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SCHMIDT RANK AND SINGULARITIES

РАНГ ШМІДТА ТА СИНГУЛЯРНОСТІ

We revisit Schmidt's theorem connecting the Schmidt rank of a tensor with the codimension of a certain variety and adapt the proof to the case of arbitrary characteristic. We also find a sharper result of this kind for homogeneous polynomials, assuming the characteristic does not divide the degree. Further, we use this to relate the Schmidt rank of a homogeneous polynomial (resp., a collection of homogeneous polynomials of the same degree) with the codimension of the singular locus of the corresponding hypersurface (resp., intersection of hypersurfaces). This gives an effective version of Ananyan–Hochster's theorem [J. Amer. Math. Soc., **33**, № 1, 291 – 309 (2020), Theorem A].

Ми знову звертаємося до теореми Шмідта, яка пов'язує ранг Шмідта тензора з корозмірністю певного многовиду, і наводимо доведення, адаптоване до випадку довільної характеристики. Крім того, отримано більш точний результат такого роду для однорідних поліномів за припущення, що характеристика не є дільником степеня. Потім ми використовуємо цей факт, щоб зв'язати ранг Шмідта однорідного полінома (відповідно, колекції однорідних поліномів однакового степеня) з корозмірністю сингулярного локуса відповідної гіперповерхні (відповідно, перетину гіперповерхонь). Це дає ефективну версію теореми Ананяна – Хохстера [J. Amer. Math. Soc., **33**, № 1, 291 – 309 (2020), теорема A].

1. Introduction. Let $\mathbf{k}$ be a field (of any characteristic) and

$$P: V_1 \times V_2 \times \ldots \times V_d \to \mathbf{k}$$

a polylinear form, where V_i are finite dimensional vector spaces over $\mathbf{k}$. Equivalently, we view P as a tensor in $V_1^* \otimes \ldots \otimes V_d^*$.

Definition 1.1. (i) *We say that $P \neq 0$ has Schmidt rank 1 if there exist a partition $[1,d] = I \sqcup J$ into two nonempty parts and polylinear forms $P_I(v_{i_1}, \ldots, v_{i_r})$, $P_J(v_{j_1}, \ldots, v_{j_s})$, where $v_a \in V_a$, $I = \{i_1 < \ldots < i_r\}$, $J = \{j_1 < \ldots < j_r\}$, such that $P = P_I \cdot P_J$. In general the Schmidt rank of P, denoted as $\mathrm{rk}^S(P)$, is the smallest number r such that $P = \sum_{i=1}^r P_i$ with P_i of Schmidt rank 1. For a collection of tensors $\overline{P} = (P_1, \ldots, P_s)$ we define the Schmidt rank $\mathrm{rk}^S(\overline{P})$ as the minimum of Schmidt ranks of nontrivial linear combinations of (P_i).*

(ii) *Given a collection of nonempty subsets $I_1, \ldots, I_r \subset [1,d]$ and a collection $(P_{I_1}, \ldots, P_{I_r})$, where P_{I_i} is a polylinear form on $\prod_{a \in I_i} V_a$, we denote by $(P_{I_1}, \ldots, P_{I_r}) \subset V_1^* \otimes \ldots \otimes V_d^*$ and call this the tensor ideal generated by $P_{I_1}, \ldots, P_{I_r}$, the subspace of polylinear forms of the form*

$$P = \sum_{i=1}^r P_{I_i} \cdot Q_{J_i},$$

for some polylinear forms Q_{J_i} on $\prod_{b \in J_i} V_b$, where $J_i = [1,d] \setminus I_i$.

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The Schmidt rank of a tensor, along with a set of related notions, such as *slice rank*, *G-rank*, *analytic rank*, as well as the version of Schmidt rank for homogeneous polynomials also known as *strength* (see below), has been a subject of study in many recent works (see [1, 4, 5, 8, 9] and references therein). One of the goals of this paper is to establish a precise relation (in the case of an algebraically closed base field $\mathbf{k}$) between this notion and the codimension of the singular locus of the corresponding hypersurface, thus giving an effective version of Ananyan – Hochster's theorem [2, Theorem A].

Let us define the subvariety $Z_P = Z_P^{V_1} \subset V_2 \times \ldots \times V_d$ as the set of $(v_2, \ldots, v_d)$ such that $P(v_1, v_2, \ldots, v_d) = 0$ for all $v_1 \in V_1$. Following Schmidt, let us set

$$g(P) := \operatorname{codim}_{V_2 \times \ldots \times V_d} Z_P.$$

In [10] (where the authors consider the case $d = 3$), this number is called *geometric rank* of P. Using [10, Theorem 3.2], one can see that it does not depend on the ordering of the variables $v_1, \ldots, v_d$.

It is easy to see that one has

$$g(P) \leq \operatorname{rk}^S(P) \tag{1.1}$$

(see Lemma 2.1(i) below or [10, Theorem 1]).

Similarly, for a collection $\overline{P} = (P_1, \ldots, P_s)$, we define $Z_{\overline{P}} \subset V_2 \times \ldots \times V_d$ by the condition on $(v_2, \ldots, v_d)$ that the corresponding map

$$V_1 \to \mathbf{k}^s : v_1 \mapsto (P_i(v_1, v_2, \ldots, v_d))_{1 \leq i \leq s}$$

has $\operatorname{rank} < s$, and we set $g(\overline{P}) := \operatorname{codim}_{V_2 \times \ldots \times V_d} Z_{\overline{P}}$.

The proof of the following theorem follows closely the proof of a similar result in the case where $\mathbf{k} = \mathbb{C}$ and P is symmetric, given in [11]. We modified the proof so that it would work in arbitrary characteristic and also streamlined some parts of the original argument. The fact that the original proof can be adapted to arbitrary characteristic was also pointed out in [11, Section 4].

Theorem 1.1. (i) *Let $g'(P)$ denote the codimension in $V_2 \times \ldots \times V_d$ of the Zariski closure of the set of $\mathbf{k}$-points in Z_P (so $g(P) \leq g'(P)$ and $g(P) = g'(P)$ if $\mathbf{k}$ is algebraically closed). Then one has*

$$\operatorname{rk}^S(P) \leq C_d g'(P),$$

where $C_d = \max(2 + \theta_{d-2}, 2^{d-2} - 1)$, and θ_n is the number of ordered collections of disjoint nonempty subsets $I_1 \sqcup \ldots \sqcup I_p \subsetneq [1, n]$ (with $p \geq 1$). In particular, we have $C_3 = 2$, $C_4 = 4$ and $C_5 = 14$.

(ii) *Assume $\mathbf{k}$ is algebraically closed. Then, for a collection $\overline{P} = (P_1, \ldots, P_s)$, one has*

$$\operatorname{rk}^S(\overline{P}) \leq C_d(g(\overline{P}) + s - 1).$$

In the appendix we prove another version of Theorem 1.1 with better bounds for $d \geq 6$. Even though Schmidt applied the above result to symmetric tensors P corresponding to homogeneous polynomials, we observe that in the symmetric case it is natural to modify the relevant variety Z_P, and that this leads to much better estimates on the rank.

Let f be a homogeneous polynomial of degree d on a finite-dimensional $\mathbf{k}$-vector space V. The *Schmidt rank* of f, denoted as $\mathrm{rk}^S(f)$, is the minimal number r such that $f = \sum_{i=1}^{r} g_i h_i$, where g_i and h_i are homogeneous polynomials of positive degrees. Note that if $\mathrm{rk}^S(f) = r$ then in terminology of [2], f has *strength* $r-1$. For a collection $\overline{f} = (f_1, \ldots, f_s)$, the Schmidt rank $\mathrm{rk}^S(\overline{f})$ is defined as the minimum of Schmidt ranks of nontrivial linear combinations of f_i.

Let $H_f(x)(\cdot,\cdot)$ denote the Hessian form of f given by the second derivatives of f. It is a symmetric bilinear form on V depending polynomially on a point $x \in V$. The symmetric analog of the variety Z_P is the subvariety $Z_f^{\mathrm{sym}} \subset V \times V$ given as

$$Z_f^{\mathrm{sym}} := \big\{(v, x) \in V \times V \mid v \in \ker H_f(x)\big\}.$$

Let us set

$$g_{\mathrm{sym}}(f) := \mathrm{codim}_{V\times V}(Z_f^{\mathrm{sym}}).$$

The symmetric analog of (1.1) is the inequality

$$g_{\mathrm{sym}}(f) \le 4\, \mathrm{rk}^S(f) \tag{1.2}$$

(see Lemma 2.1(ii)).

Similarly, for a collection $\overline{f} = (f_1, \ldots, f_s)$ of homogeneous polynomials of degree d, we define the subvariety $Z_{\overline{f}}^{\mathrm{sym}} \subset V \times V$ as the set of (v, x) such that the map

$$V \to \mathbf{k}^s : v' \mapsto (H_{f_i}(x)(v', v))_{1\le i\le s}$$

has $\mathrm{rank} < s$. We denote by $g_{\mathrm{sym}}(\overline{f})$ the codimension of $Z_{\overline{f}}^{\mathrm{sym}}$ in $V \times V$.

