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SOME NEW ESTIMATES OF INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

ДЕЯКІ НОВІ ОЦІНКИ ІНТЕГРАЛЬНИХ НЕРІВНОСТЕЙ ТА ЇХ ЗАСТОСУВАННЯ

We obtain several new integral inequalities in terms of fractional integral operators for the functions whose first derivatives satisfy either the conditions of the Lagrange theorem or the Lipschitz condition. In some special cases, the results obtained provide better upper estimates than those known in the literature for Bullen-type inequality and Hadamard-type right-hand side inequality. Finally, some error estimates for the trapezoidal formula are discussed.

Отримано кілька нових інтегральних нерівностей у термінах дробових інтегральних операторів для функцій, перші похідні яких задовольняють умови теореми Лагранжа або умову Ліпшиця. У деяких частинних випадках отримані результати дають кращі верхні оцінки, ніж відомі в літературі для нерівності типу Буллена та правосторонньої нерівності типу Адамара. Насамкінець обговорено деякі оцінки похибки для формули трапеції.

1. Introduction. In applied problems, particularly in optimization problems, the role of convexity theory is well-known. In this theory, particular importance is attached to integral inequalities in particular the Hermite–Hadamard inequality, which allows us to estimate the upper bound for the mean value of a function on a closed interval. Along with the classical calculus, fractional integro-differential calculus plays an important role in the solution of problems related to applied science and technology. A lot of research was devoted to the refinement of the upper bound for the Hadamard inequality in terms of operators of fractional calculus.

Let us remember some definitions that are widely known in the literature.

Definition 1. *The function $\phi: [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is said to be convex if we have*

$$\phi(\lambda\theta_1 + (1-\lambda)\theta_2) \le \lambda\phi(\theta_1) + (1-\lambda)\phi(\theta_2) \quad \forall \theta_1, \theta_2 \in [\vartheta_1, \vartheta_2] \quad and \quad \lambda \in [0,1].$$

Definition 2. *The function $\phi: I = [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is said to be quasiconvex if we have*

$$\phi(\lambda\theta_1 + (1-\lambda)\theta_2) \le \max\{\phi(\theta_1), \phi(\theta_2)\} \quad \forall \theta_1, \theta_2 \in I \quad and \quad \lambda \in [0,1].$$

Definition 3. *Let $\phi \in L[\vartheta_1, \vartheta_2]$. The right- and left-hand sides of the Riemann–Liouville fraction integrals in order $\alpha > 0$ are defined by*

$$J_{\vartheta_1^+}^{\alpha} \phi(\theta) = \frac{1}{\Gamma(\alpha)} \int_{\vartheta_1}^{\theta} (\theta - t)^{\alpha - 1} \phi(t) dt, \quad \theta > \vartheta_1,$$

and

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$$J_{\vartheta_2^-}^{\alpha} \phi(\theta) = \frac{1}{\Gamma(\alpha)} \int\limits_{\theta}^{\vartheta_2} (t - \theta)^{\alpha-1} \phi(t)dt, \quad \theta < \vartheta_2,$$

respectively, where $\Gamma(\alpha)$ *is Euler's gamma function. Here* $J_{\vartheta_1^+}^0 \phi(\theta) = J_{\vartheta_2^-}^0 \phi(\theta) = \phi(\theta)$. *And, for* $\alpha = 1$, *the fractional integral turns into the Riemann integral.*

The classical Hermite – Hadamard inequality which was first published in [17, 18] gives us an estimate of the mean value of a convex function $\phi \colon I = [\vartheta_1, \vartheta_2] \to \mathbb{R}$,

$$\phi\left(\frac{\vartheta_2 + \vartheta_1}{2}\right) \leq \frac{1}{\vartheta_2 - \vartheta_1} \int\limits_{\vartheta_1}^{\vartheta_2} \phi(\theta)d\theta \leq \frac{\phi(\vartheta_1) + \phi(\vartheta_2)}{2}.$$

In [7], Bullen proved the following inequality which is known as Bullen's inequality for convex function ϕ:

$$\frac{1}{\vartheta_2 - \vartheta_1} \int\limits_{\vartheta_1}^{\vartheta_2} \phi(\theta)d\theta \leq \frac{1}{2}\left[\phi\left(\frac{\vartheta_2 + \vartheta_1}{2}\right) + \frac{\phi(\vartheta_1) + \phi(\vartheta_2)}{2}\right]. \tag{1}$$

In the last few decades, many researchers in the field of inequalities have refined, extended, and obtained new inequalities of the Hadamard and Bullen for the convex functions (see, for example, [1, 2, 5, 11, 19, 22, 23, 25, 28, 31, 35 – 38, 40, 42] and the references therein).

Nápoles et al. [27] presented a survey study of various classes of convex functions. A number of articles were devoted to obtaining new Hadamard-type inequalities for the functions whose derivatives belong to different convexity classes (for example, in [3, 8, 26, 30, 32, 41]). Bayraktar in [3] for the s-Godunov – Levin convex and Butt et al. in [8] η-quasiconvex functions obtained Hadamard-type inequalities in terms of fractional integral operators. Butt et al. in [32, 41] also obtained some new generalized variants of Hadamard-type inequalities for differentiable convex mappings in terms of several fractional integral operators. Some recent research on fractional Hermite – Jensen – Mercer inequalities with several applications to special functions can be observed in [9, 10, 13]. In [25], Çakmak presented the Bullen inequalities of using s-convexity via fractional integrals involving hypergeometric function. Nápoles et al. in [26] established new Hermite – Hadamard inequalities for h-convex functions by using a generalized integral. Özdemir et al. [30] and Bayraktar et al. [4] obtained new integral inequalities for (α, s, m)-convex and (s, m_1, m_2)-convex functions, respectively. In [35], Samet and Sarıkaya presented the Bullen inequalities for via fractional integrals for the Lipschitzian functions.

Dragomir and Agarwal in [16], by using the convexity of the first derivatives of the function, obtained the following estimate for the trapezoid inequality.

Theorem 1. *Let* $\vartheta_1, \vartheta_2 \in \mathbb{R}$ *with* $\vartheta_1 < \vartheta_2$ *and* $\phi \colon I = [\vartheta_1, \vartheta_2] \to \mathbb{R}$. *If* $\phi \in C^1[\vartheta_1, \vartheta_2]$ *and* $|\phi'|$ *is a convex on* $[\vartheta_1, \vartheta_2]$, *then the inequality*

$$\left| \frac{\phi(\vartheta_1) + \phi(\vartheta_2)}{2} - \frac{1}{\vartheta_2 - \vartheta_1} \int\limits_{\vartheta_1}^{\vartheta_2} \phi(\theta)d\theta \right| \leq \frac{\vartheta_2 - \vartheta_1}{8}\left(\left|\phi'(\vartheta_1)\right| + \left|\phi'(\vartheta_2)\right|\right) \tag{2}$$

is true.

This inequality was confirmed in some recent articles (see, for example, Remark 3.6 (Theorem 1.5 for $\alpha = 1$) in [11], Theorem 1.1 in [12], Corollary 3.1(4) in [24], Remark 2.10 in [26], Remark 1(ii) in [29], Remarks 1 and 2 in [34], Remark 3.7 in [36], Remark 2.5 in [37], Remark 3 in [38] and Remark 2.2 (for $\alpha = 1$) in [40]).

The main result for quasiconvex functions was formulated by Ion in [21]:

Theorem 2. *Assume that $\vartheta_1, \vartheta_2 \in \mathbb{R}$ with $\vartheta_1 < \vartheta_2$ and $\phi \colon [\vartheta_1, \vartheta_2] \longrightarrow \mathbb{R}$ and $\phi \in C^1[\vartheta_1, \vartheta_2]$. If $|\phi'|$ is quasiconvex on $[\vartheta_1, \vartheta_2]$, then the inequality*

$$\left| \frac{\phi(\vartheta_1) + \phi(\vartheta_2)}{2} - \frac{1}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \phi(\theta) d\theta \right| \le \frac{\vartheta_2 - \vartheta_1}{4} \max\{|\phi'(\vartheta_1)|, |\phi'(\vartheta_2)|\} \tag{3}$$

is true.

This inequality as an estimate of the Hadamard-type trapezoid inequality was given in a number of studies (see, for example, [19, 23, 31, 42]).

