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NEW QUANTUM HERMITE – HADAMARD-TYPE INEQUALITIES FOR p-CONVEX FUNCTIONS INVOLVING RECENTLY DEFINED QUANTUM INTEGRALS

НОВІ КВАНТОВІ НЕРІВНОСТІ ЕРМІТА – АДАМАРА ДЛЯ p-ОПУКЛИХ ФУНКЦІЙ, ЩО ВКЛЮЧАЮТЬ НЕЩОДАВНО ВИЗНАЧЕНІ КВАНТОВІ ІНТЕГРАЛИ

We develop new Hermite – Hadamard-type integral inequalities for p-convex functions in the context of q-calculus by using the concept of recently defined T_q-integrals. Then the obtained Hermite – Hadamard inequality for p-convex functions is used to get new Hermite – Hadamard inequality for coordinated p-convex functions. Furthermore, we present some examples to demonstrate the validity of our main results. We hope that the ideas and techniques of this study may stimulate further research in this field.

Основною метою цього дослідження є розробка нових інтегральних нерівностей типу Ерміта – Адамара для p-опуклих функцій в контексті q-числення за допомогою концепції T_q- інтегралів, що були нещодавно визначені. Далі отриману нерівність Ерміта – Адамара для p-опуклих функцій використано для виведення нової нерівності Ерміта – Адамара для координованих p-опуклих функцій. Крім того, наведено кілька прикладів, щоб продемонструвати достовірність отриманих основних результатів. Ми сподіваємося, що ідеї та методи, запропоновані в цій роботі, можуть стимулювати подальші дослідження в цій галузі.

1. Introduction. A function $\mathcal{F} : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex, if it satisfies the inequality

$$\mathcal{F}(t\sigma + (1-t)\rho) \le t\mathcal{F}(\sigma) + (1-t)\mathcal{F}(\rho),$$

where $\sigma, \rho \in I$ and $t \in [0,1]$.

Convex functions have potential applications in many fascinating and delightful fields of research. Furthermore, have played a notable role in innumerable areas, such as coding theory, optimization, physics, information theory, engineering, and inequality theory. Various new classes of classical convexity have been proved in the history (see [1, 2]). Many researchers strived, attempted and maintained their work on the notion of convex functions and generalized its variant forms in different ways using novel ideas and advantageous techniques [3, 4]. Many mathematicians always kept continually hardworking in the field of inequalities and have conspired with different ideas and notions in the theory of inequalities and its applications (see [5 – 13]). Many inequalities are proved

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for convex functions, but the most famous and well-known from the related literature is the Hermite – Hadamard inequality. The Hermite – Hadamard inequality introduced by Hermite and Hadamard (see also [2] and [14, p. 137]) is one of the most prominent inequalities in the theory of convex functional analysis. It has a appealing geometrical interpretation with diverse applications

$$\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\kappa)d\kappa \leq \frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2},$$

where $\mathcal{F}\colon I \to \mathbb{R}$ is a convex function and $\sigma, \rho \in I$ with $\sigma < \rho$. Convexity is also assimilated with the concept of quantum and post quantum calculus [15].

The concept of p-convexity is defined as follows.

Definition 1 [16]. *Let I be a p-convex set. A function $\mathcal{F}\colon I \to \mathbb{R}$ is said to be a p-convex function, if*

$$(t\sigma^p + (1-t)\rho^p)^{\frac{1}{p}} \leq t\mathcal{F}(\sigma) + (1-t)\mathcal{F}(\rho) \tag{1.1}$$

for all $\sigma, \rho \in I$, $p \in \mathbb{R}\setminus\{0\}$ and for $t \in [0,1]$. If the inequality in (1.1) *is reversed, then $\mathcal{F}$ is said to be p-concave.*

In [18, Theorem 5], if we take $h(t) = t$, then we have following theorem.

Theorem 1. *Let $\mathcal{F}\colon I \subseteq (0,\infty) \to R$ be a differentiable function on I^o, σ, $\rho \in I$ with $\sigma < \rho$ and $p > 0$. If $\mathcal{F}' \in L[\sigma,\rho]$, then*

$$\mathcal{F}\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}} \leq \frac{p}{\rho^p-\sigma^p}\int_{\sigma}^{\rho}\frac{\mathcal{F}(\kappa)}{\kappa^{1-p}}d\kappa \leq \frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}. \tag{1.2}$$

Definition 2 [19]. *A function $\mathcal{F}\colon \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is called coordinated p-convex functions on Δ, if the following inequality*:

$$\begin{aligned}\mathcal{F}((t\sigma^p + (1-t)\rho^p)^{\frac{1}{p}}, (\lambda\varrho^p + (1-\lambda)d^p)^{\frac{1}{p}}) &\leq t\lambda\mathcal{F}(\sigma,\varrho) + t(1-\lambda)\mathcal{F}(\sigma,d) \\ &\quad + (1-t)\lambda\mathcal{F}(\rho,\varrho) + (1-t)(1-\lambda)\mathcal{F}(\rho,d)\end{aligned}$$

holds for all $t, \lambda \in [0,1]$, $p > 0$ and $(\sigma,\varrho), (\sigma,d), (\rho,\varrho), (\rho,d) \in \Delta$, where Δ is bi-dimensional real interval.

Here, if we put $p = 1$, then coordinated p-convexity reduces to coordinated convexity.

Zhang [16] introduced the notion of p-convex functions. It is worth to mention here that besides the classical convex functions the class of p-convex functions also includes the class of harmonically convex functions introduced and studied by İşcan [17]. For some recent investigations on p-convex functions, see [18].

On the other hand, quantum calculus is the analysis of calculus without limits, sometime called q-calculus. Quantum calculus is regarded as an incorporation subject between mathematics and physics, and many researchers have a notable concentration in this subject. Historically the subject of quantum calculus can be traced back to Euler and Jacobi, but in recent decades it has experienced a rapid development. It is also pertinent to mention here that quantum calculus is a subfield of time scale calculus. In quantum calculus, we are concerned with a specific time scale, called the q-time scale. In the twentieth century Jackson et al. introduced the notion of q-definite integrals in quantum calculus. This inspired many quantum calculus analysis, and consequently a number of articles have been written in this area. It is worth to mention here for interested readers that it is possible that some times more than one q-analogue exists. Many integral inequalities have been studied using quantum integrals for various types of functions, for example, in [20–26]. In [27], Alp and Sarikaya presented a new definition of q-integral by using trapezoid pieces and established some new q-Hermite–Hadamard-type inequalities for convex functions. In 2020, Alp and Sarikaya according to the definition of q-integral by using trapezoid pieces Young integral inequality, Hölder integral inequality, Minkowski integral inequality, and Ostrowski type integral inequalities in [27]. Subsequently, in [28], again utilize the trapezoidal parts, Kara et al. defined new genre of quantum integral given by Alp and Sarikaya and established the corresponding Hermite–Hadamard inequalities. In [29], Kara and Budak defined new T_q-integrals for the functions of two variables.

