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# BOUNDS ON THE PARAMETERS OF NON-$L$-BORDERENERGETIC GRAPHS

# ОБМЕЖЕННЯ НА ПАРАМЕТРИ НЕ $L$-ГРАНИЧНИХ ЕНЕРГЕТИЧНИХ ГРАФІВ

We consider graphs such that their Laplacian energy is equivalent to the Laplacian energy of the complete graph of the same order, which is called an $L$-borderenergetic graph. Firstly, we study the graphs with degree sequence consisting of at most three distinct integers and give new bounds for the number of vertices of these graphs to be non-$L$-borderenergetic. Second, by using Koolen – Moulton and McClelland inequalities, we give new bounds for the number of edges of a non-$L$-borderenergetic graph. Third, we use recent bounds given by Milovanovic, et al. on Laplacian energy to get similar conditions for non-$L$-borderenergetic graphs. Our bounds depend only on the degree sequence of a graph, which is much easier than computing the spectrum of the graph. In other words, we developed a faster approach to exclude non-$L$-borderenergetic graphs.

Ми розглядаємо графи, лапласівська енергія яких еквівалентна лапласівській енергії повного графа такого ж порядку, який називається $L$-граничним енергетичним графом. По-перше, ми вивчаємо графи, для яких послідовність степенів складається не більше ніж з трьох різних цілих чисел, і наводимо нові оцінки для кількості вершин цих графів, за яких вони не є $L$-гранично енергетичними. По-друге, використовуючи нерівності Кулена – Моултона та МакКлелланда, наводимо нові оцінки для кількості ребер не $L$-граничного енергетичного графа. По-третє, використовуємо оцінки, що були нещодавно отримані Міловановичем та ін. для лапласівської енергії, щоб отримати подібні умови для не $L$-граничних енергетичних графів. Наші оцінки залежать лише від послідовності степенів графа, що набагато зручніше, ніж обчислювати спектр графа. Іншими словами, розроблено більш швидкий підхід для вилучення не $L$-граничних енергетичних графів.

**1. Introduction.** A graph $G$ of order $n$ consists of the set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and edges $E$. It is said that $v_i$ is *adjacent to* $v_j$ if there exists an edge between $v_i$ and $v_j$ for $i, j \in \{1, 2, \ldots, n\}$. The number of vertices adjacent to $v_i$ is called as the degree of $v_i$, and it is shown as $d_i$ for $i = 1, 2, \ldots, n$. A graph with no vertices adjacent to itself and no multiple edges between vertices is called as *simple graph*. Moreover, if the edges of a graph has no direction, it is called as *undirected graph*. In this paper, we consider only simple and undirected graphs.

The *adjacency matrix* $A(G)$ of a graph $G$ with order $n$ has the entry $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and 0 otherwise for $i, j = 1, 2, \ldots, n$. The diagonal matrix $D(G)$ associated with $G$ is defined as $D(G) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, where $d_i$ is the degree of the vertex $v_i$ of $G$ for $i = 1, 2, \ldots, n$. In this paper, we study the *Laplacian matrix* $L(G)$ of $G$, which is defined as $L(G) = D(G) - A(G)$. For more details and references on Laplacian matrix, see [18].

Similar to adjacency and Laplacian matrices one can associate a real symmetric matrix $M$ with a graph $G$ of order $n$. The set $\operatorname{Spec}(M) = \{\lambda_i(M),\ i = 1, 2, \ldots, n\}$ of eigenvalues of $M$ is called the $M$-spectrum of $G$. Then the $M$-energy of $G$ is defined as

$$E_M(G) = \sum_{i=1}^{n} \left| \lambda_i(M) - \frac{\operatorname{tr}(M)}{n} \right|. \tag{1.1}$$

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For further details on the theory of graph energy, see [12, 17, 20], and for its applications in chemistry, see [16, 17].

In particular, the Laplacian energy of $G,$ introduced by Gutman and Zhou [13] is

$$LE(G) = \sum_{i=1}^{n} |\mu_i - \bar{d}|, \tag{1.2}$$

where $\mu_i$ is Laplacian eigenvalues of $G$ for $i = 1, 2, \ldots, n$ and $\bar{d}$ is the average degree of $G.$ The general theory of Laplacian energy is studied heavily in the literature (see, for instance, [3, 4, 8 – 10, 21]).

Furthermore, if a graph $G$ of order $n$ has the energy equivalent to the energy of a complete graph of order $n,$ it is called as *borderenergetic*, introduced by [11] and studied by [7, 15, 25]. In particular, a graph of order $n$ with Laplacian energy $2n - 2$ is called Laplacian borderenergetic, and it is first studied by Tura [24] and then in [6, 14]. In other words, a graph is called $L$-borderenergetic if it satisfies $LE(G) = LE(K_n) = 2n-2,$ where $K_n$ is the complete graph of order $n.$ A lot of families of $L$-borderenergetic graphs are presented by the authors in [5].

In this paper, we study the Laplacian energy of graphs. It is needed to find all eigenvalues of a graph to calculate its energy and check whether it is $L$-borderenergetic. We obtain results to make this decision faster for non-$L$-borderenergetic graphs. In other words, we prove sufficient conditions on the parameters of a non-$L$-borderenergetic graph.

Let $G$ be a graph with degree sequence consisting of at most three distinct integers. We first prove a simplified upper bound on the Laplacian energy of $G$ (see Lemma 2.1). Then, by using this result, we prove a lower and upper bound on the number of vertices of $G.$ These bounds can be calculated by using the degree sequence of $G$ (see Theorem 3.1 and its corollaries). For example, a graph with one vertex of degree 5 and two vertices of degree 4 and $s \geq 14$ vertices of degree 3 can not be $L$-borderenergetic. In the second part of the paper, we prove two lower bounds and one upper bound on the number of edges of $G$ so that is not $L$-borderenergetic (see Theorems 4.1, 4.2, and 4.3), where the bounds depend on the degree sequence of $G.$ In addition, we prove a lower bound on the number of the vertex with maximum degree and on the sum of signless Laplacian eigenvalues of a graph (see, resp., Theorems 4.4 and 5.1). We plot an illustrating graph by using SageMath [23] for each of our results to show an existence of a graph satisfying our results and to clearly give our point.

This paper is organized as follows. In Section 2, a preliminary result which gives a McClelland type of upper bound on Laplacian energy of a graph having at most 3 distinct degrees. Then, by this result, we prove some sufficient conditions on the number of vertices to be non-$L$-borderenergetic in Section 3. Next, the conditions on the number of edges of a non-$L$-borderenergetic graphs are given in Section 4. In Section 5, the singless Laplacian spectrum of a non-$L$-borderenergetic graph is studied. Finally, we conclude our paper in Section 6.