Remark 1. In [20, Remark 4.2], Hwang et al. obtained the following estimate for the upper bound for the Bullen inequality:

$$\left| \frac{1}{2} \left[\frac{\phi(\vartheta_2) + \phi(\vartheta_1)}{2} + \phi\left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right] - \frac{1}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \phi(\theta) d\theta \right| \le \frac{\vartheta_2 - \vartheta_1}{16} \left[|\phi'(\vartheta_2)| + |\phi'(\vartheta_1)| \right]. \tag{4}$$

Theorem 3 (Lagrange's theorem on the mean value of a function). *Let $\phi \colon [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be a continuous function on the closed interval $[\vartheta_1, \vartheta_2]$ and differentiable on the open interval $(\vartheta_1, \vartheta_2)$. Then there exists some ξ in $(\vartheta_1, \vartheta_2)$ such that*

$$\phi'(\xi) = \frac{\phi(\vartheta_2) - \phi(\vartheta_1)}{\vartheta_2 - \vartheta_1}.$$

Definition 4 [33]. *A function $\phi \colon [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is said to satisfy a Lipschitz condition on interval $[\vartheta_1, \vartheta_2]$ (M-Lipschitzian) if there exists positive constant M such that, for any two points $\theta_1, \theta_2 \in [\vartheta_1, \vartheta_2], |\phi(\theta_1) - \phi(\theta_2)| \le M|\theta_1 - \theta_2|$.*

Remark 2. It is known that if a function is differentiable, then $M \le \max(|\phi'(\theta)|)$. Also, if a function ϕ has a continuous derivative on the interval $[\vartheta_1, \vartheta_2]$, then it satisfies the Lipschitz condition on this interval with the constant $M = \max(|\phi'(\theta)|)$.

In [39], Sarıkaya et al. established some inequalities including (2) for fractional integrals by proving the following identity:

Lemma 1. *Let $\vartheta_1, \vartheta_2 \in \mathbb{R}$ with $\vartheta_1 < \vartheta_2$, $\phi \colon [\vartheta_1, \vartheta_2] \to \mathbb{R}$, and $\phi \in C^1(\vartheta_1, \vartheta_2)$. If $\phi' \in L[\vartheta_1, \vartheta_2]$, then the following equality holds:*

$$\frac{\phi(\vartheta_1) + \phi(\vartheta_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\vartheta_2 - \vartheta_1)^\alpha} \left[J_{\vartheta_1^+}^\alpha \phi(\vartheta_2) + J_{\vartheta_2^-}^\alpha \phi(\vartheta_1) \right]$$

$$= \frac{\vartheta_2 - \vartheta_1}{2} \int_0^1 [(1 - \theta)^\alpha - \theta^\alpha] \phi'(\theta \vartheta_1 + (1 - \theta)\vartheta_2) d\theta.$$

The motive for this study was the article by Delavar and Dragomir [15], in which the authors obtained an estimate in terms of the Lipschitz constant.

The purpose of this article is to obtain new estimates for a Hadamard-type inequality for functions whose first derivatives satisfy the conditions of the Lagrange theorem or the Lipschitz condition.

2. Main results. The new estimates we obtained in this article are based on the following lemma.

Lemma 2. *Let $\vartheta_1, \vartheta_2 \in \mathbb{R}$ with $\vartheta_1 < \vartheta_2$ and $\phi \colon [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be a differentiable function on $(\vartheta_1, \vartheta_2)$. If $\phi' \in L[\vartheta_1, \vartheta_2]$ and ℓ is nonnegative real number, then the equality*

$$\left[\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell + 1} \right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell + 1} \right) + \phi(\vartheta_2) + \phi(\vartheta_1) \right] - \frac{(\ell + 1)^\alpha \Gamma(\alpha + 1)}{(\vartheta_2 - \vartheta_1)^\alpha}(\mathbf{F_1} + \mathbf{F_2})$$

$$= \frac{\vartheta_2 - \vartheta_1}{\ell + 1} \int\limits_0^1 [(1 - \theta)^\alpha - \theta^\alpha]\left[\phi'\left(\frac{\theta}{\ell + 1}\vartheta_1 + \left(1 - \frac{\theta}{\ell + 1} \right)\vartheta_2 \right) \right.$$

$$\left. - \phi'\left(\frac{\theta}{\ell + 1}\vartheta_2 + \left(1 - \frac{\theta}{\ell + 1} \right)\vartheta_1 \right) \right] d\theta$$

is true, where

$$\mathbf{F_1} = J^\alpha_{\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell + 1} \right)^+}\phi(\vartheta_2) + J^\alpha_{\vartheta_2^-}\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell + 1} \right),$$

$$\mathbf{F_2} = J^\alpha_{\left(\frac{\ell\vartheta_1 + \vartheta_2}{\ell + 1} \right)^-}\phi(\vartheta_1) + J^\alpha_{\vartheta_1^+}\phi\left(\frac{\ell\vartheta_1 + \vartheta_2}{\ell + 1} \right).$$

Proof. By integrating the first integral (I_1) by parts, we obtain

$$I_1 = \int\limits_0^1 [(1 - \theta)^\alpha - \theta^\alpha]\phi'\left(\frac{\theta\vartheta_1}{\ell + 1} + \left(1 - \frac{\theta}{\ell + 1} \right)\vartheta_2 \right) d\theta$$

$$= \frac{\ell + 1}{\vartheta_2 - \vartheta_1}\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell + 1} \right) + \frac{\ell + 1}{\vartheta_2 - \vartheta_1}\phi(\vartheta_2)$$

$$- \frac{\alpha(\ell + 1)}{\vartheta_2 - \vartheta_1} \int\limits_0^1 [(1 - \theta)^{\alpha - 1} + \theta^{\alpha - 1}]\phi\left(\frac{\theta\vartheta_1}{\ell + 1} + \left(1 - \frac{\theta}{\ell + 1} \right)\vartheta_2 \right) d\theta.$$

After replacing the integration variable $\frac{\theta\vartheta_1}{\ell + 1} + \frac{(\ell + 1 - \theta)\vartheta_2}{\ell + 1} = z$, we have

$$\text{for} \quad \theta = 0 \Longrightarrow z = \vartheta_2 \quad \text{and} \quad \theta = 1 \Longrightarrow z = \frac{\vartheta_1 + \ell\vartheta_2}{\ell + 1},$$

$$\theta = \frac{\ell + 1}{\vartheta_2 - \vartheta_1}(\vartheta_2 - z), \quad d\theta = -\frac{\ell + 1}{\vartheta_2 - \vartheta_1}dz,$$

$$1 - \theta = \frac{\ell + 1}{\vartheta_2 - \vartheta_1}\left(z - \frac{\vartheta_1 + \ell\vartheta_2}{\ell + 1} \right).$$

Thus, for the integral on the right-hand side of I_1, we can write

$$\frac{\alpha(\ell+1)}{\vartheta_2 - \vartheta_1} \int_0^1 [(1-\theta)^{\alpha-1} + \theta^{\alpha-1}]\phi\left(\frac{\theta\vartheta_1}{\ell+1} + \frac{(\ell+1-\theta)\vartheta_2}{\ell+1}\right)d\tau$$

$$= \frac{\alpha(\ell+1)^{\alpha+1}}{(\vartheta_2-\vartheta_1)^{\alpha+1}} \int_{\frac{\vartheta_1+\ell\vartheta_2}{\ell+1}}^{\vartheta_2} \left[\left(z - \frac{\vartheta_1+\ell\vartheta_2}{\ell+1}\right)^{\alpha-1} + (\vartheta_2 - z)^{\alpha-1}\right]\phi(z)dz$$

$$= \frac{\alpha(\ell+1)^{\alpha+1}}{(\vartheta_2-\vartheta_1)^{\alpha+1}} \left[\int_{\frac{\vartheta_1+\ell\vartheta_2}{\ell+1}}^{\vartheta_2} \left(z - \frac{\vartheta_1+\ell\vartheta_2}{\ell+1}\right)^{\alpha-1}\phi(z)dz + \int_{\frac{\vartheta_1+\ell\vartheta_2}{\ell+1}}^{\vartheta_2} (\vartheta_2 - z)^{\alpha-1}\phi(z)dz\right].$$