Inspired by the ongoing studies, we prove some new Hermite–Hadamard-type inequalities for p-convex and coordinated p-convex functions. The fundamental benefit of these inequalities is that these can be turned into quantum Hermite–Hadamard inequalities for convex and coordinated convex functions, classical Hermite–Hadamard inequalities for convex and coordinated convex functions without having to prove each one separately.

This paper is organized as follows. Sections 2 and 3 provides a brief overview of the fundamentals of q-calculus as well as other related studies in this field. In Sections 4 and 5, we establish some new Hermite–Hadamard-type inequalities for p-convex and coordinated p-convex functions using the notions of q-integrals. The relationship between the findings reported here and similar findings in the literature are also considered. In Section 6, we prove that the newly established inequalities for p-convex functions hold using some mathematical examples. Section 7 concludes with some recommendations for future research.

2. q-Integrals and related inequalities. In this section, we first present the definitions and some properties of quantum integrals. We also mention some well-known inequalities for quantum integrals. Throughout this paper, let $0 < q < 1$ be a constant.

The q-number or q-analogue of $n \in \mathbb{N}$ is given by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}.$$

Jackson derived the q-Jackson integral in [31] from 0 to ρ as follows:

$$\int_0^\rho \mathcal{F}(\kappa) d_q\kappa = (1-q)\rho \sum_{n=0}^{\infty} q^n \mathcal{F}(\rho q^n)$$

provided the sum converges absolutely.

The q-Jackson integral in a generic interval $[\sigma, \rho]$ was given by in [30] and defined as follows:

$$\int_\sigma^\rho \mathcal{F}(\kappa) d_q\kappa = \int_0^\rho \mathcal{F}(\kappa) d_q\kappa - \int_0^\sigma \mathcal{F}(\kappa) d_q\kappa.$$

The quantum integrals on the interval $[\sigma, \rho]$ is defined as follows.

Definition 3 [31]. *Let $\mathcal{F}\colon [\sigma,\rho]\to\mathbb{R}$ be a continuous function. Then the q_σ-definite integral on $[\sigma,\rho]$ is defined as*

$$\int_{\sigma}^{\kappa} \mathcal{F}(t) \quad {}_{\sigma}d_qt \quad = (1-q)(\kappa-\sigma)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\kappa+(1-q^n)\sigma)$$

for $\kappa\in[\sigma,\rho]$.

In [15], Alp et al. proved the corresponding Hermite – Hadamard inequalities for convex functions, by using q_σ-integrals, as follows.

Theorem 2. *If $\mathcal{F}\colon [\sigma,\rho]\to\mathbb{R}$ be a convex differentiable function on $[\sigma,\rho]$, then q-Hermite – Hadamard inequalities is defined as*

$$\mathcal{F}\left(\frac{q\sigma+\rho}{1+q}\right)\le\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\kappa)_{\sigma}d_q\kappa\le\frac{q\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{1+q}.$$

On the other hand, Bermudo et al. gave the following new definition of quantum integral on the interval $[\sigma,\rho]$.

Definition 4 [32]. *Let $\mathcal{F}\colon [\sigma,\rho]\to\mathbb{R}$ be a continuous function. Then the q^ρ-definite integral on $[\sigma,\rho]$ is defined as*

$$\int_{\kappa}^{\rho} \mathcal{F}(t) \quad {}^{\rho}d_qt \quad = (1-q)(\rho-\kappa)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\kappa+(1-q^n)\rho)$$

for $\kappa\in[\sigma,\rho]$.

Bermudo et al. proved the corresponding Hermite – Hadamard inequalities for convex functions, by using q^ρ-integrals, as follows.

Theorem 3 [32]. *If $\mathcal{F}\colon [\sigma,\rho]\to\mathbb{R}$ be a convex differentiable function on $[\sigma,\rho]$, then q-Hermite – Hadamard inequalities is defined as*

$$\mathcal{F}\left(\frac{\sigma+q\rho}{1+q}\right)\le\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\kappa)^{\rho}d_q\kappa\le\frac{\mathcal{F}(\sigma)+q\mathcal{F}(\rho)}{1+q}.$$

From Theorems 2 and 3, one can write the following inequalities.

***Corollary* 1** [32]. *For any convex function $\mathcal{F}\colon [\sigma,\rho]\to\mathbb{R}$ and $0<q<1$, we have*

$$\mathcal{F}\left(\frac{q\sigma+\rho}{1+q}\right)+\mathcal{F}\left(\frac{\sigma+q\rho}{1+q}\right)\le\frac{1}{\rho-\sigma}\left\{\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\quad{}_{\sigma}d_q\kappa\quad+\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\quad{}^{\rho}d_q\kappa\quad\right\}\le\mathcal{F}(\sigma)+\mathcal{F}(\rho)$$

and

$$\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\le\frac{1}{2(\rho-\sigma)}\left\{\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\quad{}_{\sigma}d_q\kappa\quad+\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\quad{}^{\rho}d_q\kappa\quad\right\}\le\frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}.$$

In [33], Latif defined $q_{\sigma\varrho}$-integral for functions of two variables and presented important properties of this integral. Latif et al. prove also a q-Hermite – Hadamard inequality for coordinated convex functions. However, in [22] Alp and Sarikaya show that the Hermite – Hadamard inequalities in [33] are not correct. Then they prove the corrected version of Hermite – Hadamard inequalities for coordinated convex functions involving $q_{\sigma\varrho}$-integrals.

On the other hand, Budak et al. [24] defined the $q_{\sigma}^{d}, q_{\varrho}^{\rho}$ and $q^{\rho d}$-integrals for functions of two variables and they also gave the corresponding Hermite – Hadamard inequalities for these defined integrals.

3. T_q-integrals and related inequalities. In this section, we present definitions and properties given by using trapezoids.

