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n-GENERALIZED SCHÜTZENBERGER-CROSSED PRODUCT OF MONOIDS

n-УЗАГАЛЬНЕНИЙ ПЕРЕХРЕСНИЙ ШУТЦЕНБЕРГЕРІВСЬКИЙ ДОБУТОК МОНОЇДІВ

We study the n-generalized Schützenberger-crossed product from the viewpoint of combinatorial group theory and define a new version of this product. For given monoids of this new product, we obtain a representation of the n-generalized Schützenberger-crossed product of arbitrary monoids. In addition, we give necessary and sufficient conditions for the regularity of this product.

Ми вивчаємо n-узагальнений перехресний шутценбергерівський добуток з точки зору теорії комбінаторних груп та вводимо нову версію цього добутку. Для заданих моноїдів цього нового добутку отримано зображення для n-узагальненого перехресного шутценбергерівського добутку довільних моноїдів. Крім того, наведено необхідні та достатні умови для регулярності цього добутку.

1. Introduction and preliminaries. *1.1. The Schützenberger product.* The Schützenberger product is an action on monoids that was first developed for automata theory problem-solving and formal language theory analysis of the syntactic properties of the concatenation product. The Schützenberger product was first introduced by Schützenberger [15] for two monoids, and it was later expanded by Straubing [16] for any number of monoids. The authors obtained a presentation for the Schützenberger product of two monoids and provided the normal form structure of this product's constituent parts [11].

Let $M_1 = \langle X_1 \mid R_1 \rangle$ and $M_2 = \langle X_2 \mid R_2 \rangle$ be monoids. For $P \subseteq M_1 \times M_2$, $u \in M_1$, $v \in M_2$, we define

$$uP = \{(uc, d) \mid (c, d) \in P\}, \quad Pv = \{(c, dv) \mid (c, d) \in P\}.$$

The *Schützenberger product* of M_1 and M_2, denoted by $M_1 \diamond M_2$, is the set $M_1 \times \mathcal{P}(M_1 \times M_2) \times M_2$ with multiplication $(u_1, P_1, v_1)(u_2, P_2, v_2) = (u_1 u_2, P_1 v_2 \cup u_1 P_2, v_1 v_2)$. The Schützenberger product of M_1 and M_2 is presented by

$$\wp_{M_1 \diamond M_2} = \Big\langle Z \mid R_1, \ R_2, \ z_{w_1, w_2}^2 = z_{w_1, w_2}, \ z_{w_1, w_2} z_{w_1', w_2'} = z_{w_1', w_2'} z_{w_1, w_2},$$

$$x_1 z_{w_1, w_2} = z_{x_1 w_1, w_2} x_1, z_{w_1, w_2} x_2 = x_2 z_{w_1, w_2 x_2}, \ x_1 x_2 = x_2 x_1 \Big\rangle,$$

where $w_i, w_i' \in M_i, x_i \in X_i, \ i \in \{1, 2\}$, and $Z = \{z_{w_1, w_2} \mid w_1 \in M_1, \ w_2 \in M_2\} \cup X_1 \cup X_2$ (see [11]).

A generalized Schützenberger product for finite monoids has investigated and found presentation of this generalized product. We are referred to [7, 10, 14, 16] for a detailed survey on n-generalized Schützenberger product.

Definition 1.1. *Let $U_1, U_2, \ldots, U_{n-1}$ and U_n be monoids. For $P_{i,i+1} \subseteq U_i \times U_{i+1}$, $1 \leq i \leq n-1$, and $u_i \in U_i$, $1 \leq i \leq n$, we define*

$$uP_{i,i+1} = \{(ux_i, x_{i+1}); (x_i, x_{i+1}) \in P_{i,i+1}\},$$

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$$P_{i,i+1}v = \{(x_i, x_{i+1}v); (x_i, x_{i+1}) \in P_{i,i+1}\}.$$

n-*Generalized Schützenberger product of monoids* $U_1, U_2, \ldots, U_{n-1}$ *and* U_n, *denoted by* $U_1 \diamond U_2 \diamond \ldots \diamond U_n$, *is the set* $U_1 \times \mathcal{P}(U_1 \times U_2) \times U_2 \times \mathcal{P}(U_2 \times U_3) \times U_3 \times \ldots \times \mathcal{P}(U_{n-1} \times U_n) \times U_n$ *with the multiplication*

$$(u_1, P_{1,2}, u_2, P_{2,3}, u_3, \ldots, P_{n-1,n}, u_n)(u'_1, P'_{1,2}, u'_2, P'_{2,3}, u'_3, \ldots, P'_{n-1,n}, u'_n)$$

$$= \Big(u_1 u'_1, u_1 P'_{1,2} \cup P_{1,2} u'_2, u_2 u'_2, u_2 P'_{2,3} \cup P_{2,3} u'_3, \ldots,$$

$$u_{n-1} u'_{n-1}, u_{n-1} P'_{n-1,n} \cup P_{n-1,n} u'_n, u_n u'_n\Big).$$

The n-generalized Schützenberger product $U_1 \diamond U_2 \diamond \ldots \diamond U_n$ defines a monoid.