By performing some simple transformations, we get

$$I_1 = \frac{\ell+1}{\vartheta_2-\vartheta_1}\left[\phi\left(\frac{\vartheta_1+\ell\vartheta_2}{\ell+1}\right) + \phi(\vartheta_2)\right]$$

$$- \frac{(\ell+1)^{\alpha+1}\Gamma(\alpha+1)}{(\vartheta_2-\vartheta_1)^{\alpha+1}}\left[J^\alpha_{\left(\frac{\vartheta_1+\ell\vartheta_2}{\ell+1}\right)^+}\phi(\vartheta_2) + J^\alpha_{\vartheta_2^-}\phi\left(\frac{\vartheta_1+\ell\vartheta_2}{\ell+1}\right)\right].$$

Similarly, for the second integral (I_2), we can write

$$I_2 = -\frac{\ell+1}{\vartheta_2-\vartheta_1}\left[\phi\left(\frac{\vartheta_2+\ell\vartheta_1}{\ell+1}\right) + \phi(\vartheta_1)\right]$$

$$+ \frac{(\ell+1)^{\alpha+1}\Gamma(\alpha+1)}{(\vartheta_2-\vartheta_1)^{\alpha+1}}\left[J^\alpha_{\left(\frac{\ell\vartheta_1+\vartheta_2}{\ell+1}\right)^-}\phi(\vartheta_1) + J^\alpha_{\vartheta_1^+}\phi\left(\frac{\ell\vartheta_1+\vartheta_2}{\ell+1}\right)\right].$$

Then, for the difference between integrals, we have

$$\frac{\vartheta_2-\vartheta_1}{\ell+1}(I_1 - I_2) = \phi\left(\frac{\vartheta_1+\ell\vartheta_2}{\ell+1}\right) + \phi\left(\frac{\vartheta_2+\ell\vartheta_1}{\ell+1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1)$$

$$- \frac{(\ell+1)^\alpha\Gamma(\alpha+1)}{(\vartheta_2-\vartheta_1)^\alpha}(\mathbf{F_1} + \mathbf{F_2}).$$

Lemma 2 is proved.

Remark 3. For $\ell = 0$, we obtain an analogue of Lemma 1.

In what follows, we need the following lemma.

Lemma 3. *For the all $\alpha > 0$ and $\ell \geq 0$, the equality*

$$\mathbf{\Psi}(\ell, \alpha) := \int_0^1 |(1-\theta)^\alpha - \theta^\alpha||\ell+1-2\theta|d\theta = \begin{cases} \dfrac{2\alpha}{(\alpha+1)(\alpha+2)} & \text{for} \quad \ell = 0, \\[4mm] \dfrac{2\ell}{\alpha+1}\left(1 - \dfrac{1}{2^\alpha}\right) & \text{for} \quad \ell \geq 1 \end{cases} \tag{5}$$

is true.

Proof. For the $\ell = 0$, we have

$$\int\limits_0^1 |(1-\theta)^\alpha - \theta^\alpha||\ell + 1 - 2\theta|d\theta = \int\limits_0^1 |(1-\theta)^\alpha - \theta^\alpha||1 - 2\theta|d\theta$$

$$= \int\limits_0^{\frac{1}{2}} [(1-\theta)^\alpha - \theta^\alpha](1 - 2\theta)d\theta + \int\limits_{\frac{1}{2}}^1 [\theta^\alpha - (1-\theta)^\alpha](2\theta - 1)d\theta$$

$$= 2\int\limits_0^{\frac{1}{2}} [(1-\theta)^\alpha - \theta^\alpha](1 - 2\theta)d\theta. \tag{6}$$

It is not difficult to verify that

$$\int\limits_0^{\frac{1}{2}} [(1-\theta)^\alpha - \theta^\alpha](1 - 2\theta)d\theta = \int\limits_0^{\frac{1}{2}} (1-\theta)^\alpha d\theta - 2\int\limits_0^{\frac{1}{2}} \theta(1-\theta)^\alpha d\theta - \int\limits_0^{\frac{1}{2}} \theta^\alpha d\theta + 2\int\limits_0^{\frac{1}{2}} \theta^{1+\alpha}d\theta$$

$$= -\frac{1}{2^{\alpha+1}(\alpha+1)} + \frac{1}{\alpha+1} - 2\int\limits_{\frac{1}{2}}^1 \theta^\alpha(1-\theta)d\theta$$

$$- \frac{1}{2^{\alpha+1}(\alpha+1)} + \frac{2}{2^{\alpha+2}(\alpha+2)}$$

$$= -\frac{2}{2^{\alpha+1}(\alpha+1)} + \frac{1}{\alpha+1} + \frac{2}{2^{\alpha+2}(\alpha+2)}$$

$$- 2\left(\frac{1}{\alpha+1} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right) + 2\left(\frac{1}{\alpha+2} - \frac{1}{2^{\alpha+2}(\alpha+2)}\right)$$

$$= \frac{\alpha}{(\alpha+1)(\alpha+2)}. \tag{7}$$

From (6) and (7) the proof of the first part of (5) follows.

For the $\ell \geq 1$, we have

$$\int\limits_0^1 |(1-\theta)^\alpha - \theta^\alpha||\ell + 1 - 2\theta|d\theta = \int\limits_0^1 |(1-\theta)^\alpha - \theta^\alpha|(\ell + 1 - 2\theta)d\theta$$

$$= \int\limits_0^{\frac{1}{2}} [(1-\theta)^\alpha - \theta^\alpha](\ell + 1 - 2\theta)d\theta$$

$$+ \int\limits_{\frac{1}{2}}^1 [\theta^\alpha - (1-\theta)^\alpha](\ell + 1 - 2\theta)d\theta.$$

For the first integral, we get

$$\int\limits_0^{\frac{1}{2}} [(1-\theta)^\alpha - \theta^\alpha](\ell+1-2\theta)d\theta = (\ell+1)\int\limits_0^{\frac{1}{2}} (1-\theta)^\alpha d\theta - 2\int\limits_0^{\frac{1}{2}} \theta(1-\theta)^\alpha d\theta$$

$$- (\ell+1)\int\limits_0^{\frac{1}{2}} \theta^\alpha d\theta + 2\int\limits_0^{\frac{1}{2}} \theta^{\alpha+1} d\theta$$

$$= -\frac{2\ell}{2^{\alpha+1}(\alpha+1)} + \frac{\ell-1}{\alpha+1} + \frac{2}{\alpha+2}. \tag{8}$$

Given that

$$\int\limits_{\frac{1}{2}}^1 [\theta^\alpha - (1-\theta)^\alpha](\ell+1-2\theta)d\theta = \int\limits_0^{\frac{1}{2}} [(1-\theta)^\alpha - \theta^\alpha](\ell-1+2\theta)d\theta,$$

similarly to the first integral, we obtain

$$\int\limits_0^{\frac{1}{2}} [(1-\theta)^\alpha - \theta^\alpha](\ell-1+2\theta)d\theta = -\frac{2\ell}{2^{\alpha+1}(\alpha+1)} + \frac{\ell+1}{\alpha+1} - \frac{2}{\alpha+2}. \tag{9}$$

From (8) and (9) the proof of the second part of (5) follows.

Lemma 3 is proved.

Theorem 4. *Let* $\phi\colon I \to \mathbb{R}$ *and* $\phi \in C^1(I^\circ)$. *For* $0 \le \vartheta_1 < \vartheta_2$, *suppose that* $\vartheta_1, \vartheta_2 \in I^\circ$ *and* $|\phi'|$ *is convex on* $[\vartheta_1, \vartheta_2]$. *Then, for* $\alpha > 0$, *the following inequality holds*:

$$\left| \left[\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell+1}\right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell+1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1) \right] - \frac{(\ell+1)^\alpha \Gamma(\alpha+1)}{(\vartheta_2 - \vartheta_1)^\alpha}[\mathbf{F_1} + \mathbf{F_2}] \right|$$

$$\le \frac{(\vartheta_2 - \vartheta_1)^2}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\frac{2\ell}{(\ell+1)^3}\left[\left|\phi'(\vartheta_2)\right| + \left|\phi'(\vartheta_1)\right|\right], \tag{10}$$

where ℓ, $\mathbf{F_1}$ *and* $\mathbf{F_2}$ *are defined in Lemma* 2.