Theorem 1.2. (i) *Assume that $d \ge 3$ and that the characteristic of $\mathbf{k}$ does not divide $(d-1)d$. Let $g'_{\mathrm{sym}}(f)$ denote the codimension in $V \times V$ of the Zariski closure of the set of $\mathbf{k}$-points in Z_f^{sym}. Then one has*

$$\mathrm{rk}^S(f) \le 2^{d-3} g'_{\mathrm{sym}}(f).$$

(ii) *With the same assumptions as in* (i), *assume also that $\mathbf{k}$ is algebraically closed. Then*

$$\mathrm{rk}^S(\overline{f}) \le 2^{d-3}\big(g_{\mathrm{sym}}(\overline{f}) + s - 1\big).$$

For $\mathbf{k}$ algebraically closed, we prove another version of Theorem 1.2 in the appendix with better bounds for $d \ge 6$. The invariant $g_{\mathrm{sym}}(f)$ can be viewed as an invariant measuring singularities of the polar map $x \mapsto (\partial_i f(x))_{1\le i\le \dim V}$ of f (see Subsection 3.3). We also prove that $g_{\mathrm{sym}}(f)$ is related to the codimension of the singular locus of the hypersurface $f = 0$. Namely, let us set

$$c(f) := \mathrm{codim}_V \mathrm{Sing}(f = 0).$$

Assuming that $\mathrm{char}(\mathbf{k})$ does not divide $2(d-1)$, we prove that

$$c(f) \le g_{\mathrm{sym}}(f) \le (d+1)c(f) \quad \text{for} \quad d \ \text{even},$$

$$c(f) \le g_{\mathrm{sym}}(f) \le dc(f) \quad \text{for} \quad d \ \text{odd}$$

(see Proposition 3.1).

More generally, for a collection $\overline{f} = (f_1, \ldots, f_s)$, let us set

$$c(\overline{f}) := \operatorname{codim}_V \operatorname{Sing}\big(V(\overline{f})\big),$$

where $V(\overline{f}) \subset V$ is the subscheme defined by the ideal $(f_1, \ldots, f_s)$. We also consider the related invariant

$$c'(\overline{f}) := \operatorname{codim}_V S(\overline{f}),$$

where $S(\overline{f}) \subset V$ is the locus where the Jacobi matrix of $(f_1, \ldots, f_s)$ has rank $< s$. It is easy to see that

$$c'(\overline{f}) \le c(\overline{f}) \le c'(\overline{f}) + s.$$

Here is our main result concerning the relation between the Schmidt rank and the codimension of the singular locus. It can be viewed as a more precise version of the corresponding result in [9] in the case of an algebraically closed field of sufficiently large (or zero) characteristic, as well as an effective version of a result of Ananyan and Hochster (see [2, Theorem A(a)]), playing a central role in their proof of Stillman's conjecture.

Theorem 1.3. *Assume that* $\operatorname{char}(\mathbf{k})$ *does not divide* d. *Let* $c_{\mathbf{k}}(f)$ *be the codimension in* V *of the Zariski closure of the* $\mathbf{k}$*-points of* $\operatorname{Sing}(f = 0)$.

(i) *We have*

$$\frac{c(f)}{2} \le \operatorname{rk}^S(f) \le (d-1)c_{\mathbf{k}}(f).$$

(ii) *Assume* $\mathbf{k}$ *is algebraically closed. Then, for a collection* $\overline{f} = (f_1, \ldots, f_s)$, *we have*

$$\operatorname{rk}^S(\overline{f}) \le (d-1)\big(c'(\overline{f}) + s - 1\big).$$

Combining Theorem 1.3(i) with [2, Theorem A(c)], we get the following result.

***Corollary* 1.1.** *Assume that* $\mathbf{k}$ *is algebraically closed and* $\operatorname{char}(\mathbf{k})$ *does not divide* $d!$. *For* $i = 2, \ldots, d$, *let* $W_i \subset \mathbf{k}[V]_i$ *be a subspace of forms of degree* i. *Set* $W = \bigoplus_i W_i$, $w = \dim W$. *Assume that, for some* $m \ge 1$, *one has*

$$\operatorname{rk}^S(W_i) \ge (i-1)(m+2) + 3(w-1) \quad \textit{for} \quad i = 3, \ldots, d,$$

$$\operatorname{rk}^S(W_2) - 1 \ge \left\lceil \frac{m+1}{2} \right\rceil + 3(w-1).$$

Then every sequence of linearly independent homogeneous forms in W *is regular and the corresponding complete intersection subscheme in* V *satisfies Serre condition* R_m.

Note that without any assumptions on the characteristic on $\mathbf{k}$ we are able to estimate in terms of $c(f)$ the rank of $H_f(x)(u,v)$ viewed as a polynomial in $(u,v,x) \in V \times V \times V$ (see Remark 3.1).

For a homogeneous polynomial $f(x)$ of degree d on V and a vector $v \in V$, we denote by $\partial_v f(x)$, the derivative of f in the direction v. Our next result concerns $\partial_v f$ for generic v.

Theorem 1.4. *Let* f *be a homogeneous polynomial of degree* $d \ge 3$. *Assume that* $\mathbf{k}$ *is algebraically closed of characteristic not dividing* $(d-1)d$.

(i) *For generic* $v \in V$, *one has* $\mathrm{rk}^S(\partial_v f) \geq 2^{2-d}\,\mathrm{rk}^S(f)$.

(ii) *For* $s \leq 2^{2-d}\,\mathrm{rk}^S(f) + \dfrac{1}{2}$ $\left(\textit{resp.}, \ s \leq 2^{2-d}\,\mathrm{rk}^S(f) - \dfrac{1}{2}\right)$, *and for generic* $v_1, \ldots, v_s \in V$, *the derivatives* $(\partial_{v_1} f, \ldots, \partial_{v_s} f)$ *define a* (*resp.*, *normal*) *complete intersection of codimension* s *in* V.

In the appendix we prove another version of Theorem 1.4 with better bounds for $d \geq 6$. In Subsection 3.4 we will also discuss the relation of the invariant $g_{\mathrm{sym}}(f)$ with the polar map of f and with the Gauss map of the corresponding projective hypersurface.

2. Schmidt rank of polylinear forms. ***2.1. Elementary observations.*** First, let us prove (1.1) and its symmetric version (1.2). We denote by $\mathbf{k}[V]$ the space of polynomial functions on a vector space V and by $\mathbf{k}[V]_d \subset \mathbf{k}[V]$ the subspace of homogeneous polynomials of degree d.

Lemma 2.1. (i) *For* $P \in V_1^* \otimes \ldots \otimes V_d^*$ *one has* $g(P) \leq \mathrm{rk}^S(P)$.

(ii) *For* $f \in \mathbf{k}[V]_d$ *one has* $g_{\mathrm{sym}}(f) \leq 4\,\mathrm{rk}^S(f)$.

Proof. (i) If $r = \mathrm{rk}^S(P)$, then there exists a decomposition

$$P = \sum_{i=1}^{r} P_{I_i} \cdot Q_{J_i}$$

as in Definition 1.1. Swapping some I_i with J_i if necessary, we can assume that $1 \in I_i$ for all i. Then the intersection of r hypersurfaces $Q_{J_i} = 0$ in $V_2 \times \ldots \times V_d$ is contained in Z_P and has codimension $\leq r$.

(ii) If we have a decomposition $f = \sum_{i=1}^{r} g_i h_i$, then over the subvariety $Y = V(g_1, \ldots, g_r, h_1, \ldots, h_r) \subset V$ the symmetric form $H_f(x)$ has rank $\leq 2r$: the subspace cut out by $dg_1|_x, \ldots, dg_r|_x, dh_1|_x, \ldots, dh_r|_x$ is contained in its kernel. Since $\mathrm{codim}_V Y \leq 2r$, the preimage of Y in Z_f^{sym} has codimension $\leq 4r$ in $V \times V$.

Lemma 2.1 is proved.

For a subset of indices $I = \{i_1 < \ldots < i_s\} \subset [1, d]$, let us set

$$V_I := V_{i_1} \otimes \ldots \otimes V_{i_s}.$$

We have the following simple observation.

Lemma 2.2. *Let* $V_1' \subset V_1$ *be a subspace of codimension* c *and* $(\ell_1, \ldots, \ell_r)$ *be a basis of the orthogonal to* V_1' *in* V_1^*. *Suppose that we have tensors*

$$P_{I_s} \in V_1^* \otimes V_{I_s}^*, \quad Q_{J_t} \in V_{J_t}^*$$

for some subsets $I_1, \ldots, I_r, J_1, \ldots, J_p \subset [2, \ldots, d]$ *such that* $P|_{V_1' \times V_2 \times \ldots V_d}$ *belongs to the tensor ideal*

$$\big(P_{I_s}|_{V_1' \otimes V_{I_s}}, \ Q_{J_t} \mid s = 1, \ldots, r; \ t = 1, \ldots, p\big).$$

Then P *belongs to the tensor ideal*

$$\big((\ell_i \mid i = 1, \ldots, c), \ (P_{I_s}, Q_{J_t} \mid s = 1, \ldots, r; \ t = 1, \ldots, p)\big).$$

In particular,

$$\mathrm{rk}^S(P) \leq \mathrm{rk}^S(P|_{V_1' \times V_2 \times \ldots \times V_d}) + c.$$

Proof. This follows immediately from the fact that the tensor ideal $(\ell_i \mid i = 1, \ldots, c)$ is exactly the kernel of the restriction map

$$(V_1 \otimes V_2 \otimes \ldots \otimes V_d)^* \to (V_1' \otimes V_2 \otimes \ldots \otimes V_d)^*.$$

2.2. Determinantal construction. Let $f : V_1 \to V_2$ be a morphism of vector bundles on a scheme X. For every $r \geq 0$, we have a natural morphism

$$\kappa_r : \bigwedge^r V_2^\vee \otimes \bigwedge^{r+1} V_1 \to V_1 : (\phi_1 \wedge \ldots \wedge \phi_r) \otimes \alpha \mapsto \iota_{f^\vee \phi_1} \ldots \iota_{f^\vee \phi_r} \alpha,$$

where for a section ψ of $V^\vee$ we denote by $\iota_\psi : \bigwedge^i V \to \bigwedge^{i-1} V$ the corresponding contraction operator.