Theorem 1.1. *We consider the monoids* $U_i = \langle X_i \mid R_i \rangle$, $1 \le i \le n$. *The* n-*generalized Schützenberger product* $U_1 \diamond U_2 \diamond \ldots \diamond U_n$ *is defined by generators*

$$Z = \{X_i, 1 \le i \le n\} \cup \{z_{u_i, u_{i+1}}, u_i \in U_i, u_{i+1} \in U_{i+1}, 1 \le i \le n-1\}$$

and the relations

$$x_i x_j = x_j x_i, \quad 1 \le i, j \le n, \quad i \ne j,$$

$$x_i z_{u_i, u_{i+1}} = z_{x_i u_i, u_{i+1}} x_i, \quad 1 \le i \le n-1, \quad u_i \in U_i, \quad u_{i+1} \in U_{i+1},$$

$$x_i z_{u_j, u_{j+1}} = z_{u_j, u_{j+1}} x_i, \quad 1 \le i \le n, \ 1 \le j \le n-1, \ i \ne j, \ u_j \in U_j, \ u_{j+1} \in U_{j+1},$$

$$z_{u_i, u_{i+1}} x_{i+1} = x_{i+1} z_{u_i, u_{i+1} x_{i+1}}, \quad 1 \le i \le n-1, \ u_i \in U_i, \ u_{i+1} \in U_{i+1},$$

$$z_{u_i, u_{i+1}}^2 = z_{u_i, u_{i+1}}, \quad 1 \le i \le n-1,$$

$$z_{u_i, u_{i+1}} z_{u_j, u_{j+1}} = z_{u_j, u_{j+1}} z_{u_i, u_{i+1}}, \quad 1 \le i, \ j \le n-1, \ i \ne j,$$

$$R_1 \cup R_2 \cup \ldots \cup R_n.$$

1.2. Crossed product. As is common knowledge, crossed product construction can be found in Lie algebras, C^*-algebras, and group theory, among other fields of algebra. There are numerous uses for this product in other branches of mathematics, such as topology and group representation theory. Here, we look at the building of crossed products from the perspective of combinatorial group theory. This product is more significant than established group structures, since it contains direct, semidirect [3], twisted [17], and knit [4] products. Let us move on by explaining what it means "crossed product of monoids". In [1, 2], there contains almost all of this product's specific information.

Let U and V be two monoids. A crossed system of these monoids is a quadruple (U, V, φ, f), where $\varphi: V \to \text{End}(U)$ and $f: V \times V \to U$ are two maps such that the following compatibility conditions hold:

$$v_1 \triangleleft_{\varphi} (v_2 \triangleleft_{\varphi} u) = f(v_1, v_2)((v_1 v_2) \triangleleft_{\varphi} u) = f(v_1, v_2)^{-1}, \tag{1}$$

$$f(v_1, v_2) f(v_1 v_2, v_3) = (v_1 \triangleleft_{\varphi} f(v_2, v_3)) f(v_1, v_2 v_3) \tag{2}$$

for all $u \in U$ and $v_1, v_2, v_3 \in V$. The crossed system (U, V, φ, f) is called normalized if $f(1, 1) = 1$. (U, V, φ, f) is normalized crossed system then $f(1, v) = f(v, 1) = 1$ and $1 \triangleleft_\varphi u = u$ for any $u \in U$ and $v \in V$. Here, the notation " $\triangleleft$ " is defined $u \triangleleft_\varphi u = \varphi_u(v)$ as semidirect product action.

Let U and V be two monoids, $\varphi \colon V \to \operatorname{End}(U)$ and $f \colon V \times V \to U$ be two maps. Let $U \#_\varphi^f V := U \times V$ as a set with a binary operation defined by the formula

$$(u_1, v_1)(u_2, v_2) = (u_1(v_1 \triangleleft_\varphi u_2)f(v_1, v_2), v_1 v_2)$$

for all $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Then $\left(U \#_\varphi^f V, \cdot\right)$ is a group with the identity element $(1_U, 1_V)$ if and only if (U, V, φ, f) is a normalized crossed system [1]. The group $U \#_\varphi^f V$ is called *the crossed product of U and V associated to the crossed system* (U, V, φ, f). For more current research on the crossed product and its derivations, see [5, 6, 12].

In [6], authors defined iterated crossed products, which are more important than known group constructions, since it possesses direct, semidirect [3], twisted [17], knit [4] and crossed products of groups. Additionally, they discovered a complete rewriting system and obtained a presentation for the iterated crossed product of cyclic groups, which allowed them to determine the normal form of the various components of this new construction.