Proof. From Lemma 2, we have

$$\left| \phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell+1}\right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell+1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1) - \frac{(\ell+1)^\alpha \Gamma(\alpha+1)}{(\vartheta_2 - \vartheta_1)^\alpha}[F_1 + F_2] \right|$$

$$\le \frac{\vartheta_2 - \vartheta_1}{\ell+1}\int\limits_0^1 |(1-\theta)^\alpha - \theta^\alpha|$$

$$\times \left[\left| \phi'\left(\frac{\theta\vartheta_1}{\ell+1} + \left(1 - \frac{\theta}{\ell+1}\right)\vartheta_2\right) \right| + \left| \phi'\left(\frac{\theta\vartheta_2}{\ell+1} + \left(1 - \frac{\theta}{\ell+1}\right)\vartheta_1\right) \right| \right] d\theta$$

$$= \frac{\vartheta_2 - \vartheta_1}{\ell+1}(|I_1| + |I_2|). \tag{11}$$

For the first integral, we obtain

$$|I_1| = \frac{\vartheta_2 - \vartheta_1}{\ell + 1} \int_0^1 |(1-\theta)^\alpha - \theta^\alpha| \left| \phi'\left(\frac{\theta\vartheta_1}{\ell+1} + \left(1 - \frac{\theta}{\ell+1}\right)\vartheta_2 \right) \right| d\theta$$

$$\leq \frac{\vartheta_2 - \vartheta_1}{\ell + 1} \int_0^1 |(1-\theta)^\alpha - \theta^\alpha| \left[\frac{\theta}{\ell+1} |\phi'(\vartheta_1)| + \left(1 - \frac{\theta}{\ell+1}\right) |\phi'(\vartheta_2)| \right] d\theta$$

$$= \frac{\vartheta_2 - \vartheta_1}{(\ell + 1)^2} \left[|\phi'(\vartheta_1)| \int_0^1 |(1-\theta)^\alpha - \theta^\alpha|\theta d\theta \right.$$

$$\left. + |\phi'(\vartheta_2)| \int_0^1 |(1-\theta)^\alpha - \theta^\alpha|(\ell+1-\theta) d\theta \right].$$

It is not difficult to verify that

$$\int_0^1 |(1-\theta)^\alpha - \theta^\alpha|\theta d\theta = \frac{1}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right),$$

$$\int_0^1 |(1-\theta)^\alpha - \theta^\alpha|(\ell+1-\theta) d\theta = \frac{1}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)(2\ell-1).$$

So, we get

$$|I_1| \leq \frac{\vartheta_2 - \vartheta_1}{(\ell + 1)^2} \frac{1}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\left[|\phi'(\vartheta_1)| + |\phi'(\vartheta_2)|(2\ell-1) \right].$$

Analogously for the second integral, we can write

$$|I_2| \leq \frac{\vartheta_2 - \vartheta_1}{(\ell + 1)^2} \frac{1}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\left[|\phi'(\vartheta_2)| + |\phi'(\vartheta_1)|(2\ell-1) \right].$$

Taking into account the last two inequalities from (11), we obtain (10).

Theorem 4 is proved.

Remark 4. If we take $\ell = 1$ and $\alpha = 1$, then, from (10), we obtain (4).

By using Lemma 2, we obtain a trapezoid-type inequality for functions whose first derivative satisfies the Lipschitz condition.

Theorem 5. *Let $\phi\colon I \to \mathbb{R}$ and $\phi \in C^1(I)$. If ϕ' satisfies a Lipschitz condition on $[\vartheta_1, \vartheta_2]$ with respect to M_1, then, for $\alpha > 0$, the following inequality holds*:

$$\left| \left[\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell+1}\right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell+1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1) \right] - \frac{(\ell+1)^\alpha \Gamma(\alpha+1)}{(\vartheta_2 - \vartheta_1)^\alpha}[\mathbf{F_1} + \mathbf{F_2}] \right|$$

$$\leq \frac{M_1(\vartheta_2 - \vartheta_1)^2}{(\ell+1)^2} \boldsymbol{\Psi}(\ell, \alpha), \tag{12}$$

where ℓ is a nonnegative real number and $\mathbf{F_1}$, $\mathbf{F_2}$, $\boldsymbol{\Psi}(\ell, \alpha)$ are defined above.

Proof. From Lemma 2, we have

$$\left| \phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell+1}\right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell+1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1) - \frac{(\ell+1)^\alpha \Gamma(\alpha+1)}{(\vartheta_2 - \vartheta_1)^\alpha}[\mathbf{F_1} + \mathbf{F_1}] \right|$$

$$= \frac{\vartheta_2 - \vartheta_1}{\ell+1} \left| \int\limits_0^1 [(1-\theta)^\alpha - \theta^\alpha] \right.$$

$$\left. \times \left[\phi'\left(\frac{\theta\vartheta_1}{\ell+1} + \left(1 - \frac{\theta}{\ell+1}\right)\vartheta_2\right) - \phi'\left(\frac{\theta\vartheta_2}{\ell+1} + \left(1 - \frac{\theta}{\ell+1}\right)\vartheta_1\right) \right] d\theta \right|$$

$$\leq \frac{\vartheta_2 - \vartheta_1}{\ell+1} \int\limits_0^1 |(1-\theta)^\alpha - \theta^\alpha|$$

$$\times \left| \phi'\left(\frac{\theta\vartheta_1}{\ell+1} + \left(1 - \frac{\theta}{\ell+1}\right)\vartheta_2\right) - \phi'\left(\frac{\theta\vartheta_2}{\ell+1} + (1 - \frac{\theta}{\ell+1})\vartheta_1\right) \right| d\theta$$

$$\leq \frac{M_1(\vartheta_2 - \vartheta_1)^2}{(\ell+1)^2} \int\limits_0^1 |(1-\theta)^\alpha - \theta^\alpha||\ell + 1 - 2\theta|d\theta = \frac{M_1(\vartheta_2 - \vartheta_1)^2}{(\ell+1)^2}\boldsymbol{\Psi}(\ell, \alpha).$$

Theorem 5 is proved.

Corollary 1. *Under the conditions of Theorem 5, the following inequality holds*:

$$\left| \left[\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell+1}\right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell+1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1) \right] \right.$$

$$\left. - \frac{2(\ell+1)}{\vartheta_2 - \vartheta_1}\left(\int\limits_{\frac{\vartheta_1 + \ell\vartheta_2}{\ell+1}}^{\vartheta_2} \phi(\theta)d\theta + \int\limits_{\vartheta_1}^{\frac{\vartheta_1\ell+\vartheta_2}{\ell+1}} \phi(\theta)d\theta \right) \right|$$

$$\leq \frac{M_1(\vartheta_2 - \vartheta_1)^2}{(\ell+1)^2}\boldsymbol{\Psi}(\ell, 1). \tag{13}$$

Proof. For $\alpha = 1$, the proof follows from (12).

Corollary 2. *Under the conditions of Theorem 5, the following inequality holds*:

$$\left| \frac{1}{2}\left[\frac{\phi(\vartheta_2) + \phi(\vartheta_1)}{2} + \phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \right] \right.$$

$$\left. - \frac{2^{\alpha-2}\Gamma(\alpha+1)}{(\vartheta_2 - \vartheta_1)^\alpha}[\mathbf{F_1} + \mathbf{F_2}] \right| \leq \frac{M_1(\vartheta_2 - \vartheta_1)^2}{16}\boldsymbol{\Psi}(1, \alpha),$$

where

$$\mathbf{F_1} = J^{\alpha}_{\left(\frac{\vartheta_1 + \vartheta_2}{2}\right)^+} \phi(\vartheta_2) + J^{\alpha}_{\vartheta_2^-} \phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right),$$

$$\mathbf{F_2} = J^{\alpha}_{\left(\frac{\vartheta_1 + \vartheta_2}{2}\right)^-} \phi(\vartheta_1) + J^{\alpha}_{\vartheta_1^+} \phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right),$$

$$\mathbf{\Psi}(1, \alpha) = \frac{2}{\alpha + 1}\left(1 - \frac{1}{2^{\alpha}}\right).$$

Proof. For $\ell = 1$ the proof follows from (12).