Lemma 2.3. (i) *Assume that* $\bigwedge^{r+1} f = 0$. *Then the image of* κ_r *is contained in* $\ker(f)$.

(ii) *Assume, in addition, that* V_1 *and* V_2 *are trivial vector bundles and, for some point* $x \in X$, *the rank of* $f(x) : V_1|_x \to V_2|_x$ *is equal to* r. *Let* $n = \operatorname{rk} V_1$. *Then there exist* $n - r$ *global sections* $s_1, \ldots, s_{n-r}$ *of* V_1 *such that* $f(s_i) = 0$ *for all* i, *and* $s_1(x), \ldots, s_{n-r}(x)$ *is a basis of* $\ker f(x)$.

Proof. (i) This is equivalent to the statement that $\iota_{f^\vee \phi_{r+1}} \kappa_r(\alpha) = 0$ for any local section ϕ_{r+1} of $V_2^\vee$. But $\iota_{f^\vee \phi_1} \ldots \iota_{f^\vee \phi_r} \iota_{f^\vee \phi_{r+1}} = 0$ since $\bigwedge^{r+1} f^\vee = 0$.

(ii) Since V_1 and V_2 are trivial, we can choose splittings $V_1 = \mathcal{K} \oplus \mathcal{W}_1$, $V_2 = \mathcal{C} \oplus \mathcal{W}_2$ into trivial subbundles such that $\mathcal{K}|_x = \ker f(x)$, $\mathcal{W}_2|_x = \operatorname{im} f(x)$, and $f(x) : \mathcal{W}_1|_x \to \mathcal{W}_2|_x$ is an isomorphism. Let us consider the composed map

$$s : \bigwedge^r \mathcal{W}_2^\vee \otimes (\bigwedge^r \mathcal{W}_1 \otimes \mathcal{K}) \to \bigwedge^r V_2^\vee \otimes \bigwedge^{r+1} V_1 \xrightarrow{\kappa_r} V_1.$$

Then $f \circ s = 0$ and the image of $s(x)$ is exactly $\ker f(x)$. Choosing a trivialization of the target of s, we can write s as a collection of global sections of V_1, which has the required properties.

Lemma 2.3 is proved.

2.3. Higher derivatives. Let V be a finite dimensional vector space and $\mathbf{k}[V]$ denote the ring of polynomial functions on V.

For each $f \in \mathbf{k}[V]$, each $n \geq 1$ and $v_0 \in V$, we define the homogeneous form of degree n on V, $f_{v_0}^{(n)}(v)$ as the nth graded component of $f(v + v_0) \in \mathbf{k}[V]$ (viewed as a function of v, for fixed v_0) with respect to the degree grading on $\mathbf{k}[V]$, so that we have (finite) Taylor's decomposition

$$f(v + v_0) = \sum_{n \geq 0} f_{v_0}^{(n)}(v).$$

We refer to $f_{v_0}^{(n)}$ as the nth derivative of f at v_0.

Lemma 2.4. *Let* $X \subset V$ *be an irreducible closed subvariety of codimension* c, $v_0 \in X$ *a smooth* $\mathbf{k}$*-point. Let* $g_1, \ldots, g_c$ *be a set of elements in the ideal* I_X *of* X, *with linearly independent differentials at* v_0. *Then, for any* $f \in I_X$ *and any* $n \geq 1$, *the form* $f_{v_0}^{(n)} \in \mathbf{k}[V]$ *belongs to the ideal in* $\mathbf{k}[V]$ *generated by* $((g_i)_{v_0}^{(j)})_{i=1,\ldots,c;1 \leq j \leq n}$.

Proof. Without loss of generality we can assume that $v_0 = 0$. Set $A = \mathbf{k}[V]$, and let $\hat{A}$ denote the completion with respect to the ideal of the origin (the ring of formal power series). Then the key point is that $I_X \cdot \hat{A}$ is generated by $g_1, \ldots, g_c$. Indeed, this follows from the fact that the local homomorphism of local regular $\mathbf{k}$-algebras $A_\mathfrak{m}/(g_1, \ldots, g_c) \to \mathcal{O}_{X,v_0}$ (where $\mathfrak{m}$ is the maximal ideal of v_0 in A) induces an isomorphism on tangent spaces, so it induces an isomorphism of completions.

Note that higher derivatives make sense for elements of $\hat{A}$ (as components in $A_n = \mathbf{k}[V]_n$), so the assertion follows once we express any element of I_X in the form $\sum_i g_i h_i$ for some $h_i \in \hat{A}$.

Lemma 2.4 is proved.

We also need to work with certain polylinear forms of mixed derivatives. Assume that we have a decomposition $V = V_1 \oplus \ldots \oplus V_n$. Then we obtain the induced direct sum decomposition

$$\mathbf{k}[V]_m = \bigoplus_{m_1+\ldots+m_n=m} \mathbf{k}[V_1]_{m_1} \otimes \ldots \otimes \mathbf{k}[V_n]_{m_n}.$$

Now, for $f \in \mathbf{k}[V]_m$ with $m \leq n$ and a subset of indices $1 \leq i_1 < \ldots < i_m \leq n$, we denote by $f^{(V_{i_1},\ldots,V_{i_m})}$ the component of f in $\mathbf{k}[V_{i_1}]_1 \otimes \ldots \otimes \mathbf{k}[V_{i_m}]_1$. In particular, when we apply this to the mth derivative of f at v_0, we get a polylinear form

$$f_{v_0}^{(V_{i_1},\ldots,V_{i_m})} := (f_{v_0}^{(m)})^{(V_{i_1},\ldots,V_{i_m})} \in V_{i_1}^* \otimes \ldots \otimes V_{i_m}^*, \tag{2.1}$$

which we call the $(V_{i_1},\ldots,V_{i_m})$*-mixed derivative of* f *at* v_0.

Lemma 2.5. *In the situation of Lemma* 2.4, *assume*, *in addition*, *that* $V = V_1 \oplus \ldots \oplus V_n$. *Then*, *for any* $f \in \mathbf{k}[V]$ *and any collection of indices* $I = \{i_1 < \ldots < i_m\} \subset [1,n]$, *the polylinear form* $f_{v_0}^{(V_{i_1},\ldots,V_{i_m})}$ *belongs to the tensor ideal generated by* $(g_i)_{v_0}^{(V_{j_1},\ldots,V_{j_s})}$ *for* $i = 1,\ldots,c$ *and* $J = \{j_1 < \ldots < j_s\} \subset I$, $J \neq \varnothing$.

Proof. This follows easily from Lemma 2.4.

2.4. Dimension count. Let us change the notation to

$$P \colon U \times V \times W_1 \times \ldots \times W_{d-2} \to \mathbf{k}.$$

We denote $W = W_1 \times \ldots \times W_{d-2}$ and consider the variety $Z_P \subset V \times W$ of all (v,w) such that $P(u,v,w) = 0$ for all $u \in U$.

Let Z be an irreducible component of the Zariski closure of the set of $\mathbf{k}$-points $Z_P(\mathbf{k})$ (with reduced scheme structure) such that $\operatorname{codim}_{V\times W} Z = g'(P)$, and let $Z_W \subset W$ denote the closure of the image of Z under the projection $\pi_W : V \times W \to W$ (also with reduced scheme structure). Then $\mathbf{k}$-points are dense in Z_W.

We can think of P as a linear map from $U \otimes V$ to the space of polynomial functions on W, hence, it gives a morphism of trivial vector bundles over W,

$$P_W : V \otimes \mathcal{O}_W \to U^* \otimes \mathcal{O}_W, \tag{2.2}$$

and for $w \in Z_W$, $\pi_W^{-1}(w) \cap Z_P$ can be identified with $\ker(P_W(w))$.