Alp and Sarikaya, by using the area of trapezoids, introduced the following generalized quantum integral which is called ${}_{\sigma}T_q$-integral.

Definition 5 [27]. *Let $\mathcal{F}\colon[\sigma,\rho]\to\mathbb{R}$ is continuous function. For $\kappa\in[\sigma,\rho]$,*

$$\int\limits_{\sigma}^{\rho}\mathcal{F}(s)\quad{}_{\sigma}d_q^T s\quad=\frac{(1-q)(\rho-\sigma)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\rho+(1-q^n)\sigma)-\mathcal{F}(\rho)\right].$$

Theorem 4 (${}_{a}T_q$-Hermite – Hadamard) [27]. *Let $\mathcal{F}\colon[\sigma,\rho]\to\mathbb{R}$ be a convex continuous function on $[\sigma,\rho]$. Then we have*

$$\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\le\frac{1}{\rho-\sigma}\int\limits_{\sigma}^{\rho}\mathcal{F}(\kappa)\quad{}_{\sigma}d_q^T\kappa\quad\le\frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}.$$

In [28], Kara et al. introduced the following generalized quantum integral which is called ${}^{\rho}T_q$-integral.

Definition 6 [28]. *Let $\mathcal{F}\colon[\sigma,\rho]\to\mathbb{R}$ is continuous function. For $\kappa\in[\sigma,\rho]$,*

$$\int\limits_{\sigma}^{\rho}\mathcal{F}(s)\quad{}^{\rho}d_q^T s\quad=\frac{(1-q)(\rho-\sigma)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\sigma+(1-q^n)\rho)-\mathcal{F}(\sigma)\right].$$

Theorem 5 (${}^{b}T_q$-Hermite – Hadamard) [28]. *Let $\mathcal{F}\colon[\sigma,\rho]\to\mathbb{R}$ be a convex continuous function on $[\sigma,\rho]$. Then we have*

$$\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\le\frac{1}{\rho-\sigma}\int\limits_{\sigma}^{\rho}\mathcal{F}(\kappa)\quad{}^{\rho}d_q^T\kappa\quad\le\frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}.$$

Kara and Budak defined T_q-integrals for two-variables functions as follows.

Definition 7 [29]. *Suppose that $\mathcal{F}\colon[\sigma,\rho]\times[\varrho,d]\subset\mathbb{R}^2\to\mathbb{R}$ is continuous function. Then the following ${}_{\sigma\varrho}T_q, {}_{\sigma}^{d}T_q, {}_{\varrho}^{\rho}T_q$ and ${}^{\rho d}T_q$-integrals on $[\sigma,\rho]\times[\varrho,d]$ are defined by*

$$\int\limits_{\sigma}^{\rho}\int\limits_{\varrho}^{d}\mathcal{F}(t,s)\quad{}_{\varrho}d_{q_2}^T s\quad{}_{\sigma}d_{q_1}^T t$$

$$= \frac{(1-q_1)(1-q_2)(\rho-\sigma)(d-\varrho)}{4q_1q_2}$$

$$\times \Bigg[(1+q_1)(1+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m\ \mathcal{F}(q_1^n\rho+(1-q_1^n)\sigma, q_2^m d+(1-q_2^m)\varrho)$$

$$-(1+q_2)\sum_{m=0}^{\infty} q_2^m\ \mathcal{F}(\rho, q_2^m d+(1-q_2^m)\varrho)$$

$$-(1+q_1)\sum_{n=0}^{\infty} q_1^n\ \mathcal{F}(q_1^n\rho+(1-q_1^n)\sigma, d)+\ \mathcal{F}(\rho,d)\Bigg],$$

$$\int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(t,s)\ \ {}^{d}d_{q_2}^{T}s\ \ \ {}_{\sigma}d_{q_1}^{T}t$$

$$= \frac{(1-q_1)(1-q_2)(\rho-\sigma)(d-\varrho)}{4q_1q_2}$$

$$\times \Bigg[(1+q_1)(1+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m\ \mathcal{F}(q_1^n\rho+(1-q_1^n)\sigma, q_2^m \varrho+(1-q_2^m)d)$$

$$-(1+q_2)\sum_{m=0}^{\infty} q_2^m\ \mathcal{F}(\rho, q_2^m \varrho+(1-q_2^m)d)$$

$$-(1+q_1)\sum_{n=0}^{\infty} q_1^n\ \mathcal{F}(q_1^n\rho+(1-q_1^n)\sigma, \varrho)+\ \mathcal{F}(\rho,\varrho)\Bigg],$$

$$\int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(t,s)\ \ {}_{\varrho}d_{q_2}^{T}s\ \ \ {}^{\rho}d_{q_1}^{T}t$$

$$= \frac{(1-q_1)(1-q_2)(\rho-\sigma)(d-\varrho)}{4q_1q_2}$$

$$\times \Bigg[(1+q_1)(1+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m\ \mathcal{F}(q_1^n\sigma+(1-q_1^n)\rho, q_2^m d+(1-q_2^m)\varrho)$$

$$-(1+q_2)\sum_{m=0}^{\infty} q_2^m\ \mathcal{F}(\sigma, q_2^m d+(1-q_2^m)\varrho)$$

$$-(1+q_1)\sum_{n=0}^{\infty} q_1^n\ \mathcal{F}(q_1^n\sigma+(1-q_1^n)\rho, d)+\ \mathcal{F}(\sigma,d)\Bigg],$$

and

$$\int_{\sigma}^{\rho}\int_{\varrho}^{d} \mathcal{F}(t,s) \quad {}^{d}d_{q_2}^{T}s \quad {}^{\rho}d_{q_1}^{T}t$$

$$= \frac{(1-q_1)(1-q_2)(\rho-\sigma)(d-\varrho)}{4q_1q_2}$$

$$\times \Bigg[(1+q_1)(1+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \; \mathcal{F}(q_1^n\sigma+(1-q_1^n)\rho, q_2^m\varrho+(1-q_2^m)d)$$

$$-(1+q_2)\sum_{m=0}^{\infty} q_2^m \; \mathcal{F}(\sigma, q_2^m\varrho+(1-q_2^m)d)$$

$$-(1+q_1)\sum_{n=0}^{\infty} q_1^n \; \mathcal{F}(q_1^n\sigma+(1-q_1^n)\rho,\varrho)+ \; \mathcal{F}(\sigma,\varrho)\Bigg],$$

respectively.