**2. Preliminary results.** Let $G$ be a graph of order $n$ and $m$ edges with its Laplacian spectrum $\{\mu_1, \ldots, \mu_n\}$ and its degree sequence $[d_1, d_2, \ldots, d_n].$ In this paper, we use the Koolen – Moulton and the McClelland type of inequalities on the Laplacian energy, respectively (see [13]):

$$LE(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2M - \left(\frac{2m}{n}\right)^2\right]} \tag{2.1}$$

and

$$LE(G) \leq \sqrt{2Mn}, \tag{2.2}$$

where $M = m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2$.

We start with a result which simplifies the McClelland inequality on Laplacian energy for a graph with at most 3 distinct elements in its degree sequence.

**Lemma 2.1.** *Let $G$ be a graph of order $n$ with $t$, $r$ and $n-t-r$ vertices of degrees $\alpha$, $\gamma$ and $\beta$, where $n \geq t+r$ and $\alpha \geq \gamma \geq \beta$, respectively. Then the Laplacian energy of $G$ satisfies*

$$[LE(G)]^2 \leq \beta n^2 + \big[(\alpha-\beta)(\alpha-\beta+1)t + (\gamma-\beta)(\gamma-\beta+1)r\big]n - \big[(\alpha-\beta)t + (\gamma-\beta)r\big]^2.$$

***Proof.*** Suppose that $G$ has the Laplacian spectrum $\{\mu_1, \ldots, \mu_n\}$ and degree sequence $[d_1, d_2, \ldots, d_n]$ with average degree $\bar{d}$. Then $LE(G) = \sum_{i=1}^{n}|\mu_i - \bar{d}|$. By Cauchy–Schwarz inequality, we get

$$\left(\sum_{i=1}^{n}|\mu_i - \bar{d}|\right)^2 \leq n\sum_{i=1}^{n}(\mu_i - \bar{d})^2 = n\sum_{i=1}^{n}(\mu_i^2 + \bar{d}^2 - 2\mu_i\bar{d})$$

$$= n\left(2m + \sum_{i=1}^{n} d_i^2 + n\bar{d}^2 - 4\bar{d}m\right), \tag{2.3}$$

where the last equality follows from $\sum_{i=1}^{n}\mu_i^2 = \sum_{i=1}^{n} d_i^2 + d_i$ and $\sum_{i=1}^{n}\mu_i = 2m$. Since $G$ has $t$, $r$ and $n-t-r$ vertices of degree $\alpha$, $\gamma$ and $\beta$, we get

$$\bar{d} = \frac{\alpha t + \gamma r + \beta(n-t-r)}{n} \qquad \text{and} \qquad m = \frac{\alpha t + \gamma r + \beta(n-t-r)}{2},$$

$$\sum_{i=1}^{n} d_i^2 = t\alpha^2 + r\gamma^2 + (n-t-r)\beta^2.$$

Thus, by (2.3), we have

$$\begin{aligned}
[LE(G)]^2 &= \left(\sum_{i=1}^{n}|\mu_i - \bar{d}|\right)^2 \\
&\leq n\Bigg[\alpha t + \gamma r + \beta(n-t-r) + t\alpha^2 + r\gamma^2 + (n-t-r)\beta^2 \\
&\quad + n\left(\frac{\alpha t + \gamma r + \beta(n-t-r)}{n}\right)^2 \\
&\quad - 4\left(\frac{\alpha t + \gamma r + \beta(n-t-r)}{n}\right)\left(\frac{\alpha t + \gamma r + \beta(n-t-r)}{2}\right)\Bigg] \\
&= \beta n^2 + \big[(\alpha-\beta)(\alpha-\beta+1)t + (\gamma-\beta)(\gamma-\beta+1)r\big]n \\
&\quad - \big[(\alpha-\beta)t + (\gamma-\beta)r\big]^2.
\end{aligned}$$

Lemma 2.1 is proved.

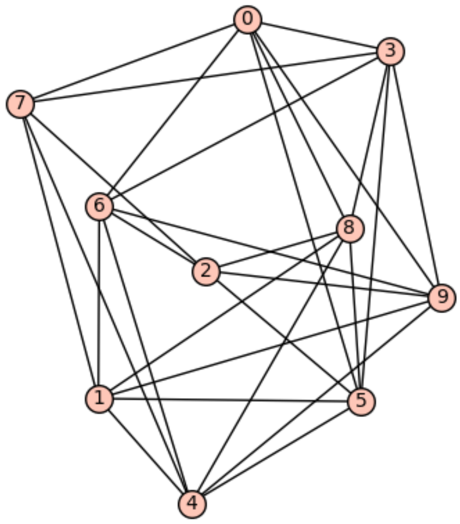



Fig. 1. An $L$-borderenergetic graph of order $n = 10$ with degree sequence $[6, 6, 6, 6, 6, 6, 6, 6, 5, 5]$ satisfying Lemma 2.1.

For $r = 0$ in Lemma 2.1, we have a graph $G$ with two distinct degrees $\alpha$ and $\beta$:

$$[LE(G)]^2 \leq \beta n^2 + (\alpha - \beta)(\alpha - \beta + 1)nt - (\alpha - \beta)^2 t^2. \tag{2.4}$$

We now give an $L$-borderenergetic graph of order 10 and we see that its parameters $n$, $\alpha$, $\beta$, $t$ satisfy (2.4).

***Example* 2.1.** The graph $G$ of order $n = 10$ with degree sequence $[6, 6, 6, 6, 6, 6, 6, 6, 5, 5]$ given in Fig. 1 satisfies the bound in Lemma 2.1. Note that the upper bound is $596$, where $t = 8$, $r = 0$, $\alpha = 6$, $\beta = 5$, $|LE(G)|^2 = 324$ and $\mathrm{Spec}(G) = \{10, 0, 7, 7, 7, 7, 5, 5, 5, 5\}$.