Let $\mathcal{U} \subset Z_W$ denote the nonempty open subset where P_W has maximal rank that we denote by r. Then over $\mathcal{U}$ the cokernel of P_W is locally free over Z_W, hence, the kernel of P_W is a subbundle $\mathcal{K} \subset V \otimes \mathcal{O}$. Denoting by $\operatorname{tot}_{\mathcal{U}}(\mathcal{K})$ the total space of the bundle $\mathcal{K}$ over $\mathcal{U}$, we have

$$\operatorname{tot}_{\mathcal{U}}(\mathcal{K}) = \pi_W^{-1}(\mathcal{U}) \cap Z_P \subset V \times W.$$

Note that $\mathbf{k}$-points are dense in $\operatorname{tot}_{\mathcal{U}}(\mathcal{K}) = \pi_W^{-1}(\mathcal{U}) \cap Z_P$, so $\pi_W^{-1}(\mathcal{U}) \cap Z$ is an irreducible component in $\pi_W^{-1}(\mathcal{U}) \cap Z_P$. Since $\operatorname{tot}_{\mathcal{U}}(\mathcal{K})$ is irreducible, we get

$$\pi_W^{-1}(\mathcal{U}) \cap Z = \operatorname{tot}_{\mathcal{U}}(\mathcal{K}).$$

Hence, we have $\dim Z = \dim Z_W + \dim V - r$ or, equivalently,

$$\operatorname{codim}_W Z_W + r = \operatorname{codim}_{V\times W} Z = g'(P). \tag{2.3}$$

***2.5. Proof of Theorem* 1.1.** *Step* 1. *Choosing a general* $\mathbf{k}$*-point*. Shrinking the open subset $\mathcal{U} \subset Z_W$ above, we can assume that $\mathcal{U}$ is smooth. Since $\mathbf{k}$-points are dense in Z_W we can choose a $\mathbf{k}$-point

$$w^0 = (w_1^0, \ldots, w_{d-2}^0) \in \mathcal{U} \subset Z_W.$$

Let us set

$$S_V := \ker(P_W(w^0) \colon V \to U^*), \qquad S_U := \ker(P_W(w^0)^* \colon U \to V^*).$$

Step 2. *The first set of key tensors*. Set

$$c := \operatorname{codim}_W Z_W.$$

Since w^0 is a smooth point of Z_W, we can choose c elements $g_1, \ldots, g_c$ in the ideal $I_{Z_W} \subset \mathbf{k}[W]$ with linearly independent derivatives at w^0. Now we recall that $W = W_1 \times \ldots \times W_{d-2}$. Thus, for each $a = 1, \ldots, c$ and each nonempty subset of indices $I = \{i_1 < \ldots < i_m\} \subset [1, d-2]$, we can consider the polylinear forms, obtained as mixed derivatives at w^0,

$$g_{a,I} := g_{a,w^0}^{(W_{i_1}, \ldots, W_{i_m})} \in W_{i_1}^* \otimes \ldots \otimes W_{i_m}^*.$$

Step 3. *Setting up the key identity.* Let us set $k = \dim V - r$. Applying Lemma 2.3(ii) to the morphism of trivial vector bundles (2.2) over Z_W, we find global sections $v_1(w), \ldots, v_k(w) \in V \otimes \mathbf{k}[Z_W]$ such that $v_1(w^0), \ldots, v_k(w^0)$ form a basis of S_V, and

$$P(u, v_i(w), w) = 0 \quad \text{for any } \ u \in U \qquad \text{and} \qquad w \in Z_W, \quad i = 1, \ldots, k.$$

Since $\mathbf{k}[W] \to \mathbf{k}[Z_W]$ is surjective we can lift $v_i(w)$ to polynomials in $V \otimes \mathbf{k}[W]$, which we denote in the same way. Now we define a collection of U^*-valued polynomials on W,

$$f_i(w) := P(u, v_i(w), w) \in U^* \otimes \mathbf{k}[W]. \tag{2.4}$$

By construction, all $f_i(w)$ belong to $U^* \otimes I_{Z_W} \subset U^* \otimes \mathbf{k}[W]$. Equation (2.4) will be the key identity that we will use.

Step 4. *The second set of key tensors*. We will consider certain mixed derivatives of $v_i(w)$, viewed as V-valued polynomials on W. Namely, for each $I = \{i_1 < \ldots < i_p\} \subset [1, d-2]$, we set

$$v_{i,I} := v_{i,w^0}^{(W_{i_1}, \ldots, W_{i_p})} \in W_I^* \otimes V = \operatorname{Hom}(W_I, V),$$

where

$$W_I := W_{i_1} \otimes \ldots \otimes W_{i_p}.$$

Since $(v_i(w^0))$ form a basis of S_V, there exists a unique operator

$$C_I \colon S_V \to \operatorname{Hom}(W_I, V) \colon v_i(w^0) \mapsto v_{i,I}.$$

We extend C_I in any way to an operator $V \to \operatorname{Hom}(W_I, V)$, which we still denote by C_I. Note that we can also view C_I as a linear map

$$C_I : V \otimes W_I \to V.$$

For an ordered collection of disjoint subsets $I_1, \ldots, I_p \subset [1, d-2]$, we consider the composition

$$C_{I_1} \ldots C_{I_p} : V \otimes W_{I_1 \sqcup \ldots \sqcup I_p} \xrightarrow{C_{I_p}} V \otimes W_{I_1 \sqcup \ldots I_{p-1}} \to \ldots \to V \otimes W_{I_1} \xrightarrow{C_{I_1}} V.$$

We allow the case of an empty collection, i.e., $p = 0$, in which case we just get the identity map $V \to V$.

Let us choose a basis $\ell_1, \ldots, \ell_r \in V^*$ in the orthogonal subspace to S_V. For ordered collections $I_1 \sqcup \ldots \sqcup I_p \subset [1, d-2]$ and for $j = 1, \ldots, r$, we consider the polylinear forms

$$\ell_j \circ C_{I_1} \ldots C_{I_p} \in V^* \otimes W^*_{I_1 \sqcup \ldots \sqcup I_p}.$$

Note that for an empty collection, i.e., for $p = 0$, we just get $\ell_j \in V^*$.

Step 5. Differentiating the key identity. For each $I = \{i_1 < \ldots < i_p\} \subset [1, d-2]$, let us consider the embedding

$$\iota(I) : W_I \to W_1 \otimes \ldots \otimes W_{d-2},$$

which completes $w_{i_1} \otimes \ldots \otimes w_{i_p}$ by the components w^0_j in the factors W_j with $j \notin I$.

Let us prove by induction on $p = 0, \ldots, d-2$ that for any $I = \{i_1 < \ldots < i_p\} \subset [1, d-2]$, one has

$$P|_{S_U \otimes V \otimes \iota(I)(W_I)} \in \Big((\ell_j \circ C_{I_1} \ldots C_{I_s} \mid I_1 \sqcup \ldots \sqcup I_s \subsetneq I, \ \ 1 \le j \le r, s \ge 0), \\ (g_{a,I'} \mid 1 \le a \le c, \ \ I' \subset I, \ \ I' \ne \varnothing) \Big),$$

where on the right we have the tensor ideal generated by the specified elements. Note that all the subsets I_t are supposed to be nonempty.

The base of induction $p = 0$ is clear, since $P(u, v, w^0_1, \ldots, w^0_{d-2}) = 0$ for any $u \in S_U$ and $v \in V$. Assume that $p > 0$ and the assertion holds for $p - 1$. Let us fix a subset $I_0 = \{i_1 < \ldots < i_p\} \subset [1, d-2]$.

Now let us equate the $(W_{i_1}, \ldots, W_{i_p})$-mixed derivatives at w^0 of both sides of the key identity (2.4). We get the following equality in $U^* \otimes W^*_{I_0}$:

$$(f_i)^{(W_{i_1}, \ldots, W_{i_p})}_{w^0} = P|_{U \otimes v_i(w^0) \otimes \iota(I_0) W_{I_0}} + \sum_{I \sqcup J = I_0, I \ne \varnothing} P|_{U \otimes C_I v_i(w^0) \otimes \iota(J) W_J}. \tag{2.5}$$

Note that by Lemma 2.5, $(f_i)^{(W_1, \ldots, W_p)}_{w^0}$ belong to the tensor ideal generated by $g_{a,I'}$ with $1 \le a \le c$ and $I' \subset I_0$, $I' \ne \varnothing$. Note also that the term in the sum in (2.5) corresponding to $J = \varnothing$ has zero restriction to S_U. Hence, we get

$$P|_{S_U \otimes S_V \otimes \iota(I_0) W_{I_0}} + \sum_{I \sqcup J = I_0; I, J \ne \varnothing} P|_{(\mathrm{id}_U \otimes C_I)(U \otimes S_V) \otimes \iota(J) W_J} \\ \in (g_{a,I'} \mid 1 \le a \le c, I' \subset I, I' \ne \varnothing).$$

Now the induction assumption implies that $P|_{S_U \otimes S_V \otimes \iota(I_0) W_{I_0}}$ belongs to the tensor ideal generated by $g_{a,I'}$ with $I' \subset I_0$, $I' \ne \varnothing$ and by the restrictions of $\ell_j \circ C_{I_1} \ldots C_{I_s}$ with $s \ge 1$ (where $I_1 \sqcup \ldots \sqcup I_s$

is a proper subset of I_0). By Lemma 2.2, adding (ℓ_j) to the generators of the tensor ideal we get the required assertion about $P|_{S_U\otimes V\otimes \iota(I_0)W_{I_0}}$.

Step 6. *Conclusion of the proof for a single tensor*. Now using the result of the previous step for $p = d-2$, we get

$$\operatorname{rk}^S P|_{S_U\otimes V\otimes W_1\otimes\ldots\otimes W_{d-2}} \le r(1+\theta_{d-2}) + c(2^{d-2}-1),$$

where θ_n is the number of ordered collections of disjoint nonempty subsets $I_1 \sqcup \ldots \sqcup I_p \subsetneq [1,n]$ (with $p \ge 1$). By Lemma 2.2, this implies that

$$\operatorname{rk}^S P \le r + r(1+\theta_{d-2}) + c(2^{d-2}-1).$$

Now we recall that $r + c = g'(P)$ (see (2.3)). Hence, we get

$$\operatorname{rk}^S P \le (r+c)\max\left(2+\theta_{d-2}, 2^{d-2}-1\right) = g'(P)C_d$$

as claimed.

