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## PRIMES OF THE FORM $[n^c]$ WITH SQUARE-FREE $n$

## ПРОСТІ ЧИСЛА ВИГЛЯДУ $[n^c]$ , ДЕ $n$ НЕ Є КВАДРАТОМ

Let  $[\cdot]$  be the floor function. We show that if  $1 < c < \frac{3849}{3334}$ , then there exist infinitely many prime numbers of the form  $[n^c]$ , where  $n$  is square-free.

Нехай  $[\cdot]$  — ціла частина числа. Показано, що для  $1 < c < \frac{3849}{3334}$  існує нескінченно багато простих чисел вигляду  $[n^c]$ , де  $n$  не є квадратом.

**1. Notations.** Let  $x$  be a sufficiently large positive number. The letter  $p$  with or without subscript will always denote prime number. By  $\varepsilon$  we denote an arbitrary small positive constant. As usual  $\mu(n)$  and  $\Lambda(n)$  denote, respectively, Möbius' function and von Mangoldt's function. The notation  $m \sim M$  means that  $m$  runs through the interval  $(M, 2M]$ . As usual  $[t]$  and  $\{t\}$  denote the integer part, respectively, the fractional part of  $t$ . Moreover,  $e(y) = e^{2\pi iy}$  and  $\psi(t) = \{t\} - 1/2$ . We denote by  $\tau_k(n)$  the number of solutions of the equation  $m_1 m_2 \dots m_k = n$  in natural numbers  $m_1, \dots, m_k$ . Instead of  $m \equiv n \pmod{k}$  we write for simplicity  $m \equiv n(k)$ . We assume that  $1 < c < \frac{3849}{3334}$  and  $\gamma = \frac{1}{c}$ . Denote

$$S_c(x) = \sum_{\substack{n \leq x \\ [n^c] = p}} \mu^2(n), \quad (1)$$

$$\tilde{S}_c(x) = \sum_{\substack{n \leq x \\ [n^c] = p}} \mu^2(n) \mu^2(n+1), \quad (2)$$

$$\sigma = \prod_p \left(1 - \frac{2}{p^2}\right), \quad (3)$$

$$z = \log^2 x. \quad (4)$$

**2. Introduction and statement of the result.** The problems for the existence of infinitely many prime numbers of a special form are as interesting as well as difficult in prime number theory. One of them is the representation of infinitely many prime numbers by polynomials. It is conjectured that if  $f(x)$  is any irreducible integer polynomial such that  $f(1), f(2), \dots$  tend to infinity and have no common factor greater than 1, then  $f(n)$  takes infinitely many prime values. This problem has been completely solved for linear polynomials by the Dirichlet theorem on primes in arithmetic progressions. Now it remains unsolved for polynomials of degree greater than 1 and it seems to be out of reach of the current state of the mathematics. For this reason, the mathematical world deals with the accessible problem of primes of the form  $[n^c]$ . Let  $\mathbb{P}$  denotes the set of all prime numbers.

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In 1953 Piatetski-Shapiro [15] has shown that for any fixed  $1 < c < \frac{12}{11}$  the set

$$\mathbb{P}_c = \{p \in \mathbb{P} \mid p = [n^c] \text{ for some } n \in \mathbb{N}\}$$

is infinite. The prime numbers of the form  $p = [n^c]$  are called Piatetski-Shapiro primes. Denote

$$\pi_c(x) = \sum_{\substack{n \leq x \\ [n^c]=p}} 1.$$

Piatetski-Shapiro's result states that

$$\pi_c(x) = \frac{x}{c \log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right) \tag{5}$$

for

$$1 < c < \frac{12}{11}.$$

Subsequently the interval for  $c$  was sharpened many times [2, 7–14, 16]. To achieve a longer interval for  $c$  the authors used the fact that the upper bound for  $c$  is closely connected with the estimate of an exponential sum over primes. The best results up to now belong to Rivat and Sargos [17] with (5) for