Similarly, for $\gamma = \alpha - 1$ and $\beta = \alpha - 2$ in Lemma 2.1, we have another simplified bound on the Laplacian energy of a graph:

$$[LE(G)]^2 \leq (\alpha - 2)n^2 + (6t + 2r)n - (r + 2t)^2.$$

**3. Results on the number of vertices of a non-$L$-borderenergetic graph.** In this section, we present our new results on non-$L$-borderenergetic graphs. In other words, we give some sufficient conditions in order a graph not to be $L$-borderenergetic. Note that we mainly extend the results in [6]. We give examples of graphs after each of our results to clarify our points. Our results mainly give conditions on the parameters of a graph so that it is not $L$-borderenergetic. Hence, by using our results, one can decide whether a given graph is $L$-borderenergetic without computing its energy spectrum. Note that computation of the spectrum of a graph is not easy, on the other hand, checking a condition on the parameters of a graph is so easy. We now present a result which extends Theorem 2 in [6].

**Theorem 3.1.** *Let $G$ be a graph of order $n$ with $t$, $r$ and $n - t - r$ vertices of degrees $\alpha$, $\gamma$ and $\beta < 4$, respectively, such that $\alpha \geq \gamma \geq \beta$ and*

$$\Delta = \big[(\alpha - \beta)(\alpha - \beta + 1)t + (\gamma - \beta)(\gamma - \beta + 1)r + 8\big]^2 - 4(4 - \beta)\big[[(\alpha - \beta)t + (\gamma - \beta)r\big]^2 + 4\big].$$

*If either of the following holds*:

(i) $\Delta < 0$,

(ii) $n < \dfrac{\big[(\alpha-\beta)(\alpha-\beta+1)t+(\gamma-\beta)(\gamma-\beta+1)r+8\big]-\sqrt{\Delta}}{2(4-\beta)}$,

(iii) $n > \dfrac{\big[(\alpha-\beta)(\alpha-\beta+1)t+(\gamma-\beta)(\gamma-\beta+1)r+8\big]+\sqrt{\Delta}}{2(4-\beta)}$,

*then $G$ is not $L$-borderenergetic.*

***Proof.*** Suppose that $G$ is $L$-borderenergetic graph, i.e., $LE(G) = 2(n-1)$. By using the upper bound for the Laplacian energy given in Lemma 2.1 for $G$, we have

$$\begin{aligned}4(n-1)^2 &= [LE(G)]^2\\ &\le \beta n^2 + \big[(\alpha-\beta)(\alpha-\beta+1)t+(\gamma-\beta)(\gamma-\beta+1)r\big]n - \big[(\alpha-\beta)t+(\gamma-\beta)r\big]^2.\end{aligned}$$

Therefore, we get the following inequality on $n$:

$$\begin{aligned}(4-\beta)n^2 - \big[(\alpha-\beta)(\alpha-\beta+1)t\\ +(\gamma-\beta)(\gamma-\beta+1)r+8\big]n + \big[(\alpha-\beta)t+(\gamma-\beta)r\big]^2+4 \le 0.\end{aligned} \tag{3.1}$$

Let $a, b, c$ and $\Delta$ be the coefficients and discriminant of this quadratic equation with respect to $n$:

$$\begin{gathered}a = 4-\beta,\\ b = -\big[(\alpha-\beta)(\alpha-\beta+1)t+(\gamma-\beta)(\gamma-\beta+1)r+8\big],\\ c = \big[(\alpha-\beta)t+(\gamma-\beta)r\big]^2+4,\\ \Delta = b^2-4ac.\end{gathered}$$

If either of $\Delta < 0$ or $n < \dfrac{-b-\sqrt{\Delta}}{2a}$ or $n > \dfrac{-b+\sqrt{\Delta}}{2a}$ holds, then (3.1) has no solution. Therefore, $G$ can not be $L$-borderenergetic. By substituting the values of $a, b, c$, we finish the proof.

***Remark* 3.1.** For $\beta = 4$ in Theorem 3.1, equation (3.1) has no solution when

$$n < \frac{\big[(\alpha-4)t+(\gamma-4)r\big]^2+4}{(\alpha-4)(\alpha-3)t+(\gamma-4)(\gamma-3)r+8}.$$

In this case we have $n < t+r$, a contradiction. Similarly, when $\beta > 4$ and

$$n < \frac{\big[(\alpha-\beta)(\alpha-\beta+1)t+(\gamma-\beta)(\gamma-\beta+1)r+8\big]-\sqrt{\Delta}}{2(4-\beta)},$$

equation (3.1) has no solution, however in this case we also get $n < t+r$, a contradiction. Therefore, we don't consider the case $\beta \ge 4$ in Theorem 3.1.

We note that Theorem 3.1 is an extension of Theorem 2 in [6], which deals with the case $\gamma = 3$, $\beta = 2$ and $t = 0$. In other words, Theorem 3.1 deals with graphs having at most three distinct arbitrary degrees, but Theorem 2 in [6] considers graphs only with degrees 2 and 3. Below we give three graph-families satisfying Theorem 3.1.

***Example* 3.1.** 1. Let $G_1$ be a graph of order $n \geq 20$ with $t = 2$, $r = 18$ and $n - t - r$ vertices of degrees $\alpha = 5$, $\gamma = 3$ and $\beta = 2$, respectively. Then we have $\Delta = -16$ in Theorem 3.1. So, we conclude that $G_1$ can not be $L$-borderenergetic as it satisfies Theorem 3.1(i).

2. Let $G_2$ be a graph of order $n = 16$ with $t = 1$ and $r = 15$ vertices of degrees $\alpha = 5$ and $\gamma = 3$, respectively. Then the bound in Theorem 3.1(ii) approximately equals to 19.33. So, we conclude that $G_2$ can not be $L$-borderenergetic as it satisfies Theorem 3.1(ii).

3. Let $G_3$ be a graph of order $n \geq 17$ with $t = 1$, $r = 2$ and $n - t - r$ vertices of degrees $\alpha = 5$, $\gamma = 4$ and $\beta = 3$, respectively. Then the bound in Theorem 3.1(iii) approximately equals to 16.81. So, we conclude that $G_3$ can not be $L$-borderenergetic as it satisfies Theorem 3.1(iii).

We plot a graph illustrating each of the cases in this example in Fig. 2.

We now present a special case of Theorem 3.1 below in which we take $\alpha = 4$, $\gamma = 3$ and $\beta = 2$. We also present its proof for completeness of the paper.