Step 7. *The case of several tensors*. Now assume that $\mathbf{k}$ is algebraically closed. Suppose we are given a collection $\overline{P} = (P_1,\ldots,P_s)$ of polylinear forms on $V_1\times\ldots\times V_d$. For a nonzero collection of coefficients $\overline{c} = (c_1,\ldots,c_s)$ in $\mathbf{k}$, we set

$$P_{\overline{c}} = c_1P_1 + \ldots + c_sP_s.$$

The key observation is that

$$Z_{\overline{P}} = \bigcup_{\overline{c}\neq 0} Z_{P_{\overline{c}}},$$

where we can consider $\overline{c}$ as points in the projective space $\mathbb{P}^{s-1}$. As we have already proved, for each $\overline{c}$,

$$\operatorname{codim}_{V_2\times\ldots\times V_d} Z_{P_{\overline{c}}} \ge C_d^{-1}\operatorname{rk}^S(P_{\overline{c}}) \ge C_d^{-1}\operatorname{rk}^S(\overline{P}).$$

After taking the union over $\overline{c}$ in $\mathbb{P}^{s-1}$, we get

$$\operatorname{codim}_{V_2\times\ldots\times V_d} Z_{\overline{P}} \ge C_d^{-1}\operatorname{rk}^S(\overline{P}) - s + 1,$$

as claimed.

3. Symmetric case. ***3.1. More on higher derivatives.*** Let $f \in \mathbf{k}[V]_d$. Thinking of the nth derivative of $f \in \mathbf{k}[V]$ (where $n \le d$) as a degree $d-n$ polynomial map

$$V \to \mathbf{k}[V]_n : v_0 \mapsto f^{(n)}_{v_0}$$

we can write it as a tensor

$$f^{(n,d-n)} \in \mathbf{k}[V]_n \otimes \mathbf{k}[V]_{d-n}.$$

By definition,

$$f(v_1+v_2) = \sum_{n=0}^{d} f^{(n,d-n)}(v_1,v_2),$$

so $f^{(n,d-n)}$ is just the component of $f(v_1+v_2)$ of bidegree $(n, d-n)$ in (v_1,v_2).

Similarly, we define an operation for $n_1 + \ldots + n_p = d$,

$$\mathbf{k}[V]_d \to \mathbf{k}[V]_{n_1} \otimes \ldots \otimes \mathbf{k}[V]_{n_p} : f \mapsto f^{(n_1,\ldots,n_p)},$$

by letting $f^{(n_1,\ldots,n_p)}$ to be the component of multidegree $(n_1,\ldots,n_p)$ in $f(v_1 + \ldots + v_p)$. For example,

$$f^{(1,1,d-2)} \in V^* \otimes V^* \otimes \mathbf{k}[V]_{d-2}$$

is exactly H_f, the Hessian symmetric form on V (depending polynomially on $x \in V$).

We will use two properties of this construction, which are easy to check:

$f^{(n_1,\ldots,n_p)}(x,\ldots,x) = \dfrac{d!}{n_1!\ldots n_p!} f(x)$;

for $m \le n_i$ the mth derivative with respect to x_i of $f^{(n_1,\ldots,n_p)}(x_1,\ldots,x_p)$ at $(x_1^0,\ldots,x_p^0)$ is equal to

$$f^{(n_1,\ldots,n_{i-1},m,n_i-m,\ldots,n_p)}\big(x_1^0,\ldots,x_{i-1}^0,v,x_i^0,\ldots,x_p^0\big).$$

***3.2. Proof of Theorem* 1.2.** It will be convenient to denote one copy of V as X in the product $V \times V = V \times X$. In addition, we view $H_f = f^{(1,1,d-2)}$ as a bilinear form on $U \times V$ where $U = V$, so that Z^{sym} consists of pairs $(v,x) \in V \times X$ such that $f^{(1,1,d-2)}(u,v,x) = 0$ for all $u \in U$.

Step 1. *Dimension count and choosing a general* $\mathbf{k}$*-point*. Let Z be an irreducible component of the Zariski closure of the set of $\mathbf{k}$-points $Z_f^{\mathrm{sym}}(\mathbf{k})$, such that $\operatorname{codim}_{V\times X} Z = g'_{\mathrm{sym}}(f)$, and let $Z_X \subset X$ denote the closure of the image of Z under the projection $p_2 \colon V \times X \to X$. As before, we choose a nonempty smooth open subset $\mathcal{U} \subset Z_X$ over which H_f has maximal $\operatorname{rank} r$, so that $p_2^{-1}(\mathcal{U}) \cap Z$ is a vector bundle of $\operatorname{rank} \dim V - r$ over $\mathcal{U}$. In particular,

$$\operatorname{codim}_X Z_X + r = g'_{\mathrm{sym}}(f).$$

We choose a $\mathbf{k}$-point x^0 in $\mathcal{U} \subset Z_X$ and set

$$S := \ker(H_f(x^0)) \subset V.$$

Step 2. *The first set of key polynomials*. Set

$$c := \operatorname{codim}_X Z_X.$$

Since x^0 is a smooth point of Z_X, we can choose c elements $g_1,\ldots,g_c$ in the ideal $I_{Z_X} \subset \mathbf{k}[X]$ with linearly independent derivatives at x^0. Thus, for each $a = 1,\ldots,c$, and for $1 \le i \le d-2$, we consider the derivatives

$$(g_a)_{x^0}^{(i)} \in \mathbf{k}[X]_i.$$

Step 3. *Setting up key identity.* Let us set $k = \dim V - r$. Applying Lemma 2.3(ii) to the morphism of trivial vector bundles $V \otimes \mathcal{O} \to V^* \otimes \mathcal{O}$ given by $H_f = f^{(1,1,d-2)}$ over Z_X, we find global sections $v_1(x),\ldots,v_k(x) \in V \otimes \mathbf{k}[Z_X]$, such that $v_1(x^0),\ldots,v_k(x^0)$ form a basis of S, and

$$f^{(1,1,d-2)}(u,v_i(x),x) = 0 \quad \text{for any } u \in U \qquad \text{and} \qquad x \in Z_X, \quad i = 1,\ldots,k.$$

We lift $v_i(x)$ to polynomials in $V \otimes \mathbf{k}[X]$, which we denote in the same way. Now we define a collection of U^*-valued polynomials on X,

$$f_i(x) := f^{(1,1,d-2)}(u,v_i(x),x) \in U^* \otimes \mathbf{k}[X]. \tag{3.1}$$

By construction, all $f_i(x)$ belong to $U^* \otimes I_{Z_X} \subset U^* \otimes \mathbf{k}[X]$.

Step 4. *The second set of key forms.* For each $1 \le m \le d-2$, we consider higher derivatives of v_i at x^0, viewed as V-valued polynomials on X.

$$(v_i)^{(m)}_{x^0} \in V \otimes \mathbf{k}[X]_m.$$

Since $(v_i(x^0))$ form a basis of S, there exists a linear operator

$$C_m : S \to V \otimes \mathbf{k}[X]_m : v_i(x^0) \mapsto (v_i)^{(m)}.$$

We extend C_m in any way to an operator $V \to V \otimes \mathbf{k}[X]_m$, which we still denote by C_m. For $m_1 + \ldots + m_p \le d-2$, we consider the composition

$$C_{m_1} \ldots C_{m_p} : V \xrightarrow{C_{m_p}} V \otimes \mathbf{k}[X]_{m_p} \to \ldots \to V \otimes \mathbf{k}[X]_{m_2+\ldots+m_p} \xrightarrow{C_{m_1}} V \otimes \mathbf{k}[X]_{m_1+\ldots+m_p}.$$

We allow the case of an empty collection, i.e., $p = 0$, in which case we just get the identity map $V \to V$.

Finally, we denote by $\ell_1, \ldots, \ell_r \in V^*$ a basis in the orthogonal subspace to S. For $m_1+\ldots+m_p \le d-2$ and for $j = 1, \ldots, r$, we consider the elements

$$\ell_j \circ C_{m_1} \ldots C_{m_p} \in V^* \otimes \mathbf{k}[X]_{m_1+\ldots+m_p}.$$

Note that for an empty collection, i.e., for $p = 0$, we just get $\ell_j \in V^*$.

Step 5. *Differentiating the key identity.* Let us prove by induction on $p = 0, \ldots, d-2$ that one has

$$\begin{aligned} &f^{(1,1,p,d-2-p)}(u, v, x, x^0)|_{S\times V\times X} \\ &\quad \in \Big(\big((\ell_j \circ C_{m_1} \ldots C_{m_s})(v, x) \mid m_1 + \ldots + m_s < p,\ \ 1 \le j \le r \big), \\ &\qquad\qquad \big((g_a)^{(m)}_{x^0}(x) \mid 1 \le a \le c,\ \ 1 \le m \le p \big) \Big), \end{aligned}$$

where on the right we have the ideal generated by the specified elements.