$$1 < c < \frac{2817}{2426}$$

and to Rivat and Wu [18] with

$$\pi_c(x) \gg \frac{x}{\log x}$$

for

$$1 < c < \frac{243}{205}.$$

As researchers in additive prime number theory have asked whether different additive questions about the primes can be resolved in prime numbers from special sets, Piatetski-Shapiro primes have become a favorite "test case" for some results. Over the last three decades, number theorists have solved various equations and inequalities with Piatetski-Shapiro primes. On the other hand, researchers in multiplicative number theory have studied arithmetic properties of primes of the form  $p = [n^c]$ . In 2014, Baker, Banks, Guo and Yeager [3] considered for the first time Piatetski-Shapiro primes  $p = [n^c]$  under imposed conditions on the numbers  $n$ . They showed that for any fixed  $1 < c < \frac{77}{76}$  there are infinitely many primes of the form  $p = [n^c]$ , where  $n$  is a natural number with at most eight prime factors. In this connection, in 2016, the article of Banks, Guo and Shparlinski [4] appeared, which contains a study of the existence of infinitely many prime numbers of the form  $[p^c]$ .

Inspired by the work of Baker, Banks, Guo, Yeager and Shparlinski, we investigate the existence of infinitely many Piatetski-Shapiro primes  $p = [n^c]$  such that  $n$  or  $n^2 + n$  runs through the set of square-free numbers. We show that for any fixed  $1 < c < \frac{3849}{3334}$  the sets

$$\mathbb{T}_c = \{p \in \mathbb{P} \mid p = [n^c], \mu^2(n) = 1\},$$

$$\tilde{\mathbb{T}}_c = \{p \in \mathbb{P} \mid p = [n^c], \mu^2(n^2 + n) = 1\}$$

are infinite. More precisely, we establish the following theorems.

**Theorem 1.** Let  $1 < c < \frac{3849}{3334}$ . Then, for the sum (1), the asymptotic formula

$$S_c(x) = \frac{6}{c\pi^2} \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right) \tag{6}$$

holds.

**Theorem 2.** Let  $1 < c < \frac{3849}{3334}$ . Then, for the sum (2), the asymptotic formula

$$\tilde{S}_c(x) = \frac{\sigma x}{c \log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right)$$

holds. Here  $\sigma$  is defined by (3).

The proof of Theorem 2 is not essentially different from the proof of Theorem 1. For this reason, we will only give the proof of Theorem 1.

**3. Preliminary lemmas.**

**Lemma 1.** Let  $|f^{(m)}(u)| \asymp YX^{1-m}$  for  $1 \leq X < u < X_0 \leq 2X$  and  $m \geq 1$ . Then

$$\left| \sum_{X < n \leq X_0} e(f(n)) \right| \ll Y^\varkappa X^\lambda + Y^{-1},$$

where  $(\varkappa, \lambda)$  is any exponent pair.

**Proof.** See [5, Chapter 3].

**Lemma 2.** Let  $H, N, M$  be positive integers and  $F$  is a real number greater than one. Let  $\alpha, \beta$  and  $\gamma$  be real numbers such that  $\alpha(\alpha - 1)\beta\gamma \neq 0$ . Set

$$\Sigma_1 = \sum_{h=H+1}^{2H} \sum_{n=N+1}^{2N} \left| \sum_{m \sim M} e\left(F \frac{m^\alpha h^\beta n^\gamma}{M^\alpha H^\beta N^\gamma}\right) \right|.$$

Then

$$\Sigma_1 \ll (HNM)^{1+\varepsilon} \left\{ \left(\frac{F}{HNM^2}\right)^{\frac{1}{4}} + \frac{1}{M^{\frac{1}{2}}} + \frac{1}{F} \right\}.$$

**Proof.** See [19, Theorem 3].

**Lemma 3.** Let  $\alpha, \alpha_1$  and  $\alpha_2$  be real numbers such that  $\alpha < 1, \alpha\alpha_1\alpha_2 \neq 0$ . Let  $M \geq 1, M_1 \geq 1, M_2 \geq 1$ . Let  $a(m)$  and  $b(m_1, m_2)$  be complex numbers with  $|a(m)| \leq 1$  and  $|b(m_1, m_2)| \leq 1$ . Set

$$\Sigma_2 = \sum_{m \sim M} \sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} a(m)b(m_1, m_2)e\left(F \frac{m^\alpha m_1^{\alpha_1} m_2^{\alpha_2}}{M^\alpha M_1^{\alpha_1} M_2^{\alpha_2}}\right),$$

where

$$F \geq M_1 M_2.$$

Then

$$\Sigma_2 \ll (MM_1M_2 \log 2M_1M_2) \left\{ \frac{1}{(M_1M_2)^{\frac{1}{2}}} + \left(\frac{F}{M_1M_2}\right)^{\frac{\varkappa}{2(1+\varkappa)}} \left(\frac{1}{M}\right)^{\frac{1+\varkappa-\lambda}{2(1+\varkappa)}} \right\},$$

where  $(\varkappa, \lambda)$  is any exponent pair.