Definition 1.2. *Let $U_1, U_2, \ldots, U_n$ be finite cyclic monoids of order $u_1, u_2, \ldots, u_n$, respectively. A crossed system of these monoids is a quadruple*

$$\left(U_i, U_{i+1} \#_{\alpha_{i+1}}^{f_{i+1}} U_{i+2} \#_{\alpha_{i+2}}^{f_{i+2}} \ldots \#_{\alpha_{n-1}}^{f_{n-1}} U_n, \alpha_i, f_i\right), \quad 1 \le i \le n - 1,$$

where $\alpha_i \colon U_{i+1} \#_{\alpha_{i+1}}^{f_{i+1}} U_{i+2} \#_{\alpha_{i+2}}^{f_{i+2}} \ldots \#_{\alpha_{n-1}}^{f_{n-1}} U_n \to \operatorname{End}(U_i)$ *and*

$$f_i \colon \left(U_{i+1} \#_{\alpha_{i+1}}^{f_{i+1}} U_{i+2} \ldots \#_{\alpha_{n-1}}^{f_{n-1}} U_n\right) \times \left(U_{i+1} \#_{\alpha_{i+1}}^{f_{i+1}} U_{i+2} \ldots \#_{\alpha_{n-1}}^{f_{n-1}} U_n\right) \to U_i$$

are maps such that (1), (2) *and the following compatibility conditions hold:*

$$u_{1,2} \triangleleft_{\alpha_1} \left(u_{2,2} \triangleleft_{\alpha_1} \left(u_{1,3} u_{2,3} \triangleleft_{\alpha_2} \left(\ldots \triangleleft_{\alpha_{n-2}} \left(u_{1,n} u_{2,n} \triangleleft_{\alpha_{n-1}} u_{3,1}\right) \ldots\right)\right) f_1(u_{2,2}, u_{3,2})\right)$$

$$= u_{1,2} \triangleleft_{\alpha_1} \left(u_{1,3} \triangleleft_{\alpha_2} \left(\ldots \triangleleft_{\alpha_{n-2}} \left(u_{1,n} \triangleleft_{\alpha_{n-1}} \left[u_{2,2} \triangleleft_{\alpha_1} \left(\ldots \triangleleft_{\alpha_{n-2}} \left(u_{2,n} \triangleleft_{\alpha_{n-1}} u_{3,1}\right) \ldots\right)\right.\right.\right.$$

$$\left.\left.\left.f_1(u_{2,2}, u_{3,2})\right]\right) \ldots\right)\right). \tag{3}$$

For the other condition, we use the notation $f_i(u_{m,i+1}, u_{n,i+1})$ *instead of*

$$f_i((u_{m,i+1}, 1_{U_{i+2}}, \ldots, 1_{U_n}), (u_{n,i+1}, 1_{U_{i+2}}, \ldots, 1_{U_n}))$$

to have more understandable expressions in multiplications. So,

$$f_i(u_{m,i+1}, u_{n,i+1}) = u_{m,i}, \quad 2 \le i \le n - 1, \tag{4}$$

where $U_{j,i}$ is the jth element of ith monoid.

The iterated crossed product of cyclic monoids $U_1, U_2, \ldots, U_n$ associated to the crossed system with respect with the actions given above is the set $U_1 \times U_2 \times \ldots \times U_n$ with the multiplication

$$(u_{1,1}, u_{1,2}, \ldots, u_{1,n})(u_{2,1}, u_{2,2}, \ldots, u_{2,n})$$

$$= \Big(u_{1,1}\big(u_{1,2} \lhd_{\alpha_1} \big(u_{1,3} \lhd_{\alpha_2} \big(\ldots \lhd_{\alpha_{n-2}} \big(u_{1,n} \lhd_{\alpha_{n-1}} u_{2,1}\big)\ldots\big)\big)\big)$$

$$f_1\big(f_2\big(\ldots\big(f_{n-1}(u_{1,n}, u_{2,n}), u_{2,n-1}\big),\ldots\big), u_{2,2}\big), u_{1,2}u_{2,2}, u_{1,3}u_{2,3},\ldots, u_{1,n}u_{2,n}\Big)$$

for all $u_{j,i} \in U_i$, $1 \le i \le n$.

This product is denoted by $U_1 \#_{\alpha_1}^{f_1} U_2 \#_{\alpha_2}^{f_2} \ldots \#_{\alpha_{n-1}}^{f_{n-1}} U_n$.

Let us think about the steps mentioned above. Then the iterated normalized crossed product $U_1 \#_{\alpha_1}^{f_1} U_2 \#_{\alpha_2}^{f_2} \ldots \#_{\alpha_{n-1}}^{f_{n-1}} U_n$ defines a monoid with the identity $(1_{U_1}, 1_{U_2}, \ldots, 1_{U_n})$ if and only if $\Big(U_i, U_{i+1}\#_{\alpha_{i+1}}^{f_{i+1}} U_{i+2}\#_{\alpha_{i+2}}^{f_{i+2}} \ldots \#_{\alpha_{n-1}}^{f_{n-1}} U_n, \alpha_i, f_i\Big)$, $1 \le i \le n-1$, is a normalized crossed system. The reader is referred to [6] for more details.

In this article, we define a new generalized monoid construction combining an iterated crossed product and the n-generalized Schützenberger product. Then we give a monoid of this new product and obtain a presentation for the n-generalized Schützenberger-crossed product of arbitrary monoids.

2. n-Generalized Schützenberger-crossed product of arbitrary monoids. We begin by defining the n-generalized Schützenberger-crossed product of arbitrary monoids.