The base of induction $p = 0$ is clear, since $f^{(1,1,d-2)}(u, v, x^0) = 0$ for any $u \in S$ and $v \in V$. Assume that $p > 0$ and the assertion holds for $p-1$. Now let us equate the pth derivatives at $x = x^0$ of both sides of (3.1). We get the following equality in $U^* \otimes \mathbf{k}[X]_p$:

$$(f_i)^{(p)}_{x^0}(x) = f^{(1,1,p,d-2-p)}(u, v_i(x^0), x, x^0) + \sum_{q=1}^{p} f^{(1,1,p-q,d-2-p+q)}(u, C_q(v_i(x^0), x), x, x^0).$$

The left-hand side belongs to the ideal generated by $(g_a)^{(m)}_{x^0}(x)$ with $1 \le a \le c$ and $1 \le m \le p$. Note also that the term corresponding to $q = p$ in the right-hand side has zero restriction to $u \in S$. Hence, we get

$$\begin{aligned} &f^{(1,1,p,d-2-p)}(u, v, x, x^0)|_{S\times S\times X} \\ &\quad + \sum_{q=1}^{p-1} f^{(1,1,p-q,d-2-p+q)}\big(u, C_q(v, x), x, x^0\big)\ \big|_{S\times S\times X} \in \big((g_a)^{(m)}_{x^0}(x)\ \big|\ 1 \le a \le c,\ \ 1 \le m \le p \big). \end{aligned}$$

Now the induction assumption implies that $f^{(1,1,p,d-2-p)}(u,v,x,x^0)|_{S\times S\times X}$ belongs to the ideal generated by $(g_a)^{(m)}_{x^0}(x)$ for $1\le a\le c$, $1\le m\le p$ and by the restrictions to $S\times X$ of $(\ell_j\circ C_{m_1}\ldots C_{m_s})(v,x)$ with $s\ge 1$, $m_1+\ldots+m_s<p$, $1\le j\le r$. By Lemma 2.2, adding (ℓ_j) to the generators of the ideal we get the required assertion about $f^{(1,1,p,d-2-p)}(u,v,x,x^0)|_{S\times V\times X}$.

Step 6. *Conclusion of the proof for a single polynomial.* Now using the result of the previous step for $p=d-2$, we have

$$f^{(1,1,d-2)}(u,v,x)|_{S\times V\times X}$$
$$\in\Big(\big((\ell_j\circ C_{m_1}\ldots C_{m_s})(v,x)\mid m_1+\ldots+m_s<d-2,\ \ 1\le j\le r\big),$$
$$\big((g_a)^{(m)}_{x^0}(x)\mid 1\le a\le c,\ \ 1\le m\le d-2\big)\Big).$$

Hence,

$$f^{(1,1,d-2)}(u,v,x)\in((\ell_j(u)\mid 1\le j\le r),$$
$$\big((\ell_j\circ C_{m_1}\ldots C_{m_s})(v,x)\mid m_1+\ldots+m_s<d-2,\ \ 1\le j\le r\big),$$
$$\big((g_a)^{(m)}_{x^0}(x)\mid 1\le a\le c,\ \ 1\le m\le d-2)\big). \tag{3.2}$$

Now plugging $u=v=x$, we obtain

$$d(d-1)\cdot f(x)\in\Big(\big(F_{j;m_1,\ldots,m_s}(x)\mid m_1+\ldots+m_s<d-2,\ \ 1\le j\le r\big),$$
$$\big((g_a)^{(m)}_{x^0}(x)\mid 1\le a\le c,\ \ 1\le m\le d-2\big)\Big),$$

where $F_{j;m_1,\ldots,m_s}(x)=(\ell_j\circ C_{m_1}\ldots C_{m_s})(x,x)$ has degree $1+m_1+\ldots+m_s<d-1$. It follows that

$$\mathrm{rk}^S(f)\le r\big(1+\theta^{\mathrm{sym}}_{d-2}\big)+c(d-2),$$

where θ^{sym}_n is the number of $(m_1,\ldots,m_s)$, with $s\ge 1$, $m_i\ge 1$, $m_1+\ldots+m_s<n$. It is easy to see that $\theta^{\mathrm{sym}}_n=2^{n-1}-1$. Since $r+c=g'_{\mathrm{sym}}(f)$, we get

$$\mathrm{rk}^S P\le (r+c)\max(2^{d-3},d-2)=g'_{\mathrm{sym}}(f)\cdot 2^{d-3},$$

as claimed.

Step 7. *The case of several polynomials.* Now assume that $\mathbf{k}$ is algebraically closed, and we are given a collection $\overline{f}=(f_1,\ldots,f_s)$ of homogeneous polynomias on V of degree d. For a nonzero collection of coefficients $\overline{c}=(c_1,\ldots,c_s)$ in $\mathbf{k}$, we set $f_{\overline{c}}=c_1f_1+\ldots+c_sf_s$. As in the nonsymmetric case, the key observation is that

$$Z^{\mathrm{sym}}_{\overline{f}}=\bigcup_{\overline{c}\ne 0}Z^{\mathrm{sym}}_{f_{\overline{c}}}, \tag{3.3}$$

where we can consider $\overline{c}$ as points in the projective space $\mathbb{P}^{s-1}$. Using the case of a single polynomial, we deduce that

$$\operatorname{codim}_{V\times V} Z_{\overline{f}}^{\mathrm{sym}} \geq 2^{-d+3}\operatorname{rk}^S(\overline{f}) - s + 1,$$

as claimed.

3.3. Relation to singularities. Now we will relate $g_{\mathrm{sym}}(f)$ to $c(f)$, the codimension in V of the singular locus of the hypersurface $f = 0$.

Proposition 3.1. (i) *The subvariety* $Z_f^{\mathrm{sym}} \subset V \times X = V \times V$ *contains the singular locus of* $f^{(2,d-2)}(v,x) = 0$.

(ii) *One has* $g_{\mathrm{sym}}(f) \leq (d+1)c(f)$ (*resp.,* $g_{\mathrm{sym}}(f) \leq dc(f)$ *if d is odd and* $\mathrm{char}(\mathbf{k}) \neq 2$).

(iii) *If* $\mathrm{char}(\mathbf{k})$ *does not divide* $d-1$, *then* $c(f) \leq g_{\mathrm{sym}}(f)$.

Proof. (i) The first derivative of $f^{(2,d-2)}(v,x)$ along v at (v^0,x^0) is $f^{(1,1,d-2)}(v,v^0,x^0)$, so if (v^0,x^0) is a singular point of $f^{(2,d-2)}(v,x) = 0$, then $f^{(1,1,d-2)}(v,v^0,x^0) = 0$ for all v, i.e., $(v^0,x^0) \in Z_f^{\mathrm{sym}}$.

(ii) Since we are comparing dimensions of algebraic varieties, without loss of generality, we can assume that $\mathbf{k}$ is algebraically closed.

By part (i), we have $g_{\mathrm{sym}}(f) \leq c(F)$, where $F = f^{(2,g-2)}$. It is easy to see that if $F(x) = F_1(x) + \ldots + F_r(x)$, then $c(F) \leq c(F_1) + \ldots + c(F_r)$. Also, if $A\colon V \to W$ is a linear surjective map and $g \in \mathbf{k}[W]$, then $c(g \circ A) = c(g)$.

Thus, it remains to check that $f^{(2,d-2)}(v,x)$ is a linear combination of $d+1$ (resp., d, if d is odd and $\mathrm{char}(\mathbf{k}) \neq 2$) polynomials of the form $f(A_i(v,x))$, for some linear surjective maps $A_i\colon V \times V \to V$.

Let us view $f(v+x)$ as a nonhomogeneous function of v, $g(v) = g_0 + g_1 + \ldots + g_d$ of degree $\leq d$ (with coefficients in $\mathbf{k}[V]$). Now picking any $d+1$ distinct elements $\lambda_0,\ldots,\lambda_d \in \mathbf{k}$, we can express $g_0,\ldots,g_d$ as linear combinations of $g(\lambda_0 v),\ldots,g(\lambda_d v)$ (since the corresponding linear change is given by the Vandermonde matrix).

In the case when d is odd and $\mathrm{char}(\mathbf{k}) \neq 2$, we can similarly express the components of even degree, $(g_{2i})_{i\leq(d-1)/2}$ as linear combinations of $g_0 = g(0)$ and $(g(\lambda_i v) + g(-\lambda_i v))/2$, for $1 \leq i \leq (d-1)/2$, where (λ_i) are nonzero constants such that (λ_i^2) are all distinct.

It remains to observe that $g_2 = f^{(2,d-2)}$ and that each $g(\lambda v) = f(\lambda v + x)$ is of the required type.

(iii) This follows from the relation

$$(d-1)f^{(1,d-1)}(v,x) = f^{(1,1,d-2)}(v,x,x).$$

Indeed, this implies that the intersection of Z_f^{sym} with the diagonal $V \subset V \times V$ is exactly the singular locus of $f = 0$, which gives the claimed inequality.

Proposition 3.1 is proved.

Now let us consider the case of a collection $\overline{f} = (f_1,\ldots,f_s)$ of homogeneous polynomials on V of degree d. We consider the corresponding family of hypersurfaces in V, $f_{\overline{c}} = 0$ parametrized by the projective space $\mathbb{P}^{s-1}$. It is clear that for the locus $S(\overline{f}) \subset V$ where the rank of Jacobi matrix of $(f_1,\ldots,f_s)$ is $< s$, we have

$$S(\overline{f}) = \bigcup_{\overline{c}\neq 0} \mathrm{Sing}(f_{\overline{c}} = 0).$$

Proposition 3.2. (i) *One has the inclusion*

$$\bigcup_{\overline{c}\neq 0} \mathrm{Sing}\left(f_{\overline{c}}^{(2,d-2)} = 0\right) \subset Z_{\overline{f}}^{\mathrm{sym}}.$$

(ii) *One has* $g_{\mathrm{sym}}(\overline{f}) \leq (d+1)c'(\overline{f}) + d(s-1)$ (*resp.,* $g_{\mathrm{sym}}(\overline{f}) \leq dc'(\overline{f}) + (d-1)(s-1)$ *if d is odd and* $\mathrm{char}(\mathbf{k}) \neq 2$).