***Corollary* 3.1.** *Let $G$ be a graph of order $n$ such that $G$ has $t$, $r$ and $n-t-r$ vertices of degree* 4, 3 *and* 2, *respectively. If either of the following holds*:

(i) $(3t + r + 4)^2 - 2[(r + 2t)^2 + 4] < 0$,

(ii) $n < \dfrac{3t + r + 4 - \sqrt{(3t + r + 4)^2 - 2[(r + 2t)^2 + 4]}}{2}$,

(iii) $n > \dfrac{3t + r + 4 + \sqrt{(3t + r + 4)^2 - 2[(r + 2t)^2 + 4]}}{2}$,

*then $G$ is not $L$-borderenergetic.*

***Proof.*** Suppose that $G$ is $L$-borderenergetic graph, i.e., $LE(G) = 2(n-1)$. By using the upper bound for the Laplacian energy given in Lemma 2.1 with $\alpha = 4$, $\gamma = 3$ and $\beta = 2$, we have

$$4(n-1)^2 = [LE(G)]^2 \leq 2n^2 + (6t + 2r)n - (r + 2t)^2.$$

Therefore, we get the following inequality on $n$:

$$2n^2 - (6t + 2r + 8)n + (r + 2t)^2 + 4 \leq 0.$$

If $(3t + r + 4)^2 - 2[(r + 2t)^2 + 4] \geq 0$, we have the following solution set for $n$:

$$n \in \left[\frac{6t + 2r + 8 - \sqrt{(6t + 2r + 8)^2 - 8[(r + 2t)^2 + 4]}}{4},\right.$$
$$\left.\frac{6t + 2r + 8 + \sqrt{(6t + 2r + 8)^2 - 8[(r + 2t)^2 + 4]}}{4}\right].$$

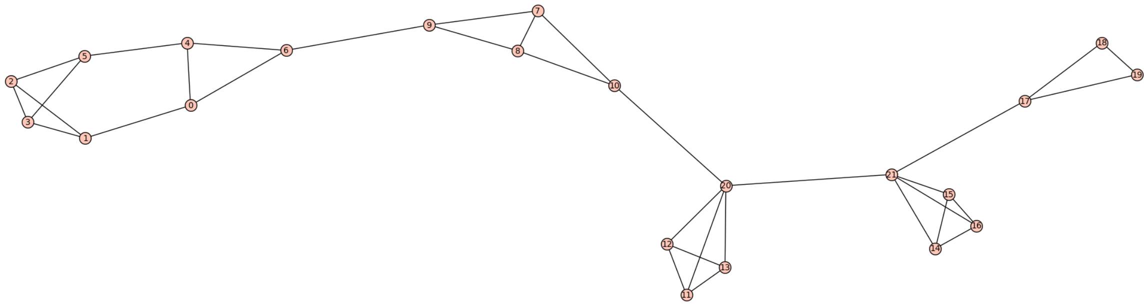
The result follows by dividing each term by 2:

$$(3t + r + 4)^2 - 2[(r + 2t)^2 + 4] < 0,$$

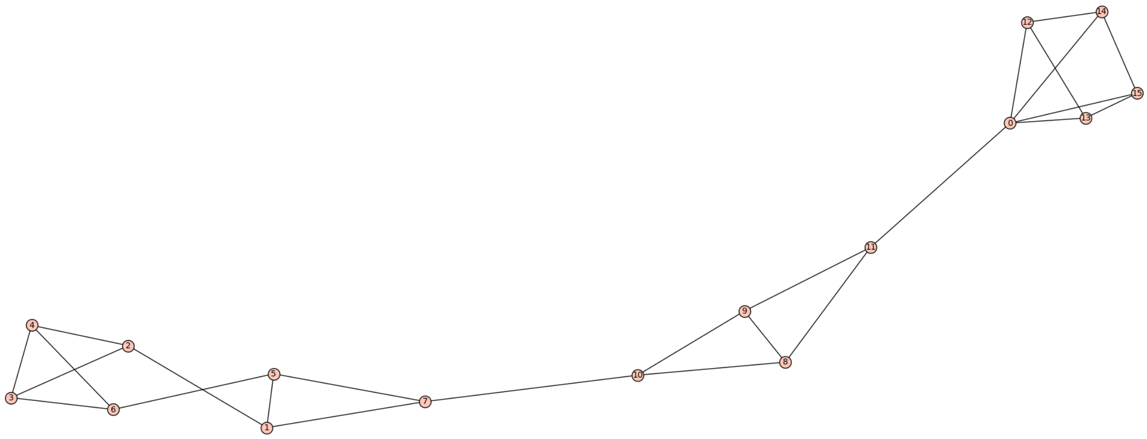
$$(3t + r + 4)^2 - 2[(r + 2t)^2 + 4] = 0,$$

$$t^2 - (2r - 24)t - r^2 + 8r + 8 = 0,$$

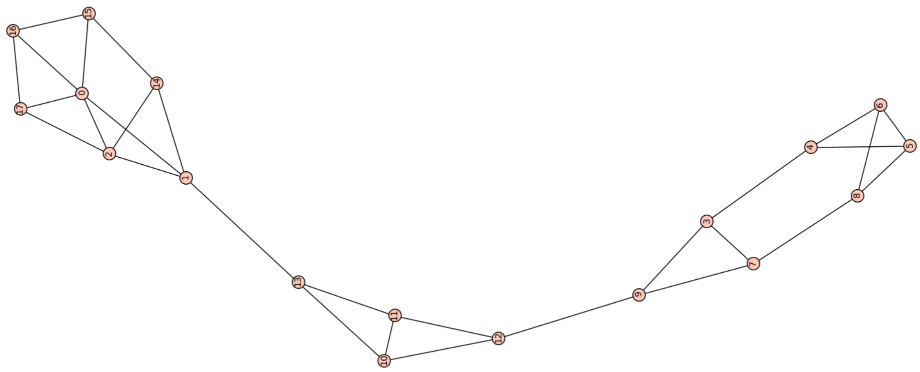
$$\Delta = 4(r - 12)^2 + 4r^2 - 32r - 32,$$



$G_1$: A non-$L$-borderenergetic graph of order 22 with degree sequence $[5, 5, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]$ satisfying Theorem 3.1(i).



$G_2$: A non-$L$-borderenergetic graph of order 16 with degree sequence $[5, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]$ satisfying Theorem 3.1(ii).



$G_3$: A non-$L$-borderenergetic graph of order 18 with degree sequence $[5, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]$ satisfying Theorem 3.1(iii).