Let $U_1, U_2, \ldots, U_{n-1}$ and U_n be monoids. For $P_{i,i+1} \subseteq U_i \times U_{i+1}$, $1 \le i \le n-1$, and $u_i \in U_i$, $1 \le i \le n$, we define

$$uP_{i,i+1} = \{(ux_i, x_{i+1}); (x_i, x_{i+1}) \in P_{i,i+1}\},$$

$$P_{i,i+1}v = \{(x_i, x_{i+1}v); (x_i, x_{i+1}) \in P_{i,i+1}\}.$$

Let us think about the following multiplication:

$$(u_1, P_1, u_2, P_2, u_3, \ldots, P_{n-1}, u_n)\big(u_1', P_1', u_2', P_2', u_3', \ldots, P_{n-1}', u_n'\big)$$

$$= \Big(u_1\big(u_2 \lhd_{\alpha_1} \big(u_3 \lhd_{\alpha_2} \big(\ldots \lhd_{\alpha_{n-2}} \big(u_n \lhd_{\alpha_{n-1}} u_1'\big)\ldots\big)\big)\big)$$

$$f_1\big(f_2\big(\ldots\big(f_{n-1}(u_n, u_n'), u_{n-1}'\big),\ldots\big), u_2'\big),$$

$$P_1u_2' \cup P_1', u_2u_2', P_2u_3' \cup P_2', u_3u_3', \ldots, P_{n-1}u_n' \cup P_{n-1}', u_nu_n'\Big)$$

on the set $U_1 \times \mathcal{P}(U_1 \times U_2) \times U_2 \times \mathcal{P}(U_2 \times U_3) \times U_3 \times \ldots \times \mathcal{P}(U_{n-1} \times U_n) \times U_n$, where $\alpha_i \colon U_{i+1}\#_{\alpha_{i+1}}^{f_{i+1}} U_{i+2}\#_{\alpha_{i+2}}^{f_{i+2}} \ldots \#_{\alpha_{n-1}}^{f_{n-1}} U_n \to \mathrm{End}(U_i)$ and

$$f_i \colon \Big(U_{i+1}\#_{\alpha_{i+1}}^{f_{i+1}} U_{i+2} \ldots \#_{\alpha_{n-1}}^{f_{n-1}} U_n\Big) \times \Big(U_{i+1}\#_{\alpha_{i+1}}^{f_{i+1}} U_{i+2} \ldots \#_{\alpha_{n-1}}^{f_{n-1}} U_n\Big) \to U_i$$

are given in above. Let us denote this new product by $U_1 \diamondsuit_{\alpha_1}^{f_1} U_2 \diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n$.

2.1. Monoid construction for $\boldsymbol{U_1 \diamondsuit_{\alpha_1}^{f_1} U_2 \diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n}$**.** In this subsection, we give the first main result of this paper.

Theorem 2.1. *Let* $U_1, U_2, \ldots, U_{n-1}$ *and* U_n *be monoids. For all* $u_i \in U_i$ *and* $P_{i,i+1} \in \mathcal{P}(U_i \times U_{i+1})$, *let us consider the actions given in* (1), (2), (3) *and* (4). *Then* n-*generalized Schützenberger-crossed product defines a monoid.*

Proof. We start by demonstrating the associative feature. To accomplish this, for any $u_i, u_i', u_i'' \in U_i$, $1 \leq i \leq n$, let $(u_1, P_1, u_2, P_2, \ldots, u_n)$, $(u_1', P_1', u_2', P_2', \ldots, u_n')$, $(u_1'', P_1'', u_2'', P_2'', \ldots, u_n'') \in U_1 \diamond_{\alpha_1}^{f_1} U_2 \diamond_{\alpha_2}^{f_2} \ldots \diamond_{\alpha_{n-1}}^{f_{n-1}} U_n$.

So, we have

$$\left[(u_1, P_1, u_2, P_2, \ldots, P_{n-1}, u_n)(u_1', P_1', u_2', P_2', \ldots, P_{n-1}', u_n') \right] (u_1'', P_1'', u_2'', P_2'', \ldots, P_{n-1}'', u_n'')$$

$$= \Big(u_1 \big(u_2 \lhd_{\alpha_1} (\ldots \lhd_{\alpha_{n-2}} (u_n \lhd_{\alpha_{n-1}} u_1') \ldots) \big) f_1 \big(f_2 (\ldots (f_{n-1}(u_n, u_n'), u_{n-1}'), \ldots), u_2' \big)$$

$$(u_2 u_2' \lhd_{\alpha_1} (\ldots \lhd_{\alpha_{n-2}} (u_n u_n' \lhd_{\alpha_{n-1}} u_1'') \ldots)) f_1 \big(f_2 (\ldots (f_{n-1}(u_n u_n', u_n'')), \ldots), u_2'' \big),$$

$$\big(P_1 u_2' \cup P_1' \big) u_2'' \cup P_1'', u_2 u_2' u_2'', \big(P_2 u_3' \cup P_2' \big) u_3'' \cup P_2'', \ldots, \big(P_{n-1} u_n' \cup P_{n-1}' \big) u_n'' \cup P_{n-1}'', u_n u_n' u_n'' \Big)$$