4. Quantum Hermite – Hadamard inequality. In this section, we show a representative application of p-convex to obtain new Hermite – Hadamard inequalities involving T_q integrals.

Theorem 6. *Let $\mathcal{F}\colon I\subseteq(0,\infty)\to R$ be a differentiable function on $I^o, \sigma^p, \rho^p \in I$ with $\sigma^p<\rho^p$ and $p>0, q\in(0,1)$. If $\mathcal{F}\in L[\sigma^p,\rho^p]$, then*

$$\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}}\right)\leq\frac{1}{2(\rho^p-\sigma^p)}\left[\int_{\sigma^p}^{\rho^p}\mathcal{F}(\;\kappa^{\frac{1}{p}})_{\sigma^p}d_q^T\;\kappa+\int_{\sigma^p}^{\rho^p}\mathcal{F}(\;\kappa^{\frac{1}{p}})^{\rho^p}d_q^T\;\kappa\right]\leq\frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}. \tag{4.1}$$

Proof. By definition of p-convexity

$$\mathcal{F}(t\kappa^p+(1-t)\gamma^p)^{\frac{1}{p}}\leq t\mathcal{F}(\kappa)+(1-t)\mathcal{F}(\gamma).$$

By taking $t=\dfrac{1}{2}$, we get

$$\mathcal{F}\left(\left(\frac{\kappa^p+\gamma^p}{2}\right)^{\frac{1}{p}}\right)\leq\frac{\mathcal{F}(\kappa)+\mathcal{F}(\gamma)}{2}.$$

Considering $\kappa^p=t\rho^p+(1-t)\sigma^p$ and $\gamma^p=t\sigma^p+(1-t)\rho^p$, we obtain

$$2\mathcal{F}\left(\left(\frac{\kappa^p+\gamma^p}{2}\right)^{\frac{1}{p}}\right)\leq\mathcal{F}\Big((t\rho^p+(1-t)\sigma^p)^{\frac{1}{p}}\Big)+\mathcal{F}\Big((t\sigma^p+(1-t)\rho^p)^{\frac{1}{p}}\Big).$$

By ${}_{\sigma}T_q$-integrating with respect to t over $[0,1]$, we have

$$2\mathcal{F}\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}}\int_0^1\;{}_0d_q^Tt\leq\int_0^1\mathcal{F}\Big((t\rho^p+(1-t)\sigma^p)^{\frac{1}{p}}\Big)\;{}_0d_q^Tt+\int_0^1\mathcal{F}\Big((t\sigma^p+(1-t)\rho^p)^{\frac{1}{p}}\Big)\;{}_0d_q^Tt.$$

From Definitions 5 and 6, we get

$$\int_0^1 \mathcal{F}(t\rho^p + (1-t)\sigma^p)^{\frac{1}{p}} \ {}_0d_q^T t = \frac{1-q}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\rho^p + (1-q^n)\sigma^p)^{\frac{1}{p}} - \mathcal{F}(\rho)\right]$$

$$= \frac{1}{\rho^p - \sigma^p}\int_{\sigma^p}^{\rho^p} \mathcal{F}(\ \kappa)^{\frac{1}{p}}{}_{\sigma^p}d_q^T\ \kappa$$

and

$$\int_0^1 \mathcal{F}(t\sigma^p + (1-t)\rho^p)^{\frac{1}{p}} \ {}_0d_q^T t = \frac{1-q}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\sigma^p + (1-q^n)\rho^p)^{\frac{1}{p}} - \mathcal{F}(\sigma)\right]$$

$$= \frac{1}{\rho^p - \sigma^p}\int_{\sigma^p}^{\rho^p} \mathcal{F}(\ \kappa^{\frac{1}{p}})^{\rho^p} d_q^T\ \kappa.$$

Thus, we obtain

$$2\mathcal{F}\left(\frac{\sigma^p + \rho^p}{2}\right)^{\frac{1}{p}} \leq \frac{1}{\rho^p - \sigma^p}\left[\int_{\sigma^p}^{\rho^p} \mathcal{F}(\ \kappa^{\frac{1}{p}})_{\sigma^p} d_q^T\ \kappa + \int_{\sigma^p}^{\rho^p} \mathcal{F}(\ \kappa^{\frac{1}{p}})^{\rho^p} d_q^T\ \kappa\right]$$

and the first inequality of (4.1) is proved.

To prove the second inequality, using the p-convexity, we have

$$\mathcal{F}\Big((t\rho^p + (1-t)\sigma^p)^{\frac{1}{p}}\Big) \leq t\mathcal{F}(\rho) + (1-t)\mathcal{F}(\sigma)$$

and

$$\mathcal{F}\Big((t\sigma^p + (1-t)\rho^p)^{\frac{1}{p}}\Big) \leq t\mathcal{F}(\sigma) + (1-t)\mathcal{F}(\rho).$$

Thus,

$$\mathcal{F}(t\rho^p + (1-t)\sigma^p)^{\frac{1}{p}} + \mathcal{F}(t\sigma^p + (1-t)\rho^p)^{\frac{1}{p}} \leq [\mathcal{F}(\sigma) + \mathcal{F}(\rho)]. \tag{4.2}$$

By taking ${}_\sigma T_q$-integral of (4.2) on $[0,1]$ and by using Definitions 5 and 6, we get

$$\frac{1}{\rho^p - \sigma^p}\left[\int_{\sigma^p}^{\rho^p} \mathcal{F}(\ \kappa^{\frac{1}{p}})_{\sigma^p} d_q^T\ \kappa + \int_{\sigma^p}^{\rho^p} \mathcal{F}(\ \kappa^{\frac{1}{p}})^{\rho^p} d_q^T\ \kappa\right] \leq \mathcal{F}(\sigma) + \mathcal{F}(\rho).$$

Thus, the proof is accomplished.

***Remark* 1.** If we set $p = 1$ in Theorem 6, then Theorem 6 reduces to [32, Theorem 20].

***Remark* 2.** In Theorem 6, if we take the limit as $q \to 1$, then inequality (4.1) becomes the inequality (1.2).

5. Quantum Hermite – Hadamard inequality on coordinates. In this section, by utilizing the inequalities given in previous section, we establish some new quantum inequalities of Hermite – Hadamard type via coordinated p-convex functions.