Fig. 2. Non-$L$-borderenergetic graphs satisfying Theorem 3.1.

$$\Delta = 8r^2 - 128r + 544,$$

$$t_1 = \frac{2r - 24 - \sqrt{8r^2 - 128r + 544}}{2},$$

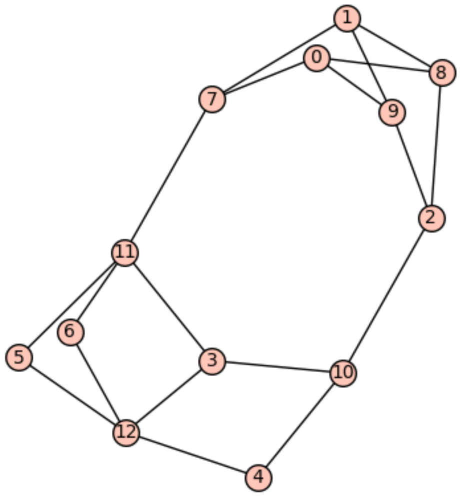



Fig. 3. A non-$L$-borderenergetic graph of order 13 having $3, 8, 2$ vertices of degree $2, 3, 4$, respectively, satisfying Corollary 3.1.

$$t_2 = \frac{2r - 24 + \sqrt{8r^2 - 128r + 544}}{2},$$

$$t_1 = r - 12 - \sqrt{2r^2 - 32r + 136},$$

$$t_2 = r - 12 + \sqrt{2r^2 - 32r + 136},$$

$$t_1 < 0 \qquad \text{or} \qquad t_2 > 0$$

is not $L$-borderenergetic. An example of a graph of order 13 is given below, where its parameters satisfy the condition of Corollary 3.1, so that we conclude that it is not $L$-borderenergetic.

***Example* 3.2.** The graph $G$ of order $n = 13$ given in Fig. 3 satisfies the condition of Corollary 3.1. Hence, we conclude that $G$ is not $L$-borderenergetic. Note that we reached this conclusion without computing the spectrum of $G$. On the other hand, we have verified that $G$ is not $L$-borderenergetic by computing its energy, i.e.,

$$LE(G) \approx 18.99 \neq 2n - 2 = 24.$$

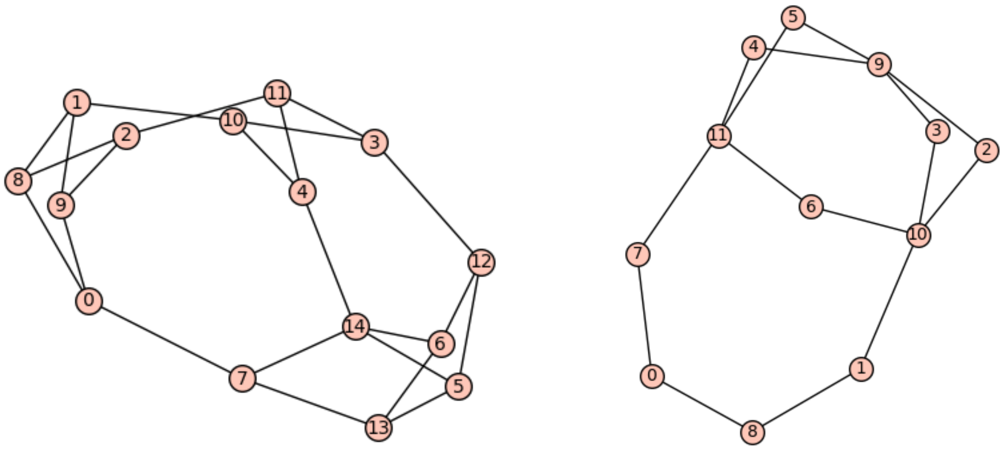
For $r = n - t$ and $r = 0$ in Corollary 3.1, we get the following results, respectively.

***Corollary* 3.2.** *If $G$ is a graph of order $n > t + 4 + \sqrt{8t + 12}$ with $n - t$ and $t$ vertices of degree 3 and 4, respectively, then $G$ is not $L$-borderenergetic.*

***Corollary* 3.3.** *Let $G$ be a graph of order $n > \dfrac{3t + 4 + \sqrt{(3t + 4)^2 - 4(2t^2 + 2)}}{2}$ with $n - t$ and $t$ vertices of degree 2 and 4, respectively. Then $G$ is not $L$-borderenergetic.*

Here we give two graphs for showing that graphs satisfying Corollaries 3.2 and 3.3 exist.

***Example* 3.3.** The graphs $G_1$ and $G_2$ of order 15 and 12 given in Fig. 4 satisfy the conditions of Corollaries 3.2 and 3.3, respectively. We conclude that $G_1$ and $G_2$ are not $L$-borderenergetic. Note that we reached this conclusion without computing their spectrum. It is easy to verify that they are not $L$-borderenergetic by checking their energies:




$G_1$ $G_2$

Fig. 4. $G_1$: A non-$L$-borderenergetic graph of order 15 having 14 and 1 vertices of degree 3 and 4, respectively, satisfying Corollary 3.2.
$G_2$: A non-$L$-borderenergetic graph of order 12 having 9 and 3 vertices of degree 2 and 4, respectively, satisfying Corollary 3.3.

$$LE(G_1) \approx 21.38 \neq 2n - 2 = 28,$$

$$LE(G_2) \approx 18.15 \neq 2n - 2 = 22.$$

**4. Results on the number of edges of a non-$L$-borderenergetic graph.** In this section, we now draw a connection between the number of edges of a graph and its Laplacian energy by using the Koolen – Moulton and McClelland inequalities on the Laplacian energy (see equations (2.1) and (2.2), respectively). In the second part of this section, we use the bounds on the Laplacian energy of a graph given in [19].

We first use the Koolen – Moulton type of inequality to get sufficient lower bound on the number of edges of a non-$L$-borderenergetic graph. In particular, we see that there exists an upper bound on the number of edges of an $L$-borderenergetic graph. This result extends Theorem 3 given in [6]. In particular, the authors in [6, Theorem 3] consider a graph with maximum degree $4$. On the other hand, Theorem 4.1 holds for a graph with any maximum degree $k$. We denote the first Zagreb index of a graph $G$ by $Z_g(G)$, which is defined as $Z_g(G) = \sum_{i=1}^{n} d_i^2$, where $[d_1, d_2, \ldots, d_n]$ is its degree sequence.