$$= \Big(u_1 \big(u_2 \lhd_{\alpha_1} (\ldots (u_n \lhd_{\alpha_{n-1}} u_1') \ldots) \big)$$

$$\underbrace{f_1 \big(u_2, u_2' \big) \big(u_2 u_2' \lhd_{\alpha_1} (\ldots \lhd_{\alpha_{n-2}} (u_n u_n' \lhd_{\alpha_{n-1}} u_1'') \ldots) \big)}_{\text{(by (1))}} f_1 \big(u_2 u_2', u_2'' \big),$$

$$P_1 u_2' u_2'' \cup P_1' u_2'' \cup P_1'', u_2 u_2' u_2'', P_2 u_3' u_3'' \cup P_2' u_3'' \cup P_2'', \ldots, P_{n-1} u_n' u_n'' \cup P_{n-1}' u_n'' \cup P_{n-1}'', u_n u_n' u_n'' \Big)$$

$$= \Big(u_1 \big(u_2 \lhd_{\alpha_1} (\ldots (u_n \lhd_{\alpha_{n-1}} u_1') \ldots) \big) \big(u_2 \lhd_{\alpha_1} (u_2' \lhd_{\alpha_1} [u_3 u_3' \lhd_{\alpha_2} (\ldots \lhd_{\alpha_{n-2}} (u_n u_n' \lhd_{\alpha_{n-1}} u_1'') \ldots)]) \big)$$

$$\underbrace{f_1 \big(u_2, u_2' \big) f_1 \big(u_2 u_2', u_2'' \big)}_{\text{(by (2))}}, P_1 u_2' u_2'' \cup P_1' u_2'' \cup P_1'', u_2 u_2' u_2'', \ldots, P_{n-1} u_n' u_n'' \cup P_{n-1}' u_n'' \cup P_{n-1}'', u_n u_n' u_n'' \Big)$$

$$= \Big(u_1 \big(u_2 \lhd_{\alpha_1} (\ldots (u_n \lhd_{\alpha_{n-1}} u_1') \ldots) \big) \big(u_2 \lhd_{\alpha_1} (u_2' \lhd_{\alpha_1} [u_3 u_3' \lhd_{\alpha_2} (\ldots \lhd_{\alpha_{n-2}} (u_n u_n' \lhd_{\alpha_{n-1}} u_1'') \ldots)]) \big)$$

$$\big(u_2 \lhd_{\alpha_1} f_1 (u_2', u_2'') \big) f_1 \big(u_2, u_2' u_2'' \big), P_1 u_2' u_2'' \cup P_1' u_2'' \cup P_1'', \ldots, P_{n-1} u_n' u_n'' \cup P_{n-1}' u_n'' \cup P_{n-1}'', u_n u_n' u_n'' \Big)$$

$$= \Big(u_1 \big(u_2 \lhd_{\alpha_1} (\ldots (u_n \lhd_{\alpha_{n-1}} u_1') \ldots) \big)$$

$$\underbrace{\big(u_2 \lhd_{\alpha_1} (u_2' \lhd_{\alpha_1} [u_3 u_3' \lhd_{\alpha_2} (\ldots (u_n u_n' \lhd_{\alpha_{n-1}} u_1'') \ldots)]) \big) f_1 \big(u_2', u_2'' \big)}_{\text{(by (3))}}$$

$$f_1 \big(u_2, u_2' u_2'' \big), P_1 u_2' u_2'' \cup P_1' u_2'' \cup P_1'', \ldots, P_{n-1} u_n' u_n'' \cup P_{n-1}' u_n'' \cup P_{n-1}'', u_n u_n' u_n'' \Big)$$

$$= \Big(u_1 \big(u_2 \lhd_{\alpha_1} (\ldots (u_n \lhd_{\alpha_{n-1}} u_1') \ldots) \big)$$

$$\Big(u_1' \lhd_{\alpha_1} (\ldots (u_n \lhd_{\alpha_{n-1}} [u_2' \lhd_{\alpha_1} (\ldots (u_n' \lhd_{\alpha_{n-1}} u_1'') \ldots) f_1 (u_2', u_2'')]) \ldots) \Big)$$

$$f_1 (u_2, u_2' u_2''), P_1 u_2' u_2'' \cup P_1' u_2'' \cup P_1'', \ldots, P_{n-1} u_n' u_n'' \cup P_{n-1}' u_n'' \cup P_{n-1}'', u_n u_n' u_n'' \Big) \Big)$$

and

$$(u_1, P_1, u_2, P_2, \ldots, P_{n-1}, u_n) \left[(u_1', P_1', u_2', P_2', \ldots, P_{n-1}', u_n')(u_1'', P_1'', u_2'', P_2'', \ldots, P_{n-1}'', u_n'') \right]$$

$$= \Big(u_1\big(u_2 \triangleleft_{\alpha_1} \big(\ldots \triangleleft_{\alpha_{n-2}} \big(u_n \triangleleft_{\alpha_{n-1}} \big[u_1'\big(u_2' \triangleleft_{\alpha_1} \big(\ldots \triangleleft_{\alpha_{n-2}} \big(u_n' \triangleleft_{\alpha_{n-1}} u_1''\big)\ldots\big)\big)$$