**Proof.** See [1, Theorem 2].

**Lemma 4.** Let  $G(n)$  be a complex valued function. Assume further that

$$P > 2, \quad P_1 \leq 2P, \quad 2 \leq U < V \leq Z \leq P, \\ U^2 \leq Z, \quad 128UZ^2 \leq P_1, \quad 2^{18}P_1 \leq V^3.$$

Then the sum

$$\sum_{P < n \leq P_1} \Lambda(n)G(n)$$

can be decomposed into  $O(\log^6 P)$  sums, each of which is either of Type I

$$\sum_{\substack{M < m \leq M_1 \\ P < ml \leq P_1}} a(m) \sum_{L < l \leq L_1} G(ml)$$

and

$$\sum_{\substack{M < m \leq M_1 \\ P < ml \leq P_1}} a(m) \sum_{L < l \leq L_1} G(ml) \log l,$$

where

$$L \geq Z, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad a(m) \ll \tau_5(m) \log P,$$

or of Type II

$$\sum_{\substack{M < m \leq M_1 \\ P < ml \leq P_1}} a(m) \sum_{L < l \leq L_1} b(l)G(ml),$$

where

$$U \leq L \leq V, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad a(m) \ll \tau_5(m) \log P, \quad b(l) \ll \tau_5(l) \log P.$$

**Proof.** See [6].

**Lemma 5.** For every  $H \geq 1$ , we have

$$\psi(t) = \sum_{1 \leq |h| \leq H} a(h)e(ht) + \mathcal{O}\left(\sum_{|h| \leq H} b(h)e(ht)\right),$$

where

$$a(h) \ll \frac{1}{|h|}, \quad b(h) \ll \frac{1}{H}. \tag{7}$$

**Proof.** This lemma is a direct result of [20]. Inspecting the arguments presented in [20, p. 206–210], the reader will easily see that the proof of Lemma 5 can be obtained immediately.

**4. Beginning of the proof.** Our first maneuvers are straightforward. By using (1), (4) and the well-known identity

$$\mu^2(n) = \sum_{d^2|n} \mu(d),$$

we write

$$S_c(x) = \sum_{\substack{n \leq x \\ [n^c]=p}} \sum_{d^2|n} \mu(d) = \sum_{d \leq \sqrt{x}} \mu(d) \sum_{\substack{n \leq x \\ [n^c]=p \\ n \equiv 0 \pmod{d^2}}} 1 = S_c^{(1)}(x) + S_c^{(2)}(x), \tag{8}$$

where

$$S_c^{(1)}(x) = \sum_{d \leq z} \mu(d) \sum_{\substack{n \leq x \\ [n^c]=p \\ n \equiv 0 \pmod{d^2}}} 1, \tag{9}$$

$$S_c^{(2)}(x) = \sum_{z < d \leq \sqrt{x}} \mu(d) \sum_{\substack{n \leq x \\ [n^c]=p \\ n \equiv 0 \pmod{d^2}}} 1. \tag{10}$$

We shall estimate  $S_c^{(1)}(x)$  and  $S_c^{(2)}(x)$ , respectively, in Sections 5 and 6. In Sections 7 we shall finalize the proof of Theorem 1.

**5. Estimation of  $S_c^{(1)}(x)$ .** From (9) we obtain

$$\begin{aligned} S_c^{(1)}(x) &= \sum_{d \leq z} \mu(d) \sum_{p \leq x^c} \sum_{\substack{n \leq x \\ [n^c]=p \\ n \equiv 0 \pmod{d^2}}} 1 = \sum_{d \leq z} \mu(d) \sum_{p \leq x^c} \sum_{\substack{n \leq x \\ p^\gamma \leq n < (p+1)^\gamma \\ n \equiv 0 \pmod{d^2}}} 1 \\ &= \sum_{d \leq z} \mu(d) \sum_{p \leq x^c} \sum_{\substack{p^\gamma \leq n < (p+1)^\gamma \\ n \equiv 0 \pmod{d^2}}} 1 + \mathcal{O}(z) \\ &= \sum_{d \leq z} \mu(d) \sum_{p \leq x^c} \sum_{p^\gamma d^{-2} \leq k < (p+1)^\gamma d^{-2}} 1 + \mathcal{O}(z) \\ &= \sum_{d \leq z} \mu(d) \sum_{p \leq x^c} \left( [-p^\gamma d^{-2}] - [-(p+1)^\gamma d^{-2}] \right) + \mathcal{O}(z) \\ &= S_c^{(3)}(x) + S_c^{(4)}(x) + \mathcal{O}(z), \end{aligned} \tag{11}$$