Remark 5. If we take $\ell = 0$, then, from (13), we get the trapezoid inequality

$$\left| \frac{\phi(\vartheta_2) + \phi(\vartheta_1)}{2} - \frac{1}{\vartheta_2 - \vartheta_1} \int\limits_{\vartheta_1}^{\vartheta_2} \phi(\theta)d\theta \right| \leq \frac{M_1(\vartheta_2 - \vartheta_1)^2}{12}. \tag{14}$$

Remark 6. If the inequality $M_1 \leq \dfrac{3(|\phi'(\vartheta_1)| + |\phi'(\vartheta_2)|)}{2(\vartheta_2 - \vartheta_1)}$ is satisfied, then estimate (14) will be better than (2).

Remark 7. If the inequality $M_1 \leq \dfrac{3\max\{|\phi'(\vartheta_1)|, |\phi'(\vartheta_2)|\}}{\vartheta_2 - \vartheta_1}$ is satisfied, then estimate (14) will be better than (3).

Example 1. Let $\phi(x) = x^2, x \in [\vartheta_1, \vartheta_2]$, where $0 \leq \vartheta_1 < \vartheta_2$.

1. For the right-hand side of the inequality (2), we get

$$\frac{\vartheta_2 - \vartheta_1}{8}(|\phi'(\vartheta_1)| + |\phi'(\vartheta_2)|) = \frac{\vartheta_2^2 - \vartheta_1^2}{4},$$

and since $M_1 = \max\{|\phi''(x)|\} = 2$, for the right-hand side of inequality (14), we have

$$\frac{M_1(\vartheta_2 - \vartheta_1)^2}{12} = \frac{(\vartheta_2 - \vartheta_1)^2}{6}.$$

Since the difference between the estimates

$$\frac{\vartheta_2^2 - \vartheta_1^2}{4} - \frac{(\vartheta_2 - \vartheta_1)^2}{6} = \frac{(\vartheta_2 - \vartheta_1)(\vartheta_2 + 5\vartheta_1)}{12} > 0 \quad \text{for all} \quad \vartheta_1, \vartheta_2 \in [0, +\infty),$$

then estimate (14) is better than (2).

2. For the right-hand side of inequality (3), we obtain

$$\frac{\vartheta_2 - \vartheta_1}{4}\max\{|\phi'(\vartheta_1)|, |\phi'(\vartheta_2)|\} = \frac{\vartheta_2(\vartheta_2 - \vartheta_1)}{2},$$

and since

$$\frac{\vartheta_2(\vartheta_2 - \vartheta_1)}{2} - \frac{(\vartheta_2 - \vartheta_1)^2}{6} = \frac{(\vartheta_2 - \vartheta_1)(2\vartheta_2 + \vartheta_1)}{6} > 0 \quad \text{for all} \quad \vartheta_1, \vartheta_2 \in [0, +\infty),$$

the estimate (14) is better than (3).

Remark 8. For the function $\phi(x) = x^2, x \in [\vartheta_1, \vartheta_2]$, where $0 \leq \vartheta_1 < \vartheta_2$, it is easy to see that (14) gives a zero error.

***Example* 2.** Let $\phi(x) = e^x$, $x \in [\vartheta_1, \vartheta_2]$, where $0 \le \vartheta_1 < \vartheta_2$.

1. For the right-hand side of inequality (2), we get

$$\frac{\vartheta_2 - \vartheta_1}{8}(|\phi'(\vartheta_1)| + |\phi'(\vartheta_2)|) = \frac{\vartheta_2 - \vartheta_1}{8}(e^{\vartheta_1} + e^{\vartheta_2}),$$

and, for the right-hand side of inequality (14), since $M_1 = \max\{|\phi''(x)|\} = e^{\vartheta_2}$, we have

$$\frac{M_1(\vartheta_2 - \vartheta_1)^2}{12} = \frac{(\vartheta_2 - \vartheta_1)^2}{12}e^{\vartheta_2}.$$

Then the difference between the estimates will be

$$\frac{\vartheta_2 - \vartheta_1}{8}(|\phi'(\vartheta_1)| + |\phi'(\vartheta_2)|) - \frac{M_1(\vartheta_2 - \vartheta_1)^2}{12}$$

$$= \frac{\vartheta_2 - \vartheta_1}{8}(e^{\vartheta_1} + e^{\vartheta_2}) - \frac{(\vartheta_2 - \vartheta_1)^2}{12}e^{\vartheta_2} = \frac{\vartheta_2 - \vartheta_1}{4}\left(\frac{e^{\vartheta_1} + e^{\vartheta_2}}{2} - \frac{\vartheta_2 - \vartheta_1}{3}e^{\vartheta_2}\right)$$

$$= \frac{(\vartheta_2 - \vartheta_1)}{4}\frac{3e^{\vartheta_1} + [3 - 2(\vartheta_2 - \vartheta_1)]e^{\vartheta_2}}{6}.$$

It is clear that, for example, if $\vartheta_2 - \vartheta_1 \le \dfrac{3}{2}$, then estimate (14) is better than (2).

2. For the right-hand side of inequality (3), we obtain

$$\frac{\vartheta_2 - \vartheta_1}{4}\max\{|\phi'(\vartheta_1)|, |\phi'(\vartheta_2)|\} = \frac{(\vartheta_2 - \vartheta_1)e^{\vartheta_2}}{4}.$$

Then the difference between the estimates will be

$$\frac{(\vartheta_2 - \vartheta_1)e^{\vartheta_2}}{4} - \frac{(\vartheta_2 - \vartheta_1)^2}{12}e^{\vartheta_2} = \frac{(\vartheta_2 - \vartheta_1)e^{\vartheta_2}}{4}\left(1 - \frac{\vartheta_2 - \vartheta_1}{3}\right).$$

Obviously, if $\vartheta_2 - \vartheta_1 \le 3$, then estimate (14) is better than (3).

***Remark* 9.** Of course, it is impossible not to notice that the implementation of the upper bound (14), with rare exceptions, requires limiting the length of the interval and the upper bounds (2) and (3) do not depend on the length of the interval.

***Remark* 10.** For $\ell = 1$, then, from (13), we get an upper bound for inequality (1):

$$\left|\frac{1}{2}\left[\frac{\phi(\vartheta_2) + \phi(\vartheta_1)}{2} + \phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right)\right] - \frac{1}{\vartheta_2 - \vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\phi(\theta)d\theta\right| \le \frac{M_1(\vartheta_2 - \vartheta_1)^2}{32}. \tag{15}$$

Theorem 6. *Let* $\phi\colon I \to \mathbb{R}$. *If* ϕ' *satisfies the conditions of the Lagrange theorem, then, for* $\alpha > 0$, *there exists a point* $\xi \in (\vartheta_1, \vartheta_2)$ *such that the following inequality holds*:

$$\left|\left[\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell + 1}\right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell + 1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1)\right] - \frac{(\ell + 1)^\alpha\Gamma(\alpha + 1)}{(\vartheta_2 - \vartheta_1)^\alpha}[\mathbf{F_1} + \mathbf{F_2}]\right|$$

$$= \frac{(\vartheta_2 - \vartheta_1)^2|\phi''(\xi)|}{(\ell + 1)^2}|\mathbf{\Psi}(\ell, \alpha)| \le \frac{(\vartheta_2 - \vartheta_1)^2\|\phi''\|_\infty}{(\ell + 1)^2}\mathbf{\Psi}(\ell, \alpha), \tag{16}$$

where $\|\phi''\|_\infty = \max_{\theta \in [\vartheta_1, \vartheta_2]} |\phi''(\theta)|$ *and* $\mathbf{F_1}$, $\mathbf{F_2}$, $\mathbf{\Psi}(\ell, \alpha)$ *are defined above*.