Theorem 7. *If* $\mathcal{F} : [\sigma, \rho] \times [\varrho, d] \to \mathbb{R}$ *is coordinated* p*-convex function on* Δ *with* $p > 0$, *then we have the following inequalities*:

$$\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}}, \left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right)$$

$$\leq \frac{1}{4(\rho^p-\sigma^p)}\left[\int_{\sigma^p}^{\rho^p} \mathcal{F}\left(\kappa^{\frac{1}{p}}, \left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right) {}_{\sigma^p}d_{q_1}^T\kappa + \int_{\sigma^p}^{\rho^p} \mathcal{F}\left(\kappa^{\frac{1}{p}}, \left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right) {}^{\rho^p}d_{q_1}^T\kappa\right]$$

$$+ \frac{1}{4(d^p-\varrho^p)}\left[\int_{\varrho^p}^{d^p} \mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}}, \gamma^{\frac{1}{p}}\right) {}_{\varrho^p}d_{q_2}^T\gamma + \int_{\varrho^p}^{d^p} \mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}}, \gamma^{\frac{1}{p}}\right) {}^{d^p}d_{q_2}^T\gamma\right]$$

$$\leq \frac{1}{4(\rho^p-\sigma^p)(d^p-\varrho^p)}\left[\int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) {}_{\varrho^p}d_{q_2}^T\gamma \, {}_{\sigma^p}d_{q_1}^T\kappa + \int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) {}^{d^p}d_{q_2}^T\gamma \, {}_{\sigma^p}d_{q_1}^T\kappa\right.$$

$$\left. + \int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) {}_{\varrho^p}d_{q_2}^T\gamma \, {}^{\rho^p}d_{q_1}^T\kappa + \int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) {}^{d^p}d_{q_2}^T\gamma \, {}^{\rho^p}d_{q_1}^T\kappa\right]$$

$$\leq \frac{1}{8(\rho^p-\sigma^p)}\left[\int_{\sigma^p}^{\rho^p} [\mathcal{F}(\kappa^{\frac{1}{p}}, \varrho) + \mathcal{F}(\kappa^{\frac{1}{p}}, d)] {}_{\sigma^p}d_{q_1}^T\kappa + \int_{\sigma^p}^{\rho^p} [\mathcal{F}(\kappa^{\frac{1}{p}}, \varrho) + \mathcal{F}(\kappa^{\frac{1}{p}}, d)] {}^{\rho^p}d_{q_1}^T\kappa\right]$$

$$+ \frac{1}{8(d^p-\varrho^p)}\left[\int_{\varrho^p}^{d^p} [\mathcal{F}(\sigma, \gamma^{\frac{1}{p}}) + \mathcal{F}(\rho, \gamma^{\frac{1}{p}})] {}_{\varrho^p}d_{q_2}^T\gamma + \int_{\varrho^p}^{d^p} [\mathcal{F}(\sigma, \gamma^{\frac{1}{p}}) + \mathcal{F}(\rho, \gamma^{\frac{1}{p}})] {}^{d^p}d_{q_2}^T\gamma\right]$$

$$\leq \frac{\mathcal{F}(\sigma, \varrho) + \mathcal{F}(\sigma, d) + \mathcal{F}(\rho, \varrho) + \mathcal{F}(\rho, d)}{8}. \tag{5.1}$$

Proof. Let $g_\kappa(\gamma) \to \mathbb{R}$ and $g_\kappa(\gamma) = \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma)$ is p-convex function on $[\varrho, d]$, by using the inequality (4.1) for the interval $[\varrho, d]$ and $q_2 \in (0, 1)$, we have

$$g_\kappa\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}} \leq \frac{1}{2(d^p-\varrho^p)}\left[\int_{\varrho^p}^{d^p} g_\kappa(\gamma^{\frac{1}{p}}) {}_{\varrho^p}d_{q_2}^T\gamma + \int_{\varrho^p}^{d^p} g_\kappa(\gamma^{\frac{1}{p}}) {}^{d^p}d_{q_2}^T\gamma\right] \leq \frac{g_\kappa(\varrho) + g_\kappa(d)}{2},$$

i.e.,

$$\mathcal{F}\left(\kappa^{\frac{1}{p}}, \left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{1}{2(d^p-\varrho^p)}\left[\int_{\varrho^p}^{d^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) {}_{\varrho^p}d_{q_2}^T\gamma + \int_{\varrho^p}^{d^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) {}^{d^p}d_{q_2}^T\gamma\right]$$

$$\leq \frac{\mathcal{F}(\kappa^{\frac{1}{p}}, \varrho) + \mathcal{F}(\kappa^{\frac{1}{p}}, d)}{2} \tag{5.2}$$

for all $\kappa \in [\sigma^p, \rho^p]$. By ${}_{\sigma^p}T_{q_1}$ integration of inequality (5.2) on $[\sigma^p, \rho^p]$, we get

$$\frac{1}{\rho^p - \sigma^p} \int_{\sigma^p}^{\rho^p} \mathcal{F}\left(\kappa^{\frac{1}{p}}, \left(\frac{\varrho^p + d^p}{2}\right)^{\frac{1}{p}}\right) {}_{\sigma^p}d_{q_1}^T\kappa$$

$$\leq \frac{1}{2(\rho^p - \sigma^p)(d^p - \varrho^p)} \left[\int_{\sigma^p}^{\rho^p} \int_{\varrho^p}^{d^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) \, {}_{\varrho^p}d_{q_2}^T\gamma \, {}_{\sigma^p}d_{q_1}^T\kappa + \int_{\sigma^p}^{\rho^p} \int_{\varrho^p}^{d^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) \, {}^{d^p}d_{q_2}^T\gamma \, {}_{\sigma^p}d_{q_1}^T\kappa \right]$$

$$\leq \frac{1}{2(\rho^p - \sigma^p)} \int_{\sigma^p}^{\rho^p} [\mathcal{F}(\kappa^{\frac{1}{p}}, \varrho) + \mathcal{F}(\kappa^{\frac{1}{p}}, d)] {}_{\sigma^p}d_{q_1}^T\kappa. \tag{5.3}$$