**Theorem 4.1.** *Let $G$ be a graph of order $n$ with $m$ edges and maximum degree $k$ such that*

$$m > \frac{4k(n-1) - 4 + Z_g(G)(n-1) + nk^2(n-2) + n(k-4)(n-1)}{4k(n-1)}.$$

*Then $G$ is not $L$-borderenergetic.*

***Proof.*** We define

$$f(x) = \frac{2x}{n} + \sqrt{(n-1)\left[2\left(x + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2x}{n}\right)^2\right) - \left(\frac{2x}{n}\right)^2\right]}.$$

As $f(x)$ is increasing in $\left[m, \dfrac{kn}{2}\right]$, we have $f(m) \leq f\left(\dfrac{kn}{2}\right)$. Hence, the Koolen – Moulton type of inequality on the Laplacian energy says

$$LE(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2M - \left(\frac{2m}{n}\right)^2\right]},$$

where $M = m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2$. Then we get

$$LE(G) = 2(n-1) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2\left(m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2\right) - \left(\frac{2m}{n}\right)^2\right]}$$

$$\leq k + \sqrt{(n-1)\left[kn + \sum_{i=1}^{n}(d_i - k)^2 - k^2\right]}.$$

Equivalently, we have the following equation arrays which follow from the equation above:

$$[2n - (k+2)]^2 \leq (n-1)\left[kn + \sum_{i=1}^{n}(d_i - k)^2 - k^2\right]$$

$$= (n-1)\left(kn + \sum_{i=1}^{n} d_i^2 - 2k\sum_{i=1}^{n} d_i + \sum_{i=1}^{n} k^2 - k^2\right)$$

$$= (n-1)\left(kn + \sum_{i=1}^{n} d_i^2 - 4km + nk^2 - k^2\right)$$

$$= (n-1)(kn + Z_g(G) - 4km + (n-1)k^2)$$

$$= (kn^2 + nZ_g(G) - 4kmn + n^2k^2 - nk^2 - kn - Z_g(G) + 4km - nk^2 + k^2,$$

$$4kmn - 4km \leq -4n^2 + 4nk - k^2 - 4k - 4 + kn^2 + n^2k^2$$

$$+ nZ_g(G) - Z_g(G) - 2nk^2 - kn + k^2 + 4n,$$

$$4km(n-1) \leq 2k(2n-2) - 4 + Z_g(G)(n-1) + nk^2(n-2) + (k-4)(n^2 - n),$$

$$m \leq \frac{4k(n-1) - 4 + Z_g(G)(n-1) + nk^2(n-2) + n(k-4)(n-1)}{4k(n-1)}.$$

Then we see that the number $m$ of edges of an $L$-borderenergetic graph is bounded by this bound.

Theorem 4.1 is proved.

The next example shows that there exist graphs satisfying Theorem 4.1.

***Example* 4.1.** The 3-regular graph $G$ of order $n = 10$ and $m = 15$ edges given in Fig. 5 is not $L$-borderenergetic and satisfies the condition of Theorem 4.1. We have $\mathrm{Spec}(G) = \{1, 0, 4, 4, 5, 5, 5, 2, 2, 2\}$ and its Laplacian energy is $LE(G) = 16 \neq 2n - 2$. We note that the right-hand side of the inequality in Theorem 4.1 is approximately 14.29.

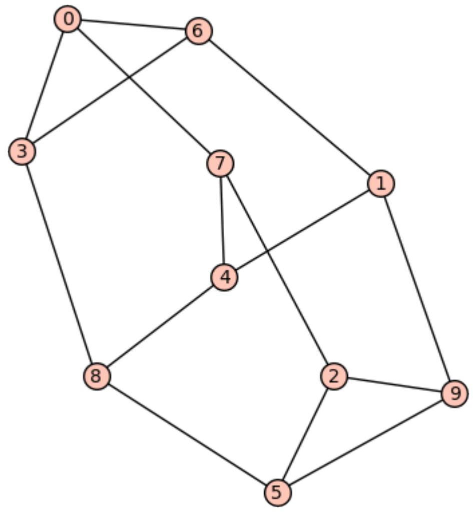



Fig. 5. A non-$L$-borderenergetic 3-regular graph of order $n = 10$ and $m = 15$ edges satisfying Theorem 4.1.

We can use the McClelland type of inequality on the Laplacian energy so that we can obtain an alternative upper bound on the number of edges of an $L$-borderenergetic graph. We note that this result extends Theorem 4 given in [6].

**Theorem 4.2.** *Let $G$ be a graph of order $n$ and $m$ edges with maximum degree $k$ such that*

$$m > \frac{kn^2 + nZ_g(G) + n^2k^2 - 4(n-1)^2}{4kn}.$$

*Then $G$ is not $L$-borderenergetic.*

***Proof.*** By using the increasing function $g(x) = \sqrt{\left(2\left(x + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2x}{n}\right)^2\right)\right)n}$ for $x \in \left[m, \frac{kn}{2}\right]$, we have $g(m) \leq g\left(\frac{kn}{2}\right)$. Thus, by the McClelland type of inequality (2.2) on the Laplacian energy, we get

$$LE(G) = 2(n-1) \leq \sqrt{2\left(m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2\right)n}$$

$$\leq \sqrt{kn^2 + n\sum_{i=1}^{n}(d_i - k)^2}.$$

Then we obtain

$$4(n-1)^2 \leq kn^2 + n\left(\sum_{i=1}^{n} d_i^2 - 2k\sum_{i=1}^{n} d_i + \sum_{i=1}^{n} k^2\right)$$

$$= kn^2 + nZ_g(G) - 4kmn + n^2k^2.$$

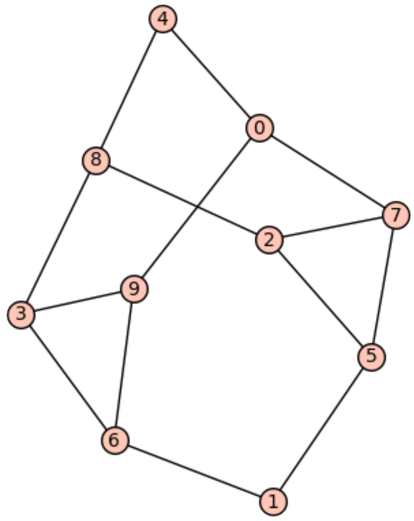



Fig. 6. A non-$L$-borderenergetic graph of order $n = 10$ and $m = 14$ edges with degree sequence $[3, 3, 3, 3, 3, 3, 3, 3, 2, 2]$ satisfying Theorem 4.2.

This gives us

$$4kmn \leq kn^2 + nZ_g(G) + n^2k^2 - 4(n-1)^2.$$

Hence, we see that

$$m \leq \frac{kn^2 + nZ_g(G) + n^2k^2 - 4(n-1)^2}{4kn}.$$

Therefore, if $G$ is $L$-borderenergetic, its number of edges is bounded by above. In other words, its number of edges cannot be greater than the right-hand side.

Theorem 4.2 is proved.