Proof. From Lemma 2, we have

$$\left|\left[\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell + 1}\right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell + 1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1)\right] - \frac{(\ell + 1)^\alpha \Gamma(\alpha + 1)}{(\vartheta_2 - \vartheta_1)^\alpha}[\mathbf{F_1} + \mathbf{F_2}]\right|$$

$$= \frac{\vartheta_2 - \vartheta_1}{\ell + 1}\left|\int_0^1 [(1 - \theta)^\alpha - \theta^\alpha]\left[\begin{array}{c}\phi'\left(\dfrac{\theta\vartheta_1}{\ell + 1} + \dfrac{(\ell + 1 - \theta)\vartheta_2}{\ell + 1}\right) \\ -\phi'\left(\dfrac{\theta\vartheta_2}{\ell + 1} + \dfrac{(\ell + 1 - \theta)\vartheta_1}{\ell + 1}\right)\end{array}\right] d\theta\right|,$$

from the Lagrange theorem, we can write

$$= \frac{(\vartheta_2 - \vartheta_1)^2}{(\ell + 1)^2}\left|\int_0^1 [(1 - \theta)^\alpha - \theta^\alpha]\phi''(\xi)(\ell + 1 - 2\theta)d\theta\right|$$

$$= \frac{(\vartheta_2 - \vartheta_1)^2}{(\ell + 1)^2}|\phi''(\xi)|\left|\int_0^1 [(1 - \theta)^\alpha - \theta^\alpha](\ell + 1 - 2\theta)d\theta\right|$$

$$\leq \frac{(\vartheta_2 - \vartheta_1)^2|\phi''|_\infty}{(\ell + 1)^2}|\mathbf{\Psi}(\ell, \alpha)|.$$

Thus, we received (16).

Theorem 6 is proved.

Corollary 3. *Under the conditions of Theorem 6, we obtain*

$$\left|\left[\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell + 1}\right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell + 1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1)\right]\right.$$

$$\left. - \frac{2(\ell + 1)}{\vartheta_2 - \vartheta_1}\left(\int_{\frac{\vartheta_1 + \ell\vartheta_2}{\ell + 1}}^{\vartheta_2} \phi(\theta)d\theta + \int_{\vartheta_1}^{\frac{\vartheta_1\ell + \vartheta_2}{\ell + 1}} \phi(\theta)d\theta\right)\right|$$

$$= \frac{(\vartheta_2 - \vartheta_1)^2|\phi''(\xi)|}{(\ell + 1)^2}\mathbf{\Psi}(\ell, 1) \leq \frac{(\vartheta_2 - \vartheta_1)^2\|\phi''\|_\infty}{(\ell + 1)^2}\mathbf{\Psi}(\ell, 1). \tag{17}$$

Proof. If we choose $\alpha = 1$ in (16), we get (17).

Corollary 4. *Under the conditions of Theorem 6, we have*

$$\left|\frac{1}{2}\left[\phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) + \frac{\phi(\vartheta_2) + \phi(\vartheta_1)}{2}\right] - \frac{2^{\alpha - 2}\Gamma(\alpha + 1)}{(\vartheta_2 - \vartheta_1)^\alpha}[F_1 + F_2]\right|$$

$$\leq \frac{(\vartheta_2 - \vartheta_1)[|\phi'(\vartheta_2)| + |\phi'(\vartheta_1)|]}{16}\mathbf{\Psi}(1, \alpha), \tag{18}$$

where

$$F_1 = J^\alpha_{\left(\frac{\vartheta_1 + \vartheta_2}{2}\right)^+}\phi(\vartheta_2) + J^\alpha_{\vartheta_2^-}\phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right),$$

$$F_2 = J^\alpha_{\left(\frac{\vartheta_1 + \vartheta_2}{2}\right)^-}\phi(\vartheta_1) + J^\alpha_{\vartheta_1^+}\phi\left(\frac{\vartheta_1 + \vartheta_2}{2}\right),$$

$$\boldsymbol{\Psi}(1, \alpha) = \frac{2}{\alpha + 1}\left(1 - \frac{1}{2^\alpha}\right).$$

Proof. If in (16) we choose $\ell = 1$, we get (18).

Remark 11. If we take $\ell = 0$, then from (17), we obtain the trapezoid inequality

$$\left|\frac{\phi(\vartheta_2) + \phi(\vartheta_1)}{2} - \frac{1}{\vartheta_2 - \vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\phi(\theta)d\theta\right| \leq \frac{(\vartheta_2 - \vartheta_1)^2}{12}\|\phi''\|_\infty. \tag{19}$$

It is clear that (14) and (19) are the estimates of the same order.

Remark 12. If we take $\ell = 1$, then, from (17), we obtain an estimate of order (15).

The following identity is obvious:

$$|[\theta\phi(\vartheta_1) + (1 - \theta)\phi(\vartheta_2)] - [(1 - \theta)\phi(\vartheta_1) + \theta\phi(\vartheta_2)]| = |1 - 2\theta||\phi(\vartheta_2) - \phi(\vartheta_1)|.$$

It was established that, for some convex functions ϕ on the interval $[\vartheta_1, \vartheta_2]$ for all $\theta \in [0, 1]$, the following inequality holds:

$$|\phi(\theta\vartheta_1 + (1 - \theta)\vartheta_2) - \phi(\theta\vartheta_2 + (1 - \theta)\vartheta_1)| \leq |1 - 2\theta||\phi(\vartheta_2) - \phi(\vartheta_1)|. \tag{20}$$

For example, it is not difficult to prove that the function $\phi(x) = x^n, x \in [0, \vartheta_2]$ satisfies inequality (20) for all $n \in \mathbb{N}$.

Indeed, for the left-hand side of inequality (20), we have

$$|\phi(t \cdot 0 + (1 - t)\vartheta_2)) - \phi((1 - t) \cdot 0 + t\vartheta_2))| = |[(1 - t)\vartheta_2]^n - (t\vartheta_2)^n| = |(1 - t)^n - t^n|\vartheta_2^n,$$

but for the right-hand side, we get

$$|1 - 2t||\phi(0) - \phi(\vartheta_2))| = |1 - 2t|\vartheta_2^n.$$

It is not difficult to verify that the function

$$\omega_1(t) = |(1 - t)^n - t^n| = \begin{cases} (1 - t)^n - t^n & \text{for} \in \left[0, \frac{1}{2}\right], \\ t^n - (1 - t)^n & \text{for} \in \left[\frac{1}{2}, 1\right] \end{cases}$$

on each of the intervals (that is, on the interval $[0, 1]$) is convex and, therefore the arcs of this function on these intervals are below the intervals of the chords of the function $\omega_2(t) = |1 - 2t|$, that is, $\omega_1(t) \leq \omega_2(t)$ for any n and for all $t \in [0, 1]$.

For many functions, it is impossible to analytically prove whether inequality (20) is satisfied or not, but this fact can be easily established by using an MS Excel spreadsheet. For example, the

functions $f(x) = -\ln(x+1), g(x) = e^x, w(x) = \sin x + \cos x + x^2$ and $h(x) = \sqrt{x^2 + 1}$ are convex on any interval from the domain of these functions.

It is numerically shown that f, g and w have intervals in which inequality (20) is satisfied. But it has also been shown that there are intervals where the w function does not satisfy this inequality. Also, no ranges were found in which the function h satisfies inequality (20).

The following theorem gives an even better estimate:

Theorem 7. *Let* $\phi\colon I \to \mathbb{R}$ *and* $\phi \in C^1(I)$. *For* $0 \leq \vartheta_1 < \vartheta_2$, *suppose that* $\vartheta_1, \vartheta_2 \in I$ *and* ϕ' *on* $[\vartheta_1, \vartheta_2]$ *satisfies* (20). *Then, for* $\alpha > 0$, *the following inequality holds*:

$$\left|\left[\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell+1}\right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell+1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1)\right] - \frac{(\ell+1)^\alpha \Gamma(\alpha+1)}{(\vartheta_2 - \vartheta_1)^\alpha}[\mathbf{F_1} + \mathbf{F_2}]\right|$$

$$\leq \frac{(\vartheta_2 - \vartheta_1)|\phi'(\vartheta_2) - \phi'(\vartheta_1)|}{(\ell+1)^2}\mathbf{\Psi}(\ell, \alpha), \tag{21}$$

where $\mathbf{F_1}$, $\mathbf{F_2}$ *and* $\mathbf{\Psi}(\ell, \alpha)$ *are defined above*.

Proof.