where

$$S_c^{(3)}(x) = \sum_{d \leq z} \frac{\mu(d)}{d^2} \sum_{p \leq x^c} ((p+1)^\gamma - p^\gamma), \tag{12}$$

$$S_c^{(4)}(x) = \sum_{d \leq z} \mu(d) \sum_{p \leq x^c} (\psi(-(p+1)^\gamma d^{-2}) - \psi(-p^\gamma d^{-2})). \tag{13}$$

**5.1. Asymptotic formula for  $S_c^{(3)}(x)$ .** Bearing in mind (4), (12) and the well-known asymptotic formulas

$$\begin{aligned} (p+1)^\gamma - p^\gamma &= \gamma p^{\gamma-1} + \mathcal{O}(p^{\gamma-2}), \\ \sum_{p \leq x^c} p^{\gamma-1} &= \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right), \end{aligned}$$

$$\sum_{d \leq z} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} + \mathcal{O}\left(\frac{1}{z}\right),$$

we derive

$$S_c^{(3)}(x) = \frac{6}{c\pi^2} \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right). \tag{14}$$

**5.2. Upper bound for  $S_c^{(4)}(x)$ .** By (13) and Abel's summation formula it follows

$$S_c^{(4)}(x) \ll zx^{\frac{5}{2}} \log^2 x + \sum_{d \leq z} \max_{2 \leq t \leq x^c} |\Sigma(t)|, \tag{15}$$

where

$$\Sigma(t) = \sum_{n \leq t} \Lambda(n) (\psi(-(n+1)^\gamma d^{-2}) - \psi(-n^\gamma d^{-2})). \tag{16}$$

Splitting the range of  $n$  into dyadic subintervals from (15) and (16), we get

$$S_c^{(4)}(x) \ll zx^{\frac{5}{2}} \log^2 x + (\log x) \sum_{d \leq z} \max_{N \leq x^c} |S_c^{(5)}(N)|, \tag{17}$$

where

$$S_c^{(5)}(N) = \sum_{n \sim N} \Lambda(n) (\psi(-(n+1)^\gamma d^{-2}) - \psi(-n^\gamma d^{-2})). \tag{18}$$

By using the trivial estimate for  $S_c^{(5)}(N)$ , from (17) and (18) we have

$$S_c^{(4)}(x) \ll zx^{\frac{5}{2}} \log^2 x + \frac{x}{\log^2 x} + (\log x) \sum_{d \leq z} \max_{\frac{x}{z \log^4 x} \leq N \leq x^c} |S_c^{(5)}(N)|. \tag{19}$$

Henceforth we will use that

$$\frac{x}{z \log^4 x} \leq N \leq x^c. \tag{20}$$

Now (18) and Lemma 5 imply

$$S_c^{(5)}(N) = S_c^{(6)}(N) + S_c^{(7)}(N) + S_c^{(8)}(N), \tag{21}$$

where

$$S_c^{(6)}(N) = \sum_{n \sim N} \Lambda(n) \sum_{1 \leq |h| \leq H} a(h) (e(-h(n+1)^\gamma d^{-2}) - e(-hn^\gamma d^{-2})), \tag{22}$$

$$S_c^{(7)}(N) \ll \sum_{n \sim N} \Lambda(n) \sum_{|h| \leq H} b(h) e(-hn^\gamma d^{-2}), \tag{23}$$

$$S_c^{(8)}(N) \ll \sum_{n \sim N} \Lambda(n) \sum_{|h| \leq H} b(h) e(-h(n+1)^\gamma d^{-2}). \tag{24}$$