Similarly, by ${}^{\rho^p}T_{q_1}$ integration of inequality (5.2) on $[\sigma^p, \rho^p]$, we obtain

$$\frac{1}{\rho^p - \sigma^p} \int_{\sigma^p}^{\rho^p} \mathcal{F}\left(\kappa^{\frac{1}{p}}, \left(\frac{\varrho^p + d^p}{2}\right)^{\frac{1}{p}}\right) {}^{\rho^p}d_{q_1}^T\kappa$$

$$\leq \frac{1}{2(\rho^p - \sigma^p)(d^p - \varrho^p)} \left[\int_{\sigma^p}^{\rho^p} \int_{\varrho^p}^{d^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) \, {}_{\varrho^p}d_{q_2}^T\gamma \, {}^{\rho^p}d_{q_1}^T\kappa + \int_{\sigma^p}^{\rho^p} \int_{\varrho^p}^{d^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) \, {}^{d^p}d_{q_2}^T\gamma \, {}^{\rho^p}d_{q_1}^T\kappa \right]$$

$$\leq \frac{1}{2(\rho^p - \sigma^p)} \int_{\sigma^p}^{\rho^p} [\mathcal{F}(\kappa^{\frac{1}{p}}, \varrho) + \mathcal{F}(\kappa^{\frac{1}{p}}, d)] {}^{\rho^p}d_{q_1}^T\kappa. \tag{5.4}$$

On the other hand, the function $g_\kappa(\gamma) \to \mathbb{R}$ and $g_\kappa(\gamma) = \mathcal{F}(\kappa, \gamma^{\frac{1}{p}})$ is p-convex function on $[\sigma, \rho]$, by using the inequality (4.1) for the interval $[\sigma, \rho]$ and $q_1 \in (0, 1)$, we have

$$g_\gamma\left(\frac{\sigma^p + \rho^p}{2}\right)^{\frac{1}{p}} \leq \frac{1}{2(\rho^p - \sigma^p)} \left[\int_{\sigma^p}^{\rho^p} g_\gamma(\kappa^{\frac{1}{p}}) \, {}_{\sigma^p}d_{q_1}^T\kappa + \int_{\sigma^p}^{\rho^p} g_\gamma(\kappa^{\frac{1}{p}}) \, {}^{\rho^p}d_{q_1}^T\kappa \right]$$

$$\leq \frac{g_\gamma(\sigma) + g_\gamma(\rho)}{2},$$

that is,

$$\mathcal{F}\left(\left(\frac{\sigma^p + \rho^p}{2}\right)^{\frac{1}{p}}, \gamma^{\frac{1}{p}}\right) \leq \frac{1}{2(\rho^p - \sigma^p)} \left[\int_{\sigma^p}^{\rho^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) \, {}_{\sigma^p}d_{q_1}^T\kappa + \int_{\sigma^p}^{\rho^p} \mathcal{F}(\kappa^{\frac{1}{p}}, \gamma^{\frac{1}{p}}) \, {}^{\rho^p}d_{q_1}^T\kappa \right]$$

$$\leq \frac{\mathcal{F}(\sigma, \gamma^{\frac{1}{p}}) + \mathcal{F}(\rho, \kappa^{\frac{1}{p}})}{2} \tag{5.5}$$

for all $\gamma \in [\varrho^p, d^p]$. By ${}_{\varrho^p}T_{q_2}$ integration of inequality (5.5) on $[\varrho^p, d^p]$, we get

$$\frac{1}{d^p-\varrho^p}\int_{\varrho^p}^{d^p}\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\gamma^{\frac{1}{p}}\right){}_{\varrho^p}d_{q_2}^T\gamma$$

$$\leq \frac{1}{2(\rho^p-\sigma^p)(d^p-\varrho^p)}\left[\int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}_{\varrho^p}d_{q_2}^T\gamma\,{}_{\sigma^p}d_{q_1}^T\kappa+\int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}_{\varrho^p}d_{q_2}^T\gamma\,{}^{\rho^p}d_{q_1}^T\kappa\right]$$

$$\leq \frac{1}{2(d^p-\varrho^p)}\int_{\varrho^p}^{d^p}[\mathcal{F}(\sigma,\gamma^{\frac{1}{p}})+\mathcal{F}(\rho,\gamma^{\frac{1}{p}})]_{\varrho^p}d_{q_2}^T\gamma. \tag{5.6}$$

Similarly, by ${}^{d^p}T_{q_2}$ integration of inequality (5.5) on $[\varrho^p, d^p]$, we obtain

$$\frac{1}{d^p-\varrho^p}\int_{\varrho^p}^{d^p}\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\gamma^{\frac{1}{p}}\right){}^{d^p}d_{q_2}^T\gamma$$

$$\leq \frac{1}{2(\rho^p-\sigma^p)(d^p-\varrho^p)}\left[\int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}^{d^p}d_{q_2}^T\gamma\,{}_{\sigma^p}d_{q_1}^T\kappa+\int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}^{d^p}d_{q_2}^T\gamma\,{}^{\rho^p}d_{q_1}^T\kappa\right]$$

$$\leq \frac{1}{2(d^p-\varrho^p)}\int_{\varrho^p}^{d^p}[\mathcal{F}(\sigma,\gamma^{\frac{1}{p}})+\mathcal{F}(\rho,\gamma^{\frac{1}{p}})]^{d^p}d_{q_2}^T\gamma. \tag{5.7}$$

By adding (5.3), (5.4), (5.6) and (5.7), we get

$$\frac{1}{\rho^p-\sigma^p}\left[\int_{\sigma^p}^{\rho^p}\mathcal{F}\left(\kappa^{\frac{1}{p}},\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right){}_{\sigma^p}d_{q_1}^T\kappa+\int_{\sigma^p}^{\rho^p}\mathcal{F}\left(\kappa^{\frac{1}{p}},\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right){}^{\rho^p}d_{q_1}^T\kappa\right]$$

$$+\frac{1}{d^p-\varrho^p}\left[\int_{\varrho^p}^{d^p}\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\gamma^{\frac{1}{p}}\right){}_{\varrho^p}d_{q_2}^T\gamma+\int_{\varrho^p}^{d^p}\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\gamma^{\frac{1}{p}}\right){}^{d^p}d_{q_2}^T\gamma\right]$$

$$\leq \frac{1}{(\rho^p-\sigma^p)(d^p-\varrho^p)}\left[\int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}_{\varrho^p}d_{q_2}^T\gamma\,{}_{\sigma^p}d_{q_1}^T\kappa+\int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}^{d^p}d_{q_2}^T\gamma\,{}_{\sigma^p}d_{q_1}^T\kappa\right.$$

$$\left.+\int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}_{\varrho^p}d_{q_2}^T\gamma\,{}^{\rho^p}d_{q_1}^T\kappa+\int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}^{d^p}d_{q_2}^T\gamma\,{}^{\rho^p}d_{q_1}^T\kappa\right]$$