***Example* 4.2.** The graph $G$ of order $n = 10$ and $m = 14$ edges with degree sequence $[3, 3, 3, 3, 3, 3, 3, 3, 2, 2]$ given in Fig. 6 is non-$L$-borderenergetic and satisfies the condition of Theorem 4.2. We have $\operatorname{Spec}(G) = \{0, 5, 5, 2, 2, 1, 1, 4, 4, 4\}$ and its Laplacian energy is $LE(G) = 16 \neq 2n - 2$. Note that the right-hand side of the inequality is approximately $13.96$, which is less than $m$.

In the following theorem we obtain a lower and an upper bound on the number of edges for an $L$-borderenergetic graph by using the bound on Laplacian energy given in [19]. Namely, we get a sufficient interval on the number of edges of a graph in order not to be $L$-borderenergetic.

**Theorem 4.3.** *Let $G$ be a graph of order $n$ and $m$ edges with maximum degree $k$ such that $-(8k-1)n^2 + 4(Z_g(G) + 2k)n > 0$ and*

$$\frac{n}{4} - \frac{1}{4}\sqrt{-(8k-1)n^2 + 4(Z_g(G) + 2k)n} < m < \frac{n}{4} + \frac{1}{4}\sqrt{-(8k-1)n^2 + 4(Z_g(G) + 2k)n}.$$

*Then $G$ is not $L$-borderenergetic.*

***Proof.*** The lower bound given in [19] for the Laplacian energy of $G$

$$LE(G) \geq \frac{2\left(Z_g(G) + 2m - \dfrac{4m^2}{n}\right)}{\mu},$$

where $\mu$ is greater than the maximum Laplacian eigenvalue of $G$ gives us

$$2n - 2 \geq \frac{2\left(Z_g(G) + 2m - \dfrac{4m^2}{n}\right)}{2k}$$

as the largest Laplacian eigenvalue can be at most $2k$ (see [1] for its proof). From the last inequality we have

$$(2n-2)kn \geq nZ_g(G) + 2mn - 4m^2,$$

and then

$$4m^2 - 2mn + (2n-2)kn - nZ_g(G) \geq 0.$$

This equation has no solution when $-(8k-1)n^2 + 4(Z_g(G) + 2k)n > 0$ and $m$ satisfies

$$\frac{n}{4} - \frac{1}{4}\sqrt{-(8k-1)n^2 + 4(Z_g(G)+2k)n} < m < \frac{n}{4} + \frac{1}{4}\sqrt{-(8k-1)n^2 + 4(Z_g(G)+2k)n}.$$

Theorem 4.3 is proved.

In the next example we see that the number of edges of an $L$-borderenergetic graph satisfies Theorem 4.3.

***Example* 4.3.** The graph $G$ of order $n = 10$ and $m = 15$ edges with degree sequence $[4, 4, 4, 4, 3, 3, 2, 2, 2, 2]$ given in Fig. 7 is $L$-borderenergetic and satisfies the bound in Theorem 4.3. Note that the right-hand side of the inequality is approximately $10.94$ for $G$. We have $\mathrm{Spec}(G) = \{0, 1, 2, 2, 2, 2, 5, 5, 5, 6\}$. Note that the Laplacian energy of $G$ is 18, so that it is $L$-borderenergetic.

There exist another upper bounds on the greatest eigenvalue $\mu$ of a graph. In the next theorem we use the result on $\mu$ given in [19] so that we get an upper bound on the number $t$ of vertices of the maximum degree of an $L$-borderenergetic graph.

**Theorem 4.4.** *Let $G$ be a graph of order $n$ and $m$ edges with maximum degree $k$ and minimum degree $k-1$ such that $t$ and $n-t$ vertices of degree $k$ and $k-1$, respectively. If*

$$t > -\frac{(2k^2 - 6k - \sqrt{4k+8m-4n+1} + 3)n^2 - 8m^2 + (2k + 4m + \sqrt{4k+8m-4n+1} - 1)n}{2(2k-1)n},$$

*then $G$ is not $L$-borderenergetic.*

***Proof.*** The lower bound given in [19] for the Laplacian energy of $G$ is

$$LE(G) \geq \frac{2\left(Z_g(G) + 2m - \dfrac{4m^2}{n}\right)}{\mu},$$

where $\mu$ is greater than the maximum Laplacian eigenvalue of $G$. We know by [22] that

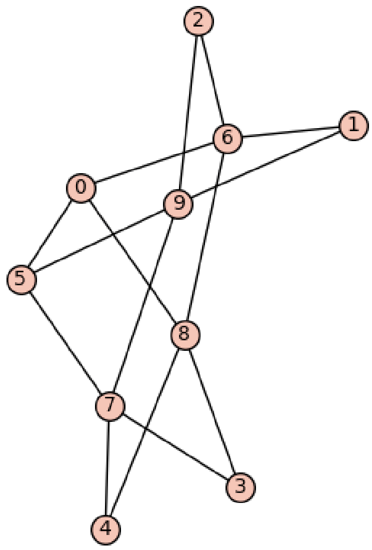



Fig. 7. An $L$-borderenergetic graph of order $n = 10$ and $m = 15$ edges with degree sequence $[4, 4, 4, 4, 3, 3, 2, 2, 2, 2]$, satisfying Theorem 4.3.

$$\mu \leq \sqrt{2m - (n-1) - \delta^2 + \left(\frac{2\Delta - 1}{2}\right)^2} + \left(\frac{2\Delta - 1}{2}\right),$$

where $G$ has $n - t$ vertices of degree $\delta = k - 1$ and $t$ vertices of degree $\Delta = k$. From the last inequality we have

$$LE(G) \geq \frac{2\left(Z_g(G) + 2m - \dfrac{4m^2}{n}\right)}{\sqrt{2m - (n-1) - \delta^2 + \left(\dfrac{2\Delta - 1}{2}\right)^2} + \left(\dfrac{2\Delta - 1}{2}\right)}.$$

Thus,

$$2n - 2 \geq \frac{2\left(tk^2 + (n-t)(k-1)^2 + 2m - \dfrac{4m^2}{n}\right)}{\sqrt{2m - (n-1) - (k-1)^2 + \left(\dfrac{2k - 1}{2}\right)^2} + \left(\dfrac{2k - 1}{2}\right)}.$$

As a result, we have

$$t \leq -\frac{(2k^2 - 6k - \sqrt{4k + 8m - 4n + 1} + 3)n^2 - 8m^2 + (2k + 4m + \sqrt{4k + 8m - 4n + 1} - 1)n}{2(2k-1)n}.$$

Theorem 4.4 is proved.

