}
&af(\xi)^2+d\big(f(\xi)\big)f(\xi)+f(\xi)d\big(f(\xi)\big)\\
&\quad=bf(\xi)bf(\xi)+cf(\xi)^2+f(\xi)cf(\xi)+bf(\xi)g(f(\xi))\\
&\qquad+g(f(\xi))bf(\xi)+g\big(f(\xi)\big)^2+\big(\alpha' d(f(\xi))+\beta' g(f(\xi))+[q',f(\xi)]\big)f(\xi)\\
&\qquad+f(\xi)\big(\alpha' d(f(\xi))+\beta' g(f(\xi))+[q',f(\xi)]\big)
\end{aligned}\tag{21}$$

for all $\xi=(\xi_1,\ldots,\xi_n)\in U^n$.

Now we have to consider the following two cases:

1. We consider that d, g are linearly C-independent modulo inner derivations of U. Since d and g are outer derivations, by Kharchenko's theorem [17], we may replace $d(f(\xi_1,\ldots,\xi_n))$ by

$$f^d(\xi_1,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,x_i,\ldots,\xi_n)$$

and $g(f(\xi_1,\ldots,\xi_n))$ by

$$f^g(\xi_1,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)$$

in (21). Then U satisfies blended component

$$\begin{aligned}
&bf(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)bf(\xi_1,\ldots,\xi_n)\\
&\quad+f^g(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f^g(\xi_1,\ldots,\xi_n)\\
&\quad+\left(\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\right)^2+\beta'\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f(\xi_1,\ldots,\xi_n)\\
&\quad+\beta' f(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)=0. \qquad (22)
\end{aligned}$$

Taking $y_i=-y_i$, we get

$$\begin{aligned}
&-bf(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)-\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)bf(\xi_1,\ldots,\xi_n)\\
&\quad-f^g(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)-\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f^g(\xi_1,\ldots,\xi_n)\\
&\quad+\left(\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\right)^2-\beta'\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f(\xi_1,\ldots,\xi_n)\\
&\quad-\beta' f(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)=0. \qquad (23)
\end{aligned}$$

Adding (22) and (23), we obtain $2\Big(\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\Big)^2=0$. In particular, for $y_1=\xi_1$ and $y_2=\ldots=y_n=0$, we have $2f(\xi_1,\ldots,\xi_n)^2=0$. Since $\mathrm{Char}R\neq 2$, so $f(\xi_1,\ldots,\xi_n)^2=0$, which implies $f(\xi_1,\ldots,\xi_n)=0$, a contradiction.

2. We consider that d, g are linearly C-dependent modulo inner derivations of U and $d(x)=\alpha_1 g(x)+[q_1,x]$ for all $x\in U$ and some $\alpha_1\in C$. Then (21) reduces to

$$\begin{aligned}
&af(\xi)^2+\big(\alpha_1 g(f(\xi))+[q_1,f(\xi)]\big)f(\xi)+f(\xi)\big(\alpha_1 g(f(\xi))+[q_1,f(\xi)]\big)\\
&\quad=bf(\xi)bf(\xi)+cf(\xi)^2+f(\xi)cf(\xi)+bf(\xi)g(f(\xi))+g(f(\xi))bf(\xi)\\
&\qquad+g\big(f(\xi)\big)^2+\big(\alpha'\alpha_1 g(f(\xi))+\alpha'[q_1,f(\xi)]+\beta' g(f(\xi))+[q',f(\xi)]\big)f(\xi)\\
&\qquad+f(\xi)\big(\alpha'\alpha_1 g(f(\xi))+\alpha'[q_1,f(\xi)]+\beta' g(f(\xi))+[q',f(\xi)]\big)
\end{aligned}$$

for all $\xi=(\xi_1,\ldots,\xi_n)\in U^n$. Since g is a outer derivations, by Kharchenko's theorem [17], we may replace $g(f(\xi_1,\ldots,\xi_n))$ by

$$f^g(\xi_1,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n).$$

Then U satisfies blended component

$$\begin{aligned}&\alpha_1\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f(\xi_1,\ldots,\xi_n)+\alpha_1 f(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\\&\quad=bf(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)bf(\xi_1,\ldots,\xi_n)\\&\qquad+f^g(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f^g(\xi_1,\ldots,\xi_n)\\&\qquad+\left(\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\right)^2+(\alpha'\alpha_1+\beta')\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f(\xi_1,\ldots,\xi_n)\\&\qquad+(\alpha'\alpha_1+\beta')f(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)=0.\end{aligned}\tag{24}$$