Henceforth we will use that

$$d \leq z. \tag{25}$$

Further we choose

$$H = x^{\varepsilon-1}Nd^2. \tag{26}$$

It is easy to see that (4), (20) and (26) lead to  $H \geq 1$ . From (4), (7), (20), (23), (25), (26) and Lemma 1 with exponent pair  $\left(\frac{1}{2}, \frac{1}{2}\right)$ , we deduce

$$\begin{aligned} S_c^{(7)}(N) &\ll (\log N) \left( b_0N + \sum_{1 \leq |h| \leq H} b(h) \sum_{n \sim N} e(-hn^\gamma d^{-2}) \right) \\ &\ll (\log N) \left( NH^{-1} + \sum_{1 \leq |h| \leq H} b(h) \left( |h|^{\frac{1}{2}} N^{\frac{\gamma}{2}} d^{-1} + |h|^{-1} N^{1-\gamma} d^2 \right) \right) \\ &\ll (\log N) \left( NH^{-1} + H^{-1} \sum_{1 \leq |h| \leq H} \left( |h|^{\frac{1}{2}} N^{\frac{\gamma}{2}} d^{-1} + |h|^{-1} N^{1-\gamma} d^2 \right) \right) \\ &\ll (\log N) \left( NH^{-1} + H^{\frac{1}{2}} N^{\frac{\gamma}{2}} d^{-1} + N^{1-\gamma} d^2 \right) \\ &\ll x^{1-\varepsilon} d^{-2} \log x. \end{aligned} \tag{27}$$

In the same way, for the sum defined by (24), we obtain

$$S_c^{(8)}(N) \ll x^{1-\varepsilon} d^{-2} \log x. \tag{28}$$

It remains to estimate the sum  $S_c^{(6)}(N)$ . By (7) and (22) we have

$$S_c^{(6)}(N) \ll \sum_{1 \leq |h| \leq H} \frac{1}{|h|} \left| \sum_{n \sim N} \Lambda(n) \Phi_h(n) e(-hn^\gamma d^{-2}) \right|, \tag{29}$$

where

$$\Phi_h(t) = e(ht^\gamma d^{-2} - h(t+1)^\gamma d^{-2}) - 1.$$

Bearing in mind the estimates

$$\Phi_h(t) \ll |h|N^{\gamma-1}d^{-2}, \quad \Phi'_h(t) \ll |h|N^{\gamma-2}d^{-2}$$

for  $t \in [N, 2N]$  and using Abel's summation formula from (29), we derive

$$\begin{aligned} S_c^{(6)}(N) &\ll \sum_{1 \leq h \leq H} \frac{1}{h} \left| \Phi_h(2N) \sum_{n \sim N} \Lambda(n) e(-hn^\gamma d^{-2}) \right| \\ &\quad + \sum_{1 \leq h \leq H} \frac{1}{h} \int_N^{2N} \left| \Phi'_h(t) \sum_{N < n \leq t} \Lambda(n) e(-hn^\gamma d^{-2}) \right| dt \ll N^{\gamma-1} d^{-2} |S_c^{(9)}(N_1)|, \end{aligned} \tag{30}$$

where

$$S_c^{(9)}(N_1) = \sum_{1 \leq h \leq H} \left| \sum_{N < n \leq N_1} \Lambda(n) e(hn^\gamma d^{-2}) \right| \tag{31}$$

for some number  $N_1 \in (N, 2N]$ .

Put

$$F = H_1 L^\gamma M^\gamma d^{-2}. \tag{32}$$

**Lemma 6.** *Assume that*

$$H_1 \leq H, \quad H_2 \sim H_1, \quad |a(m)| \leq 1, \quad LM \asymp N, \quad L \gg N^{\frac{1}{2}-\gamma} H_1^{-\frac{1}{2}} x^{1-3\varepsilon}. \quad (33)$$

Set

$$S_I = \sum_{H_1 \leq h \leq H_2} \left| \sum_{m \sim M} a(m) \sum_{l \sim L} e(hm^\gamma l^\gamma d^{-2}) \right|. \quad (34)$$

Then

$$S_I \ll x^{1-2\varepsilon} N^{1-\gamma}.$$

**Proof.** By (4), (20), (25), (32) and (33) it follows  $F \geq 1$ . Now (4), (20), (25), (26), (32), (33), (34) and Lemma 2 yield

$$\begin{aligned} S_I &\ll (H_1 ML)^{1+\varepsilon} \left\{ \left( \frac{F}{H_1 ML^2} \right)^{\frac{1}{4}} + \frac{1}{L^{\frac{1}{2}}} + \frac{1}{F} \right\} \\ &\ll x^\varepsilon \left( HN^{\frac{\gamma+3}{4}} L^{-\frac{1}{4}} d^{-\frac{1}{2}} + HNL^{-\frac{1}{2}} + d^2 N^{1-\gamma} \right) \\ &\ll x^{1-2\varepsilon} N^{1-\gamma}. \end{aligned}$$

The lemma is proved.