(iii) *Assume that* $(f_1, \ldots, f_s)$ *define a complete intersection* $V(\overline{f}) \subset V$, *i.e.*, $\operatorname{codim}_V V(\overline{f}) = s$. *Then*

$$c'(\overline{f}) \leq c(\overline{f}) \leq c'(\overline{f}) + s.$$

Assume, *in addition*, *that* $\operatorname{char}(\mathbf{k})$ *does not divide* $d-1$. *Then*

$$c'(\overline{f}) \leq g_{\text{sym}}(\overline{f}).$$

Proof. (i) This follows from Proposition 3.1(i) due to (3.3).

(ii) Since $S(\overline{f})$ has codimension $c'(\overline{f})$ in V, it follows that for some $a \leq s-1$, there exists an a-dimensional subvariety $X \subset \mathbb{P}^{s-1}$ such that

$$c(f_{\overline{c}}) = \operatorname{codim}_V \operatorname{Sing}(f_{\overline{c}}) \leq c'(\overline{f}) + a \quad \text{for} \quad \overline{c} \in X.$$

Applying Proposition 3.1(ii), we see that, for each $\overline{c} \in X$, one has

$$\operatorname{codim}_{V \times V} Z^{\text{sym}}_{f_{\overline{c}}} \leq (d+1)(c'(\overline{f}) + a)$$

(resp., $\leq d(c'(\overline{f}) + a)$ if d is odd). Hence, by using (3.3), we get

$$\operatorname{codim}_{V \times V} Z^{\text{sym}}_{\overline{f}} \leq (d+1)(c'(\overline{f}) + a) - a$$

(resp., $\leq d(c'(\overline{f}) + a) - a$ if d is odd). Since $a \leq s-1$, this implies the assertion.

(iii) If $(f_1, \ldots, f_s)$ define a complete intersection, then, by the Jacobi criterion of smoothness, we obtain

$$\operatorname{Sing} V(\overline{f}) = S(\overline{f}) \cap V(\overline{f}).$$

In particular, we have an inclusion $\operatorname{Sing} V(\overline{f}) \subset S(\overline{f})$, so

$$c'(\overline{f}) = \operatorname{codim}_V S(\overline{f}) \leq c(\overline{f}).$$

Also, we get

$$c(\overline{f}) - s = \operatorname{codim}_{V(\overline{f})} \operatorname{Sing} V(\overline{f}) \leq \operatorname{codim}_V S(\overline{f}) = c'(\overline{f}).$$

If we assume in addition that $\operatorname{char}(\mathbf{k})$ does not divide $d-1$, then the intersection of $Z^{\text{sym}}_{\overline{f}}$ with the diagonal $V \subset V \times V$ is exactly $S(\overline{f})$. Hence, we obtain

$$c'(\overline{f}) = \operatorname{codim}_V S(\overline{f}) \leq g_{\text{sym}}(\overline{f}).$$

Proposition 3.2 is proved.

***Proof of Theorem* 1.3.** (i) If $f(x) = \sum_{i=1}^{r} h_i(x) g_i(x)$ then the locus $h_i(x) = g_i(x) = 0$, for $i = 1, \ldots, r$, is contained in the singular locus of $f(x) = 0$, so $c(f) \leq 2r$.

Now for the other inequality, let $c = c_{\mathbf{k}}(f)$ and X an irreducible component of codimension c of the Zariski closure of the $\mathbf{k}$-points of $\operatorname{Sing}(f = 0)$. Let $v_0 \in X$ be a smooth $\mathbf{k}$-point and $g_1, \ldots, g_c \in I(X)$ defined over $\mathbf{k}$ with linearly independent differentials at v_0. For all $k \in [n], \partial_k f \in I(X)$, so Lemma 2.4 yields

$$\partial_k f = (\partial_k f)^{(d-1)}_{v_0} \in \left((g_i)^{(j)}_{v_0} \right)_{i \in [c], j \in [d-1]}.$$

By Euler's formula,

$$f = \frac{1}{d}\sum_{k=1}^{n} x_k \partial_k f \in \Big((g_i)_{v_0}^{(j)}\Big)_{i\in[c], j\in[d-1]}.$$

This gives $\mathrm{rk}^S(f) \le (d-1)\cdot c$.

(ii) We deduce this from the result for a single form as in the proof of Theorem 1.2.

***Proof of Corollary* 1.1.** In the notation of [2, Theorem A] (recalling that the strength of f is $\mathrm{rk}^S(f)-1$), the inequality of Theorem 1.3(i) implies that one can take

$${}^m A(d) = (d-1)(m+2) - 1.$$

It is also well-known that for $d = 2$, one can take

$${}^m A(2) = \left\lceil \frac{m+1}{2} \right\rceil$$

(see, e.g., [3, Proposition 4.10]). Now the assertion follows from [2, Theorem A(c)].

***Remark* 3.1.** For $\mathbf{k}$ algebraically closed of *arbitrary* characteristic, Eq. (3.2) shows that

$$\mathrm{rk}^S f^{(1,1,d-2)}(u,v,x) \le (2^{d-3}+1)\cdot g_{\mathrm{sym}}(f) \le (2^{d-3}+1)\cdot(d+1)\cdot c(f).$$

The proof of Theorem 1.4 is based on the following geometric observation.

Lemma 3.1. *For generic* $v_1,\ldots,v_s \in V$, *where* $s < \dim V$, *we have*

$$c'(\partial_{v_1} f,\ldots,\partial_{v_s} f) \ge g_{\mathrm{sym}}(f) - s + 1.$$

Proof. Let us denote by $Z^{(s)} \subset V^s \times V$ the locally closed subvariety consisting of $(v_1,\ldots,v_s,x)$, such that $v_1,\ldots,v_s$ are linearly independent and $\dim \mathrm{span}\big(H_f(x)(\cdot,v_1),\ldots,H_f(x)(\cdot,v_s)\big) < s$ (here we consider $H_f(\cdot,v)$ as a linear form on V). We want to estimate the dimension of $Z^{(s)}$. We get a surjective map (with at least 1-dimensional fibers) $\widetilde{Z}^{(s)} \to Z^{(s)}$, where $\widetilde{Z}^{(s)} \subset V \times V^s \times V$ is given by

$$\widetilde{Z}^{(s)} = \Big\{(v,v_1,\ldots,v_s,x) \mid v \in \ker H_f(x),\ \ v \ne 0,\ \ (v_1,\ldots,v_s)$$
$$\text{linearly independent},\ \ v \in \mathrm{span}(v_1,\ldots,v_s)\Big\}.$$

We have a natural projection

$$\widetilde{Z}^{(s)} \to Z_f^{\mathrm{sym}} : (v,v_1,\ldots,v_s,x) \mapsto (v,x),$$

which is a locally trivial fibration whose fibers are irreducible of dimension $n(s-1)+s$, where $n = \dim V$. It follows that

$$\dim Z^{(s)} \le \dim \widetilde{Z}^{(s)} - 1 \le \dim Z_f^{\mathrm{sym}} + n(s-1) + s - 1.$$

Hence,

$$\mathrm{codim}_{V^s\times V} Z^{(s)} \ge g_{\mathrm{sym}}(f) - s + 1.$$

Next, we observe that $H_f(x)(\cdot,v) = f^{(1,1,d-2)}(\cdot,v,x) = (\partial_v f)^{(1,d-2)}(\cdot,x)$, so $S(\partial_{v_1} f,\ldots,\partial_{v_s} f)$ is exactly the fiber over $(v_1,\ldots,v_s)$ of the projection $Z^{(s)} \to V^s$. For generic $v_1,\ldots,v_s$, only the components of $Z^{(s)}$ dominant over V^s will play a role, and we deduce that

$$c'(\partial_{v_1} f, \ldots, \partial_{v_s} f) = \mathrm{codim}_V\, S(\partial_{v_1} f, \ldots, \partial_{v_s} f) \geq \mathrm{codim}_{V^s \times V}\, Z^{(s)} \geq g_{\mathrm{sym}}(f) - s + 1.$$

Lemma 3.1 is proved.

***Proof of Theorem* 1.4.** (i) By Lemma 3.1 with $s = 1$, $c(\partial_v f) \geq g_{\mathrm{sym}}(f)$. Hence, by Theorems 1.3(i) and 1.2,

$$\mathrm{rk}^S(\partial_v f) \geq \frac{1}{2} c(\partial_v f) \geq \frac{1}{2} g_{\mathrm{sym}}(f) \geq 2^{2-d}\, \mathrm{rk}^S(f).$$

(ii) If $c'(\partial_{v_1} f, \ldots, \partial_{v_s} f) \geq s$ (resp., $c'(\partial_{v_1} f, \ldots, \partial_{v_s} f) \geq s + 2$), then $(\partial_{v_1} f, \ldots, \partial_{v_s} f)$ define a (resp., normal) complete intersection of codimension s. Hence, the assertion follows from Theorem 1.2 and Lemma 3.1.

3.4. Singularities of the polar map. Let $f \in \mathbf{k}[V]_d$. Note that $H_f(x)$ can be identified with the tangent map to the *polar map* $\phi_f \colon V \to V^*$ of f sending x to $f_x^{(1)} = df|_x$. Thus, $g_{\mathrm{sym}}(f)$ measures the degeneracy of this map.