From Lemma 2 and by taking into account inequality (20), we have

$$\left|\left[\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell+1}\right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell+1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1)\right] - \frac{(\ell+1)^\alpha \Gamma(\alpha+1)}{(\vartheta_2 - \vartheta_1)^\alpha}[\mathbf{F_1} + \mathbf{F_2}]\right|$$

$$\leq \frac{\vartheta_2 - \vartheta_1}{\ell+1}\int_0^1 |(1-\theta)^\alpha - \theta^\alpha|\left|\phi'\left(\frac{\theta\vartheta_1}{\ell+1} + \left(1 - \frac{\theta}{\ell+1}\right)\vartheta_2\right)\right.$$

$$\left. - \phi'\left(\frac{\theta\vartheta_2}{\ell+1} + \left(1 - \frac{\theta}{\ell+1}\right)\vartheta_1\right)\right|d\theta$$

$$\leq \frac{\vartheta_2 - \vartheta_1}{(\ell+1)^2}|\phi'(\vartheta_2) - \phi'(\vartheta_1)|\int_0^1 |(1-\theta)^\alpha - \theta^\alpha||\ell+1 - 2\theta|d\theta$$

$$= \frac{\vartheta_2 - \vartheta_1}{(\ell+1)^2}|\phi'(\vartheta_2) - \phi'(\vartheta_1)|\mathbf{\Psi}(\ell, \alpha).$$

Thus, we have (21).

Theorem 7 is proved.

Corollary 5. *Under the conditions of Theorem 7, the following inequality holds*:

$$\left|\left[\phi\left(\frac{\vartheta_1 + \ell\vartheta_2}{\ell+1}\right) + \phi\left(\frac{\vartheta_2 + \ell\vartheta_1}{\ell+1}\right) + \phi(\vartheta_2) + \phi(\vartheta_1)\right]\right.$$

$$\left. - \frac{2(\ell+1)}{\vartheta_2 - \vartheta_1}\left(\int_{\frac{\vartheta_1 + \ell\vartheta_2}{\ell+1}}^{\vartheta_2} \phi(\theta)d\theta + \int_{\vartheta_1}^{\frac{\vartheta_1\ell + \vartheta_2}{\ell+1}} \phi(\theta)d\theta\right)\right|$$

$$\leq \frac{\vartheta_2 - \vartheta_1}{(\ell+1)^2}[|\phi'(\vartheta_1)| + |\phi'(\vartheta_2)|]\mathbf{\Psi}(\ell, 1). \tag{22}$$

Proof. For $\alpha = 1$ and given the fact that $|\phi'(\vartheta_2) - \phi'(\vartheta_1)| \leq |\phi'(\vartheta_1)| + |\phi'(\vartheta_2)|$, the proof follows from (21).

Remark 13. If we take $\ell = 0$, then, from (22), we get the trapezoid inequality

$$\left| \frac{\phi(\vartheta_2) + \phi(\vartheta_1)}{2} - \frac{1}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \phi(\theta) d\theta \right| \leq \frac{\vartheta_2 - \vartheta_1}{12} [|\phi'(\vartheta_1)| + |\phi'(\vartheta_2)|]. \tag{23}$$

For the trapezoid inequality, we got an estimate which is 1.5 times better than (2).

Example 3. Let $\phi(x) = x^n, x \in [0, \vartheta_2], n \in \mathbb{N}$. Then, for the left-hand side of inequality (23), we get

$$\left| \frac{\phi(0) + \phi(\vartheta_2)}{2} - \frac{1}{\vartheta_2} \int_0^{\vartheta_2} \phi(x) dx \right| = \left| \frac{\vartheta_2^n}{2} - \frac{1}{\vartheta_2} \int_0^{\vartheta_2} x^n dx \right| = \left| \frac{\vartheta_2^n}{2} - \frac{\vartheta_2^{n+1}}{\vartheta_2(n+1)} \right|$$

$$= \vartheta_2^n \left(\frac{1}{2} - \frac{1}{n+1} \right) = \frac{(n-1)\vartheta_2^n}{2(n+1)},$$

but, for the right-hand side, we have

$$\frac{\vartheta_2 - 0}{12} (|\phi'(0)| + |\phi'(\vartheta_2)|) = \frac{\vartheta_2}{12} n\vartheta_2^{n-1} = \frac{n\vartheta_2^n}{12}.$$

Then the difference between the right- and the left-hand sides will

$$\frac{n\vartheta_2^n}{12} - \frac{(n-1)\vartheta_2^n}{2(n+1)} = \frac{\vartheta_2^n}{2} \left(\frac{n}{6} - \frac{n-1}{n+1} \right) = \frac{\vartheta_2^n}{2} \frac{n^2 - 5n + 6}{6(n+1)}$$

$$= \frac{\vartheta_2^n}{2} \frac{(n-2)(n-3)}{6(n+1)} \geq 0 \quad \forall n \in \mathbb{N}.$$

This means that $\frac{n\vartheta_2^n}{12}$ is the upper bound of the trapezoid inequality. By formula (2), the estimate is equal to $\frac{n\vartheta_2^n}{8}$. As one can see, the new estimate is one and a half times smaller.

Corollary 6. *From* (23), *we have*

$$\left| \frac{\phi(\vartheta_2) + \phi(\vartheta_1)}{2} - \frac{1}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \phi(\theta) d\theta \right| \leq \frac{\vartheta_2 - \vartheta_1}{6} \max\{|\phi'(\vartheta_1)|, |\phi'(\vartheta_2)|\} \tag{24}$$

and

$$\left| \frac{\phi(\vartheta_2) + \phi(\vartheta_1)}{2} - \frac{1}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \phi(\theta) d\theta \right| \leq \frac{M_2(\vartheta_2 - \vartheta_1)}{6}, \tag{25}$$

where $M_2 = \sup_{\theta \in [\vartheta_1, \vartheta_2]} |\phi'(\theta)|$.

Proof. It is easy to see that inequality (23) implies (24). Since the function is continuous on the interval $[\vartheta_1, \vartheta_2]$, it reaches its largest and smallest value. Then, from (23), follows inequality (25).

***Remark* 14.** Obviously, the estimate (24) is better than estimate (3).

***Remark* 15.** If we take $\ell = 1$, then, from (22), we have an upper bound for inequality (1):

$$\left| \frac{1}{2}\left[\frac{\phi(\vartheta_2) + \phi(\vartheta_1)}{2} + \phi\left(\frac{\vartheta_1 + \vartheta_2}{2} \right) \right] - \frac{1}{\vartheta_2 - \vartheta_1} \int\limits_{\vartheta_1}^{\vartheta_2} \phi(\theta)d\theta \right| \leq \frac{\vartheta_2 - \vartheta_1}{32} \big[|\phi'(\vartheta_2)| + |\phi'(\vartheta_1)| \big].$$

Obviously, this estimate for the upper bound is much better than estimate (4).

***Remark* 16.** If we choose $\alpha = 1$ and $\ell = 0$ then, from (21), we get the trapezoid inequality

$$\left| \frac{\phi(\vartheta_2) + \phi(\vartheta_1)}{2} - \frac{1}{\vartheta_2 - \vartheta_1} \int\limits_{\vartheta_1}^{\vartheta_2} \phi(\theta)d\theta \right| \leq \frac{\vartheta_2 - \vartheta_1}{12} \big| \phi'(\vartheta_2) - \phi'(\vartheta_1) \big|. \tag{26}$$

Obviously, the estimate (26) is better than all the previously obtained estimates.

It can be easily proved analytically that the considered 2nd and 3rd degree functions satisfy inequality (20) on any interval with the positive semiaxis.

***Example* 4.** Let $\phi(x) = x^n, x \in [\vartheta_1, \vartheta_2]$ with $0 < \vartheta_1 < \vartheta_2$. Then, from right-hand side of inequality (26), we have the estimate

$$E_n = \frac{\vartheta_2 - \vartheta_1}{12}\big(|\phi'(\vartheta_2)| - |\phi'(\vartheta_1)|\big) = \frac{n(\vartheta_2 - \vartheta_1)}{12}\big(\vartheta_2^{n-1} - \vartheta_1^{n-1}\big),$$

and, for the left-hand side, we obtain

$$T_n = \left| \frac{\phi(\vartheta_1) + \phi(\vartheta_2)}{2} - \frac{1}{\vartheta_2 - \vartheta_1} \int\limits_{\vartheta_1}^{\vartheta_2} \phi(x)dx \right| = \left| \frac{\vartheta_1^n + \vartheta_2^n}{2} - \frac{\vartheta_2^{n+1} - \vartheta_1^{n+1}}{(\vartheta_2 - \vartheta_1)(n+1)} \right|.$$