Taking $y_i=-y_i$, we get

$$\begin{aligned}&-\alpha_1\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f(\xi_1,\ldots,\xi_n)-\alpha_1 f(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\\&\quad=-bf(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)-\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)bf(\xi_1,\ldots,\xi_n)\\&\qquad-f^g(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)-\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f^g(\xi_1,\ldots,\xi_n)\\&\qquad+\left(\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\right)^2-(\alpha'\alpha_1+\beta')\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)f(\xi_1,\ldots,\xi_n)\\&\qquad-(\alpha'\alpha_1+\beta')f(\xi_1,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)=0.\end{aligned}\tag{25}$$

Adding (24) and (25), we obtain $2\Big(\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\Big)^2=0$. In particular, for $y_1=\xi_1$ and $y_2=\ldots=y_n=0$, we have $2f(\xi_1,\ldots,\xi_n)^2=0$. Since $\mathrm{Char}R\neq 2$, so $f(\xi_1,\ldots,\xi_n)^2=0$, which implies $f(\xi_1,\ldots,\xi_n)=0$, a contradiction.

Theorem 1.1 is proved.

Acknowledgements. The first author expresses his thanks to the University Grants Commission, New Delhi, for its SRF awarded to him under UGC-Ref. No. 1133/(CSIR-UGC NET JUNE 2018) dated April 16, 2019.

We are highly thankful to the learned referee(s) whose comments and suggestions enhanced the paper and made it more readable.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. E. Albas, *Generalized derivations on ideals of prime rings*, Miskolc Math. Notes, **14**, № 1, 3 – 9 (2013).
2. E. F. Alharfie, N. M. Muthana, *On homoderivations and commutativity of rings*, Bull. Inst. Math. Virtual Inst., **9**, 301 – 304 (2019).
3. E. F. Alharfie, N. M. Muthana, *The commutativity of prime rings with homoderivations*, Int. J. Adv. and Appl. Sci., **5**, № 5, 79 – 81 (2018).
4. A. Al-Kenani, A. Melaibari, N. Muthana, *Homoderivations and commutativity of* $*$*-prime rings*, East-West J. Math., **17**, № 2, 117 – 126 (2015).
5. A. Ali, N. Rehman, S. Ali, *On Lie ideals with derivations as homomorphisms and antihomomorphisms*, Acta Math. Hungar., **101**, № 1-2, 79 – 82 (2003).
6. K. I. Beidar, W. S. Martindale III, A. V. Mikhalev, *Rings with generalized identities*, Pure and Appl. Math., **196**, Marcel Dekker, New York (1996).
7. C. L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc., **103**, № 3, 723 – 728 (1988).
8. V. De Filippis, O. M. Di Vincenzo, *Vanishing derivations and centralizers of generalized derivations on multilinear polynomials*, Comm. Algebra, **40**, № 6, 1918 – 1932 (2012).
9. V. De Filippis, *Generalized derivations as Jordan homomorphisms on Lie ideals and right ideals*, Acta Math. Sin. (Engl. Ser.), **25**, № 2, 1965 – 1974 (2009).
10. B. Dhara, *Generalized derivations acting on multilinear polynomials in prime rings*, Czech. Math. J., **68**, № 1, 95 – 119 (2018).
11. B. Dhara, S. Sahebi, V. Rehmani, *Generalized derivations as a generalization of Jordan homomorphisms acting on Lie ideals and right ideals*, Math. Slovaca, **65**, № 5, 963 – 974 (2015).
12. B. Dhara, *Generalized derivations acting as a homomorphism or antihomomorphism in semiprime rings*, Beitr. Algebra und Geom., **53**, 203 – 209 (2012).
13. M. M. El Sofy, *Rings with some kinds of mappings*, M. Sci. Thesis, Cairo Univ., Branch of Fayoum, Cairo, Egypt (2000).
14. C. Faith, Y. Utumi, *On a new proof of Litoff's theorem*, Acta Math. Acad. Sci. Hung., **14**, 369 – 371 (1963).
15. I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc., **8**, 1104 – 1110 (1957).
16. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., **37** (1964).
17. V. K. Kharchenko, *Differential identity of prime rings*, Algebra and Logic, **17**, 155 – 168 (1978).
18. T. K. Lee, *Generalized derivations of left faithful rings*, Comm. Algebra, **27**, № 8, 4057 – 4073 (1999).
19. T. K. Lee, *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sin., **20**, № 1, 27 – 38 (1992).
20. W. S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra, **12**, 576 – 584 (1969).
21. A. Melaibari, N. Muthana, A. Al-Kenani, *Centrally-extended homoderivations on rings*, Gulf J. Math., **4**, № 2, 62 – 70 (2016).
22. T. L. Wong, *Derivations with power central values on multilinear polynomials*, Algebra Colloq., **3**, 369 – 478 (1996).
23. Y. Wang, H. You, *Derivations as homomorphisms or antihomomorphisms on Lie ideals*, Acta Math. Sin. (Engl. Ser.), **23**, № 6, 1149 – 1152 (2007).

Received 17.06.22