$$f_1\big(f_2\big(\ldots \big(f_{n-1}\big(u_n', u_n''\big), u_{n-1}''\big), \ldots\big), u_2''\big)\big]\big)\ldots\big)\big)f_1\big(f_2\big(\ldots f_{n-1}\big(x_n, x_n' x_n''\big), \ldots\big), x_2' x_2''\big),$$

$$P_1\big(u_2' u_2''\big) \cup \big(P_1' u_2'' \cup P_1''\big), u_2 u_2' u_2'', P_2\big(u_3' u_3''\big) \cup \big(P_2' u_3'' \cup P_2''\big), \ldots,$$

$$P_{n-1}\big(u_n' u_n''\big) \cup \big(P_{n-1}' u_n'' \cup P_{n-1}''\big), u_n u_n' u_n''\Big)$$

$$= \Big(u_1\big(u_2 \triangleleft_{\alpha_1} \big(\ldots \triangleleft_{\alpha_{n-2}} \big(u_n \triangleleft_{\alpha_{n-1}} u_1'\big)\ldots\big)\big)$$

$$\big(u_1' \triangleleft_{\alpha_1} \big(\ldots \triangleleft_{\alpha_{n-2}} \big(u_n \triangleleft_{\alpha_{n-1}} \big[u_2' \triangleleft_{\alpha_1} \big(\ldots \triangleleft_{\alpha_{n-2}} \big(u_n' \triangleleft_{\alpha_{n-1}} u_1''\big)\ldots\big)f_1\big(x_2', x_2''\big)\big]\big)\ldots\big)f_1\big(u_2, u_2' u_2''\big),$$

$$P_1 u_2' u_2'' \cup P_1' u_2'' \cup P_1'', u_2 u_2' u_2'', \ldots, P_{n-1} u_n' u_n'' \cup P_{n-1}' u_n'' \cup P_{n-1}'', u_n u_n' u_n''\Big).$$

Let $1_{U_1}, 1_{U_2}, \ldots, 1_{U_n}$ be the identity elements of monoids $U_1, U_2, \ldots, U_n$, respectively. So, n-generalized Schützenberger-crossed product $U_1 \diamondsuit_{\alpha_1}^{f_1} U_2 \diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n$ is a monoid with the identity $(1_{U_1}, \emptyset, 1_{U_2}, \emptyset, \ldots, \emptyset, 1_{U_n})$.

Theorem 2.1 is proved.

2.2. A presentation for $U_1 \diamondsuit_{\alpha_1}^{f_1} U_2 \diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n$. In this subsection, we give the second main result of this paper, which gives a presentation of the n-generalized Schützenberger-crossed product of monoids.

Theorem 2.2. *Let the monoids U_i, $1 \leq i \leq n$, be defined by the presentations $\langle X_i \mid R_i \rangle$. Then the n-generalized Schützenberger-crossed product $U_1 \diamondsuit_{\alpha_1}^{f_1} U_2 \diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n$ is defined by generators*

$$Z = \{X_i, \ 1 \leq i \leq n\} \cup \big\{z_{u_i, u_{i+1}}; \ u_i \in U_i, u_{i+1} \in U_{i+1}, \ 1 \leq i \leq n-1\big\}$$

and the relations

$$R_i, \quad 1 \leq i \leq n-1, \tag{5}$$

$$R_n = W, \tag{6}$$

$$x_i x_j = x_j x_i, \quad 1 \leq i \leq n, \ 2 \leq j \leq n, \ i \neq j, \tag{7}$$

$$x_i x_1 = \big(x_i \triangleleft_{\alpha_{n-1}} x_1\big) x_i, \quad 2 \leq i \leq n, \tag{8}$$

$$x_i z_{u_i, u_{i+1}} = z_{x_i u_i, u_{i+1}} x_i, \quad 1 \leq i \leq n-1, \ u_i \in U_i, \ u_{i+1} \in U_{i+1}, \tag{9}$$

$$z_{u_i, u_{i+1}} x_j = x_j z_{u_i, u_{i+1}}, \quad 1 \leq i \leq n-1, \ 1 \leq j \leq n, \ i+1 \neq j, \tag{10}$$

$$x_i z_{u_j, u_{j+1}} = z_{u_j, u_{j+1}} x_i, \quad 1 \leq i \leq n, \ 1 \leq j \leq n-1, \ i \neq j, \ u_j \in U_j, \ u_{j+1} \in U_{j+1}, \tag{11}$$

$$z_{u_i, u_{i+1}} x_{i+1} = x_{i+1} z_{u_i, u_{i+1} x_{i+1}}, \quad 1 \leq i \leq n-1, \ u_i \in U_i, \ u_{i+1} \in U_{i+1}, \tag{12}$$