We show an $L$-borderenergetic graph of order 10 in the next example such that it has $t = 2$ vertices with maximum degree 6 which is less than the bound given in Theorem 4.4.

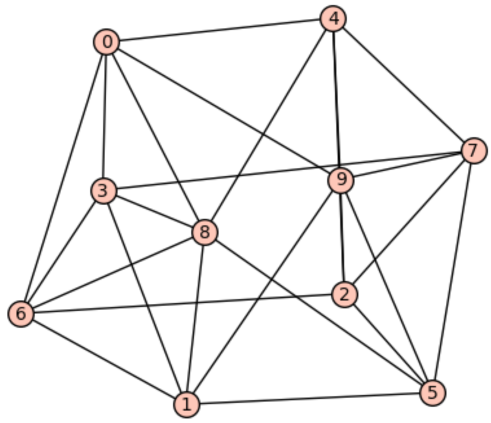



Fig. 8. An $L$-borderenergetic graph of order $n = 10$ and $m = 26$ edges with degree sequence $[6, 6, 5, 5, 5, 5, 5, 5, 5, 5]$ satisfying Theorem 4.4.

***Example* 4.4.** The graph $G$ of order $n = 10$ and $m = 26$ edges with degree sequence $[6, 6, 5, 5, 5, 5, 5, 5, 5, 5]$ given in Fig. 8 is $L$-borderenergetic and satisfies the bound in Theorem 4.4. We have $\mathrm{Spec}(G) = \{7, 4, 3, 0, 8, 8, 6, 6, 5, 5\}$. Note that the right-hand side of the inequality is approximately $7.31$ for $G$, where $t = 2$ and $k = 6$.

**5. Results on the signless Laplacian spectrum of a non-$L$-borderenergetic graph.** It is well-known that the *signless Laplacian matrix* $Q(G)$ of $G$ is defined as $Q(G) = D(G) + A(G)$. Let $0 = \mu_n \leq \mu_{n-1} \leq \ldots \leq \mu_1$ and $0 \leq q_n \leq q_{n-1} \leq \ldots \leq q_1$ be the Laplacian spectrum and signless Laplacian spectrum of $G$, respectively. If $G$ is an $r$-regular graph of order $n$, then its signless Laplacian spectrum is given in [2] as $\{2r = q_1, 2r - \mu_{n-1} = q_2, \ldots, 2r - \mu_2 = q_{n-1}, 2r - \mu_1 = q_n\}$.

It is also well-known that the signless Laplacian energy of $G$ is defined as

$$QE(G) = \sum_{i=1}^{n} |q_i - \bar{d}|,$$

where $\bar{d}$ is the average degree of $G$.

Let $\sigma$, $1 \leq \sigma \leq n$, be the number of signless Laplacian eigenvalues greater than or equal to $\bar{d} = \dfrac{2m}{n}$. We denote $S_\sigma^+$ to be the sum of signless Laplacian eigenvalues greater than or equal to $\bar{d}$ as $S_\sigma^+ = \sum_{i=1}^{\sigma} q_i$.

Finally, we give a condition for a regular graph to be $L$-borderenergetic or not.

**Theorem 5.1.** *Let $G$ be an $r$-regular graph of order $n$ and $m$ edges such that $n = S_\sigma^+ - r\sigma + 1$. Then $G$ is $L$-borderenergetic*; *otherwise $G$ is non-$L$-borderenergetic.*

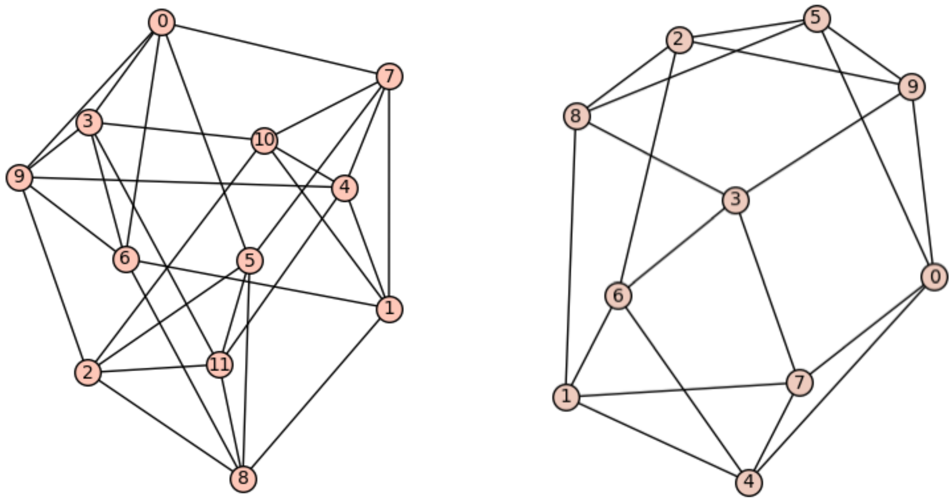
***Proof.*** By [9], we know that

$$QE(G) = \sum_{i=1}^{n} |q_i - r| = 2S_\sigma^+ - \frac{4m\sigma}{n} = 2S_\sigma^+ - 2r\sigma.$$

Since $G$ is regular, we get $LE(G) = QE(G) = 2(S_\sigma^+ - r\sigma)$. So, the result is clear.

We give an example for Theorem 5.1.

***Example* 5.1.** An $L$-borderenergetic 5-regular graph $G_1$ and a non-$L$-borderenergetic 4-regular graph $G_2$ of order 12 and 10 given in Fig. 9 are examples for Theorem 5.1, respectively.




$G_1$ $G_2$

Fig. 9. $G_1$ : An $L$-borderenergetic 5-regular graph of order 12 satisfying Theorem 5.1.
$G_2$ : A non-$L$-borderenergetic 4-regular graph of order 10 satisfying Theorem 5.1.

**6. Conclusion.** In this paper we proved results on the parameters of a graph $G$ so that it is not $L$-borderenergetic. By using these results one can rapidly decide whether a given graph is non-$L$-borderenergetic without computing its Laplacian energy spectrum. We focus on a subset of graphs, namely graphs having vertices with degrees at most three distinct integers. Then the sufficient conditions on the number of vertices and edges of graphs to be not $L$-borderenergetic are given. It could be a good future work to consider new upper bounds on the Laplacian energy of a graph to find new conditions on a graph to be a non-$L$-borderenergetic. Also, it can be searched more useful conditions to determine regular graphs to be $L$-borderenergetic or not.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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