More precisely, for any morphism $\phi \colon X \to Y$ between smooth connected varieties, let us define the *Thom – Boardman rank*[2] of ϕ, denoted as $\mathrm{rk}^{TB}(\phi)$, as follows. Consider the subvariety Z_ϕ in the tangent bundle TX of X consisting of (x, v) such that $d\phi_x(v) = 0$. Then we set

$$\mathrm{rk}^{TB}(\phi) = \mathrm{codim}_{TX}\, Z_\phi.$$

Note that $\mathrm{rk}^{TB}(\phi) \leq r$, where r is the generic rank of the differential of ϕ, however, the inequality can be strict.

By definition,

$$g_{\mathrm{sym}}(f) = \mathrm{rk}^{TB}(\phi_f).$$

As is well-known, the generic rank of $d\phi_f = H_f$ is related to the dimension of the projective dual variety X^* of the projective hypersurface associated with f (more precisely, $\dim X^* + 2$ is the generic rank of H_f over the hypersurface $f = 0$). However, it is easy to see that $g^{\mathrm{sym}}(f)$ can be much smaller than the generic rank of ϕ_f. For example, if $q_1(x)$ and $q_2(y)$ are nondegenerate quadratic forms in two different groups of variables $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$, then $\mathrm{rk}^S(q_1(x) q_2(y)) = 1$, so $g_{\mathrm{sym}}(q_1(x) q_2(y)) \leq 4$. On the other hand, the generic rank of $\phi_{q_1(x) q_2(y)}$ is $2n$ (assuming the characteristic of $\mathbf{k}$ is $\neq 2, 3$).

***Example* 3.1.** In the case $d = 3$, the Schmidt rank of f is equal to its slice $\mathrm{rank}\, s(f)$, i.e., the minimal s such that there exists a linear subspace $L \subset V$ of codimension s contained in $(f = 0)$. Thus, for a cubic form f, assuming that $\mathbf{k}$ is algebraically closed of characteristic $\neq 2, 3$, we get from Theorem 1.2 and from (1.2) that

$$s(f) \leq \mathrm{rk}^{TB}(\phi_f) \leq 4 s(f).$$

If f is a general homogeneous polynomial of degree d, then we still have $\mathrm{rk}^S(f) = s(f)$ (see [4]). So, for such f, assuming $\mathbf{k}$ to be algebraically closed of characteristic not dividing $(d-1)d$, we obtain

$$2^{3-d} s(f) \leq \mathrm{rk}^{TB}(\phi_f) \leq 4 s(f).$$

It seems that the invariant $\mathrm{rk}^{TB}(\phi)$ deserves to be studied more. For example, we do not know whether it is always true that $\mathrm{rk}^{TB}(\phi) = \dim X$ for a *finite* morphism ϕ between smooth projective varieties in characteristic zero. Note the following corollary from Proposition 3.1(iii).

[2] The name is due to the relation with Thom – Boardman stratification in singularity theory, see [6].

***Corollary* 3.1.** *Assume that* $\mathrm{char}(\mathbf{k})$ *does not divide* $d-1$. *Then*

$$\mathrm{rk}^{TB}(\phi_f) \geq c(f).$$

In particular, if the projective hypersurface associated with f *is smooth then*

$$\mathrm{rk}^{TB}(\phi_f) = \dim V.$$

Let $V_f \subset \mathbb{P}V$ denote the projective hypersurface associated with f. In [7] the authors consider (for $\mathbf{k} = \mathrm{C}$) the closed locus $S_{\geq r} \subset V_f$ where the co-rank of the Hessian H_f is $\geq r$. They prove that if V_f is smooth then for $r(r+1) \leq \dim V$, the subvariety $S_{\geq r}(V)$ is nonempty and

$$\mathrm{codim}_{V_f} S_{\geq r}(V) \leq r(r+1)/2.$$

By using Corollary 3.1, we get the inequality

$$\mathrm{codim}_{V_f} S_{\geq r}(V) \geq r-1.$$

If V_f is smooth then the projectivization of the restriction of ϕ_f to $(f=0)$ can be identified with the *Gauss map*

$$\gamma \colon V_f \to \mathbb{P}V^*.$$

It is easy to check that if $\mathrm{char}(k)$ does not divide $d(d-1)$, then for any point $x \in (f=0) \subset V$ one has $\ker(d(\phi_f)_x) \subset T_x(f=0)$ and the natural projection

$$\ker(d(\phi_f)_x) \to \ker(d\gamma_x)$$

is an isomorphism. Thus, the above inequalities can be viewed as restrictions on possible degeneracies of the Gauss map of V_f (which is finite by a result of Zak in [12]).

Appendix A. This appendix gives alternative versions of Theorems 1.1, 1.2, and 1.4, with better bounds for $d \geq 6$. The second version of Theorem 1.1 is the following.

Theorem A.1. (i) *Let* $g'(P)$ *denote the codimension in* $V_2 \times \ldots \times V_d$ *of the Zariski closure of* $Z_P(\mathbf{k})$. *Then one has*

$$\mathrm{rk}^S(P) \leq (2^{d-1}-1)g'(P).$$

(ii) *Assume* $\mathbf{k}$ *is algebraically closed. Then for a collection* $\overline{P} = (P_1, \ldots, P_s)$, *one has*

$$\mathrm{rk}^S(\overline{P}) \leq (2^{d-1}-1)(g(\overline{P}) + s - 1).$$

For algebraically closed fields the above result matches the one obtained by Cohen and Moshkowitz [13], but we give a very short proof.

Proof. (i) The proof will mimic that of Theorem 1.3. Write $g = g'(P)$ and let X be an irreducible component of the Zariski closure of $Z_P(\mathbf{k})$ such that $\mathrm{codim}_{V_1 \times V_2 \times \ldots \times V_{d-1}} X = g$. Let $x_1, \ldots, x_n$ be a basis for V_d^*. Write $P = \sum_{k=1}^{n} x_k \cdot Q_k$, where $Q_k \colon V_1 \times V_2 \times \ldots \times V_{d-1} \to \mathbf{k}$ are polylinear forms. Let $v_0 \in X$ be a smooth $\mathbf{k}$-point and $h_1, \ldots, h_g \in I(X)$ defined over $\mathbf{k}$ with linearly independent differentials at v_0. For all $k \in [n]$, $Q_k = (Q_k)_{v_0}^{(V_1, V_2, \ldots, V_{d-1})} \in I(X)$, so by Lemma 2.5 it is in the tensor ideal generated by

$$\left((h_i)_{v_0}^{(V_{j_1}, \ldots, V_{j_s})}\right)_{i \in [g], \varnothing \neq \{j_1 < \ldots < j_s\} \subset [d-1]}.$$

By definition, P is in the tensor ideal generated by the Q_k, so $\mathrm{rk}^S(P) \leq (2^{d-1}-1) \cdot g$.

(ii) We deduce this from the result for a single tensor as in the proof of Theorem 1.1.

The second version of Theorem 1.2 is the following.

Theorem A.2. *Assume that* $\mathbf{k}$ *is algebraically closed of characteristic not dividing* $(d-1)d$.

(i) *For a single form* f *of degree* d,

$$\mathrm{rk}^S(f) \leq (d-1)g_{\mathrm{sym}}(f).$$

(ii) *If* $(f_1, \ldots, f_s)$ *define a complete intersection of codimension* s *in* V, *then*

$$\mathrm{rk}^S(\overline{f}) \leq (d-1)\big(g_{\mathrm{sym}}(\overline{f}) + s - 1\big).$$

Proof. (i) Combine Theorem 1.3(i) and Proposition 3.1(iii).

(ii) Combine Theorem 1.3(ii) and Proposition 3.2(iii).

The second version of Theorem 1.4 is the following.

Theorem A.3. *Let* f *be a homogeneous polynomial of degree* d. *Assume that* $\mathbf{k}$ *is algebraically closed of characteristic not dividing* $(d-1)d$.

(i) *For generic* $v \in V$, *one has* $\mathrm{rk}^S(\partial_v f) \geq \dfrac{1}{2d-2}\,\mathrm{rk}^S(f)$.

(ii) *For* $s \leq \dfrac{1}{2d-2}\,\mathrm{rk}^S(f) + \dfrac{1}{2}$ $\left(\textit{resp.}, s \leq \dfrac{1}{2d-2}\,\mathrm{rk}^S(f) - \dfrac{1}{2}\right)$, *and for generic* $v_1, \ldots, v_s \in V$, *the derivatives* $(\partial_{v_1} f, \ldots, \partial_{v_s} f)$ *define a* (*resp., normal*) *complete intersection of codimension* s *in* V.

Proof. (i) By Lemma 3.1 with $s = 1$, $c(\partial_v f) \geq g_{\mathrm{sym}}(f)$. Hence, by Theorems 1.3(i) and A.2,

$$\mathrm{rk}^S(\partial_v f) \geq \frac{1}{2}c(\partial_v f) \geq \frac{1}{2}g_{\mathrm{sym}}(f) \geq \frac{1}{2d-2}\,\mathrm{rk}^S(f).$$

(ii) If $c'(\partial_{v_1} f, \ldots, \partial_{v_s} f) \geq s$ (resp., $c'(\partial_{v_1} f, \ldots, \partial_{v_s} f) \geq s+2$), then $(\partial_{v_1} f, \ldots, \partial_{v_s} f)$ define a (resp., normal) complete intersection of codimension s. Hence, the assertion follows from Theorem A.2 and Lemma 3.1.

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