$$z_{u_i, u_{i+1}}^2 = z_{u_i, u_{i+1}}, \quad 1 \leq i \leq n-1, \tag{13}$$

$$z_{u_i, u_{i+1}} z_{u_j, u_{j+1}} = z_{u_j, u_{j+1}} z_{u_i, u_{i+1}}, \quad 1 \leq i, j \leq n-1, \ i \neq j, \tag{14}$$

$$z_{u_i, u_{i+1}} z_{u_i', u_{i+1}'} = z_{u_i', u_{i+1}'} z_{u_i, u_{i+1}}, \quad u_i, u_i' \in U_i, \ u_{i+1}, u_{i+1}' \in U_{i+1}), \tag{15}$$

where $W = f_1(x_{n,1}, x_{n,2}) f_1(x_{n,1} x_{n,2}, x_{n,3}) \ldots f_1(x_{n,1} x_{n,2} \ldots x_{n,n-1}, x_{n,n})$ *is the reduced word on* X_1.

Proof. Let us denote the set of all words in Z by Z^*. Let θ be a homomorphism defined by

$$\theta : Z^* \to U_1 \diamondsuit_{\alpha_1}^{f_1} U_2 \diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n,$$

$$x_i \mapsto \theta(x_i) = (1_{U_1}, \emptyset, 1_{U_2}, \emptyset, \ldots, \emptyset, x_i, \emptyset, \ldots, 1_{U_n}),$$

$$z_{u_i, u_{i+1}} \mapsto \theta(z_{u_i, u_{i+1}}) = (1_{U_1}, \emptyset, 1_{U_2}, \emptyset, \ldots, 1_{U_i}, \{u_i, u_{i+1}\}, 1_{U_{i+1}}, \ldots, 1_{U_n}).$$

Then θ is onto by considering the generator set. Now we show that $U_1 \diamondsuit_{\alpha_1}^{f_1} U_2 \diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n$ satisfies relations (5)–(15), respectively. For $x_{i,j} \in U_j$, $1 \le i, \ j \le n$, we have

$$\left(x_{1,1}, \emptyset, 1_{U_2}, \ldots, \emptyset, 1_{U_n}\right)\left(x_{1,2}, \emptyset, 1_{U_2}, \ldots, \emptyset, 1_{U_n}\right) \ldots \left(x_{1,n}, \emptyset, 1_{U_2}, \ldots, \emptyset, 1_{U_n}\right)$$

$$= \left(x_{1,1}\left(1_{U_2} \triangleleft_{\alpha_1} \left(\ldots \triangleleft_{\alpha_{n-2}} \left(1_{U_n} \triangleleft_{\alpha_{n-1}} x_{1,2}\right)\ldots\right)\right)\right) f(1_{U_n}, 1_{U_n}), \emptyset, 1_{U_2}, \ldots, \emptyset, 1_{U_n}) \ldots$$

$$(x_{1,n-1}, \emptyset, 1_{U_2}, \ldots, \emptyset, 1_{U_n})(x_{1,n}, \emptyset, 1_{U_2}, \ldots, \emptyset, 1_{U_n})$$

$$= \ldots = (x_{1,1} x_{1,2} \ldots x_{1,n-1} x_{1,n}, \emptyset, 1_{U_2}, \ldots, \emptyset, 1_{U_n})$$

and

$$(1_{U_1}, \emptyset, x_{2,1}, \ldots, \emptyset, 1_{U_n})(1_{U_1}, \emptyset, x_{2,2}, \ldots, \emptyset, 1_{U_n}) \ldots (1_{U_1}, \emptyset, x_{1,n}, \ldots, \emptyset, 1_{U_n})$$

$$= \left(1_{U_1}\left(x_{2,1} \triangleleft_{\alpha_1} \left(\ldots \triangleleft_{\alpha_{n-2}} \left(1_{U_n} \triangleleft_{\alpha_{n-1}} x_{1,2}\right)\ldots\right)\right)\right) f(1_{U_n}, 1_{U_n}), \emptyset, 1_{U_2}, \ldots, \emptyset, 1_{U_n}) \ldots$$

$$(1_{U_1}, \emptyset, x_{2,n-1}, \ldots, \emptyset, 1_{U_n})(1_{U_1}, \emptyset, x_{2,n}, \ldots, \emptyset, 1_{U_n})$$

$$= \ldots = (1_{U_1}, \emptyset, x_{2,1} x_{2,2} \ldots x_{2,n-1} x_{2,n}, \ldots, \emptyset, 1_{U_n}).$$

By continuing the above multiplication for the elements of the monoids $U_3, U_4, \ldots, U_{n-1}$ and U_n this equation gives relation (5). Now for relation (6), we have the following:

$$(1_{U_1}, \emptyset, 1_{U_2}, \ldots, \emptyset, x_{n,1})(1_{U_1}, \emptyset, 1_{U_2}, \ldots, \emptyset, x_{n,2}) \ldots (1_{U_1}, \emptyset, 1_{U_2}, \ldots, \emptyset, x_{n,n})$$

$$= \left(1_{U_1}\left(1_{U_2} \triangleleft_{\alpha_1} \left(\ldots \triangleleft_{\alpha_{n-2}} \left(x_{n,1} \triangleleft_{\alpha_{n-1}} 1_{U_1}\right)\ldots\right)\right)\right) f(x_{n,1}, x_{n,2}), \emptyset, 1_{U_2}, \ldots, \emptyset, x_{n,1} x_{n,2}) \ldots$$