$$\leq \frac{1}{2(\rho^p-\sigma^p)}\left[\int_{\sigma^p}^{\rho^p}[\mathcal{F}(\kappa^{\frac{1}{p}},\varrho)+\mathcal{F}(\kappa^{\frac{1}{p}},d)]_{\sigma^p}d_{q_1}^T\kappa+\int_{\sigma^p}^{\rho^p}[\mathcal{F}(\kappa^{\frac{1}{p}},\varrho)+\mathcal{F}(\kappa^{\frac{1}{p}},d)]^{\rho^p}d_{q_1}^T\kappa\right]$$

$$+\frac{1}{2(d^p-\varrho^p)}\left[\int\limits_{\varrho^p}^{d^p}[\mathcal{F}(\sigma,\gamma^{\frac{1}{p}})+\mathcal{F}(\rho,\gamma^{\frac{1}{p}})]_{\varrho^p}d_{q_2}^T\gamma+\int\limits_{\varrho^p}^{d^p}[\mathcal{F}(\sigma,\gamma^{\frac{1}{p}})+\mathcal{F}(\rho,\gamma^{\frac{1}{p}})]^{d^p}d_{q_2}^T\gamma\right]. \quad (5.8)$$

This completes the proof of second and third inequality of (5.1). From left-hand side of inequality (4.1), we have

$$\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right)$$
$$\le\frac{1}{2(\rho^p-\sigma^p)}\left[\int\limits_{\sigma^p}^{\rho^p}\mathcal{F}\left(\kappa^{\frac{1}{p}},\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right){}_{\sigma^p}d_{q_1}^T\kappa+\int\limits_{\sigma^p}^{\rho^p}\mathcal{F}\left(\kappa^{\frac{1}{p}},\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right){}^{\rho^p}d_{q_1}^T\kappa\right] \quad (5.9)$$

and

$$\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right)$$
$$\le\frac{1}{2(d^p-\varrho^p)}\left[\int\limits_{\varrho^p}^{d^p}\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\gamma^{\frac{1}{p}}\right){}_{\varrho^p}d_{q_2}^T\gamma+\int\limits_{\varrho^p}^{d^p}\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\gamma^{\frac{1}{p}}\right){}^{d^p}d_{q_2}^T\gamma\right]. \quad (5.10)$$

Using (5.9) and (5.10) in (5.8), we get first inequality of (5.1). By applying right-hand side of the inequality (4.1), we obtain

$$\frac{1}{2(\rho^p-\sigma^p)}\left[\int\limits_{\sigma^p}^{\rho^p}[\mathcal{F}(\kappa^{\frac{1}{p}},\varrho)+\mathcal{F}(\kappa^{\frac{1}{p}},d)]_{\sigma^p}d_{q_1}^T\kappa+\int\limits_{\sigma^p}^{\rho^p}[\mathcal{F}(\kappa^{\frac{1}{p}},\varrho)+\mathcal{F}(\kappa^{\frac{1}{p}},d)]^{\rho^p}d_{q_1}^T\kappa\right]$$
$$\le\frac{\mathcal{F}(\sigma,\varrho)+\mathcal{F}(\rho,\varrho)+\mathcal{F}(\sigma,d)+\mathcal{F}(\rho,d)}{2} \quad (5.11)$$

and

$$\frac{1}{2(d^p-\varrho^p)}\left[\int\limits_{\varrho^p}^{d^p}[\mathcal{F}(\sigma,\gamma^{\frac{1}{p}})+\mathcal{F}(\rho,\gamma^{\frac{1}{p}})]_{\varrho^p}d_{q_2}^T\gamma+\int\limits_{\varrho^p}^{d^p}[\mathcal{F}(\sigma,\gamma^{\frac{1}{p}})+\mathcal{F}(\rho,\gamma^{\frac{1}{p}})]^{d^p}d_{q_2}^T\gamma\right]$$
$$\le\frac{\mathcal{F}(\sigma,\varrho)+\mathcal{F}(\rho,\varrho)+\mathcal{F}(\sigma,d)+\mathcal{F}(\rho,d)}{2}. \quad (5.12)$$

Using (5.11) and (5.12) in (5.8), we obtain the last inequality of (5.1).

Thus, the proof is accomplished.

***Corollary* 2.** *If we take the limit* $q_1, q_2 \to 1$ *in Theorem* 7, *then inequality* (5.1) *becomes the Hermite–Hadamard inequality for coordinated* p*-convex functions which is given by*

$$\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right)$$

$$\leq \frac{1}{2(\rho^p-\sigma^p)}\int_{\sigma^p}^{\rho^p}\mathcal{F}\left(\kappa^{\frac{1}{p}},\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right)d\kappa$$

$$+\frac{1}{2(d^p-\varrho^p)}\int_{\varrho^p}^{d^p}\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\gamma^{\frac{1}{p}}\right)d\gamma$$

$$\leq \frac{1}{(\rho^p-\sigma^p)(d^p-\varrho^p)}\int_{\sigma^p}^{\rho^p}\int_{\varrho^p}^{d^p}\mathcal{F}\left(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}}\right)d\kappa d\gamma$$

$$\leq \frac{1}{4(\rho^p-\sigma^p)}\left[\int_{\sigma^p}^{\rho^p}[\mathcal{F}(\kappa^{\frac{1}{p}},\varrho)+\mathcal{F}(\kappa^{\frac{1}{p}},d)]d\kappa\right]$$

$$+\frac{1}{4(d^p-\varrho^p)}\left[\int_{\varrho^p}^{d^p}[\mathcal{F}(\sigma,\gamma^{\frac{1}{p}})+\mathcal{F}(\rho,\gamma^{\frac{1}{p}})]d\gamma\right]$$

$$\leq \frac{\mathcal{F}(\sigma,\varrho)+\mathcal{F}(\sigma,d)+\mathcal{F}(\rho,\varrho)+\mathcal{F}(\rho,d)}{4}$$

which is given by Yang in [19, *Theorem* 2.2 (*for* $h_1(t)=h_2(t)=t$ *and* $p_1=p_2=p$)].