1. For $n = 2$, we get

$$E_2 = \frac{(\vartheta_2 - \vartheta_1)^2}{6}, \quad T_2 = \frac{\vartheta_1^2 + \vartheta_2^2}{2} - \frac{\vartheta_2^3 - \vartheta_1^3}{3(\vartheta_2 - \vartheta_1)} = \frac{(\vartheta_2 - \vartheta_1)^2}{6},$$

i.e., we got a zero error $(P_{21} = E_2 - T_2 = 0)$. Inequality (23) gives us the error

$$P_{22} = \left| \frac{\vartheta_2^2 - \vartheta_1^2}{6} - T_2 \right| = \left| \frac{\vartheta_2^2 - \vartheta_1^2}{6} - \frac{(\vartheta_2 - \vartheta_1)^2}{6} \right| = \frac{\vartheta_1(\vartheta_2 - \vartheta_1)}{3},$$

and, from inequality (2), we have

$$P_{23} = \left| \frac{\vartheta_2^2 - \vartheta_1^2}{4} - T_2 \right| = \left| \frac{\vartheta_2^2 - \vartheta_1^2}{4} - \frac{(\vartheta_2 - \vartheta_1)^2}{6} \right| = \frac{(\vartheta_2 - \vartheta_1)(\vartheta_2 + 5\vartheta_1)}{12}.$$

It is easy to check that $P_{21} < P_{22} < P_{23}$ for all $\vartheta_1, \vartheta_2 \in \mathbb{R}^+$ with $\vartheta_1 < \vartheta_2$.

2. For $n = 3$, from (26), we get

$$E_3 = \frac{(\vartheta_2 - \vartheta_1)\big(\vartheta_2^2 - \vartheta_1^2\big)}{4},$$

$$|T_3| = \left| \frac{\vartheta_1^3 + \vartheta_2^3}{2} - \frac{\vartheta_2^4 - \vartheta_1^4}{4(\vartheta_2 - \vartheta_1)} \right| = \frac{(\vartheta_2 - \vartheta_1)(\vartheta_2^2 - \vartheta_1^2)}{4},$$

i.e., we got a zero error $(P_{31} = 0)$. Inequality (23) gives us the error

$$P_{32} = \left| \frac{(\vartheta_2^2 + \vartheta_1^2)(\vartheta_2 - \vartheta_1)}{4} - T_3 \right| = \frac{\vartheta_1^2(\vartheta_2 - \vartheta_1)}{2},$$

and, from inequality (2), we obtain

$$P_{33} = \left| \frac{3(\vartheta_2 - \vartheta_1)(\vartheta_2^2 + \vartheta_1^2)}{8} - T_3 \right| = \frac{(\vartheta_2 - \vartheta_1)(\vartheta_2^2 + 5\vartheta_1^2)}{8}.$$

It is easy to check that $P_{31} < P_{32} < P_{33}$ for all $\vartheta_1, \vartheta_2 \in \mathbb{R}^+$ with $\vartheta_1 < \vartheta_2$.

3. Applications of results. Let $m \in \mathbb{N}, \vartheta_1, \vartheta_2 \in \mathbb{R}, \vartheta_1 < \vartheta_2, m \geq 1$ and P be a division of the interval $[\vartheta_1, \vartheta_2]$ into m subintervals, that is, $P : \vartheta_1 = x_0 < x_1 < x_2 < \ldots < x_{m-1} < x_m = \vartheta_2$. Then, for the trapezoid formula, we can write

$$\int_{\vartheta_1}^{\vartheta_2} \phi(x)dx = T(\phi, P) + E(\phi, P) = \sum_{i=0}^{m-1} \frac{\phi'(x_{i+1}) + \phi'(x_i)}{2}(x_{i+1} - x_i) + E(\phi, P),$$

where $E(\phi, P)$ is the error estimation in numerical integration.

Proposition 1. *Under the conditions of Theorem* 5, *the following inequality is true*:

$$E(\phi, P) \leq \frac{M_1}{6} \sum_{i=0}^{m-1} (x_{i+1} - x_i)^3,$$

where M_1 is the Lipschitz constant of the function ϕ' on the interval $[a, b]$.

Proof. Let M_{1i} is the Lipschitz constant of the function ϕ' on the interval $[x_i, x_{i+1}]$. Then, for $E(\phi, P)$ from (14), we can write

$$\left| \frac{\phi'(x_{i+1}) + \phi'(x_i)}{2}(x_{i+1} - x_i) - \int_{x_i}^{x_i+1} \phi(x)dx \right| \leq \frac{(x_{i+1} - x_i)^3 M_{1i}}{6}$$

or, by summing over i, we get

$$\left| T(\phi, P) - \int_a^b \phi(x)dx \right| \leq \frac{1}{6} \sum_{i=0}^{m-1} (x_{i+1} - x_i)^3 M_{1i} \leq \frac{M_1}{6} \sum_{i=0}^{m-1} (x_{i+1} - x_i)^3,$$

where $M_1 = \max_{\forall i}\{M_{1i}\}$.

Proposition 2. *Under the conditions of Corollary* 5, *the following inequality is true*:

$$E(\phi, P) = \frac{1}{12} \sum_{i=0}^{m-1} \left\{ (x_{i+1} - x_i)^2 \left[|\phi'(x_{i+1})| + |\phi'(x_i)| \right] \right\}$$

$$\leq \frac{\max\{|\phi'(a)|, |\phi'(b)|\}}{6} \sum_{i=0}^{m-1} (x_{i+1} - x_i)^2. \tag{27}$$

Proof. For each subinterval $[x_i, x_{i+1}]$, by applying Corollary 6, we obtain

$$\left| \frac{\phi'(x_{i+1}) + \phi'(x_i)}{2}(x_{i+1} - x_i) - \int\limits_{x_i}^{x_i+1} \phi(x)dx \right| \leq \frac{(x_{i+1} - x_i)^2 \max\{|\phi'(x_{i+1})||\phi'(x_i)|\}}{6}$$

or, by summing over i, we have

$$\left| T(\phi, P) - \int\limits_a^b \phi(x)dx \right| \leq \frac{1}{6}\sum_{i=0}^{m-1}\left[(x_{i+1} - x_i)^2 \max\{|\phi'(x_{i+1})|, |\phi'(x_i)|\} \right]$$

$$\leq \frac{\max\{|\phi'(b)|, |\phi'(a)|\}}{6}\sum_{i=0}^{\ell-1}(x_{i+1} - x_i)^2.$$

Remark 17. The estimate (27) is better than the estimate obtained in the study [16] (see Proposition 4.1).

Proposition 3. *Under the conditions of Theorem 6, the following inequality is true*:

$$E(\phi, P) = \frac{1}{12}\sum_{i=0}^{m-1}(x_{i+1} - x_i)^3|\phi''(\xi_i)| \leq \frac{|\phi''(\xi_{i^*})|}{12}\sum_{i=0}^{m-1}(x_{i+1} - x_i)^3, \qquad (28)$$

where $|\phi''(\xi_{i^*})| = \max_i\{|\phi''(\xi_i)|\}$.

Proof. Let $\xi_i \in [x_i, x_{i+1}]$. For each subinterval $[x_i, x_{i+1}]$, by applying Corollary 3 from (17) we obtain

$$\left| \frac{\phi'(x_{i+1}) + \phi'(x_i)}{2}(x_{i+1} - x_i) - \int\limits_{x_i}^{x_i+1} \phi(x)dx \right| \leq \frac{(x_{i+1} - x_i)^3[|\phi''(\xi_i)|\,]}{12}$$

or, by summing over i, we get

$$\left| T(\phi, P) - \int\limits_{\vartheta_1}^{\vartheta_2} \phi(x)dx \right| \leq \sum_{i=0}^{m-1}\frac{(x_{i+1} - x_i)^3|\phi''(\xi_i)|}{12} \leq \frac{|\phi''(\xi_{i^*})|}{12}\sum_{i=0}^{m-1}(x_{i+1} - x_i)^3.$$

Remark 18. With a uniform partition of the interval of integration, from inequality (28), we obtain an estimate $E(\phi, P) = \max_{[a,b]}|\phi''(x)|\dfrac{(b-a)^3}{12\ell^2}$ existing in the literature (see, for example, [6, p. 285] and [14]).

Proposition 4. *Under the conditions of Theorem 7, the following inequality is true*:

$$E(\phi, P) = \frac{1}{12}\sum_{i=0}^{m-1}(x_{i+1} - x_i)^2|\phi'(x_{i+1}) - \phi'(x_i)|$$

Proof. The proof follows from inequality (26).

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