**Lemma 7.** *Assume that*

$$\begin{aligned} H_1 \leq H, \quad H_2 \sim H_1, \quad |a(m)| \leq 1, \quad |b(l)| \leq 1, \\ LM \asymp N, \quad N^{2\gamma} H_1 x^{6\varepsilon-2} \ll L \ll N^{\frac{1}{3}}. \end{aligned} \quad (35)$$

Set

$$S_{II} = \sum_{H_1 \leq h \leq H_2} \left| \sum_{m \sim M} a(m) \sum_{l \sim L} b(l) e(hm^\gamma l^\gamma d^{-2}) \right|. \quad (36)$$

Then

$$S_{II} \ll x^{1-2\varepsilon} N^{1-\gamma}.$$

**Proof.** By (4), (20), (25), (32) and (35) it follows  $F \geq LH_1$ . Now (4), (20), (25), (26), (32), (35), (36), Lemma 1 with exponent pair

$$BA^5 BA^2 BA^2 B(0, 1) = \left( \frac{480}{1043}, \frac{528}{1043} \right)$$

and Lemma 2 give us

$$\begin{aligned} S_{II} &\ll (H_1 ML \log 2LH_1) \left\{ \frac{1}{(LH_1)^{\frac{1}{2}}} + \left( \frac{F}{LH_1} \right)^{\frac{240}{1523}} \left( \frac{1}{M} \right)^{\frac{995}{3046}} \right\} \\ &\ll x^\varepsilon \left\{ H^{\frac{1}{2}} NL^{-\frac{1}{2}} + HN \left( \frac{N^\gamma}{Ld^2} \right)^{\frac{240}{1523}} \left( \frac{1}{M} \right)^{\frac{995}{3046}} \right\} \\ &\ll x^{1-2\varepsilon} N^{1-\gamma}. \end{aligned}$$

The lemma is proved.

**Lemma 8.** For the sum denoted by (31) we have

$$S_c^{(9)}(N_1) \ll x^{1-\varepsilon} N^{1-\gamma}.$$

**Proof.** Splitting the range of  $h$  into dyadic subintervals, from (31), we get

$$S_c^{(9)}(N_1) \ll |S_c^{(10)}(N_1)| \log x, \quad (37)$$

where

$$S_c^{(10)}(N_1) = \sum_{h \sim H_1} \left| \sum_{N < n \leq N_1} \Lambda(n) e(hn^\gamma d^{-2}) \right| \quad (38)$$

and  $H_1 \leq H/2$ . Put

$$U = N^{2\gamma} H_1 x^{6\varepsilon-2}, \quad V = N^{\frac{1}{3}}, \quad Z = \left[ N^{\frac{1}{2}-\gamma} H_1^{-\frac{1}{2}} x^{1-3\varepsilon} \right] + \frac{1}{2}. \quad (39)$$

By using (38), (39), Lemmas 4, 6 and 7, we deduce

$$S_c^{(10)}(N_1) \ll x^{1-\frac{3\varepsilon}{2}} N^{1-\gamma}. \quad (40)$$

Now (37) and (40) imply the proof of the lemma.

Taking into account (4), (19), (21), (27), (28), (30) and Lemma 8, we find

$$S_c^{(4)}(x) \ll \frac{x}{\log^2 x}. \quad (41)$$

**5.3. Asymptotic formula for  $S_c^{(1)}(x)$ .** From (4), (11), (14) and (41) it follows

$$S_c^{(1)}(x) = \frac{6}{c\pi^2} \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right). \quad (42)$$

**6. Upper bound for  $S_c^{(2)}(x)$ .** By (10) we obtain

$$S_c^{(2)}(x) \ll \sum_{z < d \leq \sqrt{x}} \sum_{\substack{n \leq x \\ n=0 \pmod{d^2}}} 1 \ll x \sum_{z < d \leq \sqrt{x}} \frac{1}{d^2} \ll xz^{-1}. \quad (43)$$

**7. The end of the proof.** Summarizing (4), (8), (42) and (43), we establish the asymptotic formula (6).

Theorem 1 is proved.

The author states that there is no conflict of interest.

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