6. Examples.

***Example* 1.** Define the function $\mathcal{F}(\kappa)=\kappa^3$ on $[0,1]$. Applying Theorem 6 for $q=\dfrac{1}{2}$ and $p=3$, we have

$$\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}}\right)=\mathcal{F}\left(\frac{1}{\sqrt[3]{2}}\right)=\frac{1}{2},$$

$$\frac{1}{2(\rho^p-\sigma^p)}\left[\int_{\sigma^p}^{\rho^p}\mathcal{F}(\kappa^{\frac{1}{p}})\,{}_{\sigma^p}d_q^T\kappa+\int_{\sigma^p}^{\rho^p}\mathcal{F}(\kappa^{\frac{1}{p}})\,{}^{\rho^p}d_q^T\kappa\right]=\left[\int_0^1\kappa\,{}_0d_{\frac{1}{2}}^T\kappa+\int_0^1\kappa\,{}^1d_{\frac{1}{2}}^T\kappa\right]=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right)=\frac{1}{2}$$

and

$$\frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}=\frac{1}{2}.$$

This demonstrates the result described in Theorem 6.

***Example* 2.** Define the function $\mathcal{F}(\kappa,\gamma)=\kappa^2\gamma^2$ on $[0,1]\times[0,1]$. Applying Theorem 7 for $q_1=q_2=\dfrac{1}{2}$ and $p=2$, we get

$$\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right)=\mathcal{F}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=\frac{1}{4},$$

$$\frac{1}{4(\rho^p-\sigma^p)}\left[\int\limits_{\sigma^p}^{\rho^p}\mathcal{F}\left(\kappa^{\frac{1}{p}},\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right){}_{\sigma^p}d_{q_1}^T\kappa+\int\limits_{\sigma^p}^{\rho^p}\mathcal{F}\left(\kappa^{\frac{1}{p}},\left(\frac{\varrho^p+d^p}{2}\right)^{\frac{1}{p}}\right){}^{\rho^p}d_{q_1}^T\kappa\right]$$

$$+\frac{1}{4(d^p-\varrho^p)}\left[\int\limits_{\varrho^p}^{d^p}\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\gamma^{\frac{1}{p}}\right){}_{\varrho^p}d_{q_2}^T\gamma+\int\limits_{\varrho^p}^{d^p}\mathcal{F}\left(\left(\frac{\sigma^p+\rho^p}{2}\right)^{\frac{1}{p}},\gamma^{\frac{1}{p}}\right){}^{d^p}d_{q_2}^T\gamma\right]$$

$$=\frac{1}{4}\left[\int\limits_0^1\frac{\kappa}{2}\,{}_0d_{\frac{1}{2}}^T\kappa+\int\limits_0^1\frac{\kappa}{2}\,{}^1d_{\frac{1}{2}}^T\kappa+\int\limits_0^1\frac{\gamma}{2}\,{}_0d_{\frac{1}{2}}^T\gamma+\int\limits_0^1\frac{\gamma}{2}\,{}^1d_{\frac{1}{2}}^T\gamma\right]$$

$$=\frac{1}{4}\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)=\frac{1}{4},$$

$$\frac{1}{4(\rho^p-\sigma^p)(d^p-\varrho^p)}\left[\int\limits_{\sigma^p}^{\rho^p}\int\limits_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}_{\varrho^p}d_{q_2}^T\gamma\,{}_{\sigma^p}d_{q_1}^T\kappa+\int\limits_{\sigma^p}^{\rho^p}\int\limits_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}^{d^p}d_{q_2}^T\gamma\,{}_{\sigma^p}d_{q_1}^T\kappa\right.$$

$$\left.+\int\limits_{\sigma^p}^{\rho^p}\int\limits_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}_{\varrho^p}d_{q_2}^T\gamma\,{}^{\rho^p}d_{q_1}^T\kappa+\int\limits_{\sigma^p}^{\rho^p}\int\limits_{\varrho^p}^{d^p}\mathcal{F}(\kappa^{\frac{1}{p}},\gamma^{\frac{1}{p}})\,{}^{d^p}d_{q_2}^T\gamma\,{}^{\rho^p}d_{q_1}^T\kappa\right]$$

$$=\frac{1}{4}\left[\int\limits_0^1\int\limits_0^1\kappa\gamma\,{}_0d_{q_2}^T\gamma\,{}_0d_{q_1}^T\kappa+\int\limits_0^1\int\limits_0^1\kappa\gamma\,{}^1d_{q_2}^T\gamma\,{}_0d_{q_1}^T\kappa+\int\limits_0^1\int\limits_0^1\kappa\gamma\,{}_0d_{q_2}^T\gamma\,{}^1d_{q_1}^T\kappa+\int\limits_0^1\int\limits_0^1\kappa\gamma\,{}^1d_{q_2}^T\gamma\,{}^1d_{q_1}^T\kappa\right]$$

$$=\frac{1}{4}\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)=\frac{1}{4},$$

$$\frac{1}{8(\rho^p-\sigma^p)}\left[\int\limits_{\sigma^p}^{\rho^p}[\mathcal{F}(\kappa^{\frac{1}{p}},\varrho)+\mathcal{F}(\kappa^{\frac{1}{p}},d)]_{\sigma^p}d_{q_1}^T\kappa+\int\limits_{\sigma^p}^{\rho^p}[\mathcal{F}(\kappa^{\frac{1}{p}},\varrho)+\mathcal{F}(\kappa^{\frac{1}{p}},d)]^{\rho^p}d_{q_1}^T\kappa\right]$$

$$+\frac{1}{8(d^p-\varrho^p)}\left[\int\limits_{\varrho^p}^{d^p}[\mathcal{F}(\sigma,\gamma^{\frac{1}{p}})+\mathcal{F}(\rho,\gamma^{\frac{1}{p}})]_{\varrho^p}d_{q_2}^T\gamma+\int\limits_{\varrho^p}^{d^p}[\mathcal{F}(\sigma,\gamma^{\frac{1}{p}})+\mathcal{F}(\rho,\gamma^{\frac{1}{p}})]^{d^p}d_{q_2}^T\gamma\right]$$

$$=\frac{1}{8}\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)=\frac{1}{4}$$

and

$$\frac{\mathcal{F}(\sigma,\varrho)+\mathcal{F}(\sigma,d)+\mathcal{F}(\rho,\varrho)+\mathcal{F}(\rho,d)}{4}=\frac{1}{4}.$$

This demonstrates the result described in Theorem 7.

7. Conclusions. In this paper, we establish new T_q-Hermite–Hadamard-type inequalities for p-convex and coordinated p-convex functions. It is also shown that some classical results can be obtained by the results presented in the current investigations by taking the limit $q \to 1$. In the upcoming directions, researchers can be obtain similar inequalities for different convexity classes of two variables functions.

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