$$(1_{U_1}, \emptyset, 1_{U_2}, \ldots, \emptyset, x_{n,n-1})(1_{U_1}, \emptyset, 1_{U_2}, \ldots, \emptyset, x_{n,n})$$

$$= \ldots = \Big(f(x_{n,1}, x_{n,2}) f(x_{n,1} x_{n,2}, x_{n,3}) \ldots f(x_{n,1} x_{n,2} \ldots$$

$$x_{n,n-1}, x_{n,n}), \emptyset, \ldots, x_{n,1} x_{n,2} \ldots x_{n,n-1} x_{n,n}\Big),$$

$$\left(1_{U_1}, \emptyset, 1_{U_2}, \ldots, 1_{U_{i-1}}, \emptyset, x_i, \emptyset, 1_{U_{i+1}}, \ldots, 1_{U_n}\right)\left(1_{U_1}, \emptyset, 1_{U_2}, \ldots, 1_{U_{j-1}}, \emptyset, x_j, \emptyset, 1_{U_{j+1}}, \ldots, 1_{U_n}\right)$$

$$= (1_{U_1}, \emptyset, 1_{U_2}, \ldots, \emptyset, x_i, \emptyset, \ldots, \emptyset, 1_{U_k}, \emptyset, \ldots, \emptyset, x_j, \emptyset, \ldots, 1_{U_n})$$

$$= (1_{U_1}, \emptyset, 1_{U_2}, \ldots, 1_{U_{j-1}}, \emptyset, x_j, \emptyset, 1_{U_{j+1}}, \ldots, 1_{U_n})$$

$$(1_{U_1}, \emptyset, 1_{U_2}, \ldots, 1_{U_{i-1}}, \emptyset, x_i, \emptyset, 1_{U_{i+1}}, \ldots, 1_{U_n}).$$

This equation gives relation (7). Now, let us show that relation (8) holds:

$$\left(1_{U_1}, \emptyset, 1_{U_2}, \ldots, 1_{U_{i-1}}, \emptyset, x_i, \emptyset, 1_{U_{i+1}}, \ldots, 1_{U_n}\right)\left(x_1, \emptyset, 1_{U_2}, \ldots, 1_{U_{i-1}}, \emptyset, 1_{U_i}, \emptyset, 1_{U_{i+1}}, \ldots, 1_{U_n}\right)$$

$$= \Big(1_{U_1}\big(1_{U_2} \triangleleft_{\alpha_1} \big(\ldots \big(1_{U_{i-1}} \triangleleft_{\alpha_{i-2}} \big(x_i \triangleleft_{\alpha_{i-1}} 1_{U_{i+1}} \big(\ldots \triangleleft_{\alpha_{n-2}} \big(1_{U_n} \triangleleft_{\alpha_{n-1}} x_1\big) \ldots\big)\big)\big)\big) \ldots\big)\big)$$

$$f(1_{U_n}, 1_{U_n}), \emptyset, 1_{U_2}, \ldots, 1_{U_n}\Big)$$

$$= \big(x_i \triangleleft_{\alpha_{n-1}} x_1, \emptyset, 1_{U_2}, \ldots, 1_{U_n}\big)\big(1_{U_1}, \emptyset, 1_{U_2}, \ldots, 1_{U_{i-1}}, \emptyset, x_i, \emptyset, 1_{U_{i+1}}, \ldots, 1_{U_n}\big).$$

Now for relation (9), we have the following:

$$\big(1_{U_1}, \emptyset, 1_{U_2}, \ldots, 1_{U_{i-1}}, \emptyset, x_i, \emptyset, 1_{U_{i+1}}, \ldots, 1_{U_n}\big)\big(1_{U_1}, \emptyset, 1_{U_2}, \ldots, 1_{U_i}, \{u_i, u_{i+1}\}, 1_{U_{i+1}}, \ldots, 1_{U_n}\big)$$

$$= \big(1_{U_1}, \emptyset, 1_{U_2}, \emptyset, \ldots, 1_{U_{i-1}}, \emptyset, x_i, \{x_i u_i, u_{i+1}\}, 1_{U_{i+1}}, \ldots, 1_{U_n}\big)$$

$$= \big(1_{U_1}, \emptyset, \ldots, 1_{U_i}, \{x_i u_i, u_{i+1}\}, 1_{U_{i+1}}, \ldots, 1_{U_n}\big)\big(1_{U_1}, \emptyset, \ldots, 1_{U_{i-1}}, \emptyset, x_i, \emptyset, 1_{U_{i+1}}, \ldots, 1_{U_n}\big).$$

It is obvious that satisfies relations (10)–(15) by applying the above operations in a similar way.

Hence, these above arguments say that θ induces an epimorphism $\overline{\theta}$ from the monoid, defined by (5)–(15), say M, onto $U_1\diamondsuit_{\alpha_1}^{f_1} U_2\diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n$.

Let $w \in Z^*$ be any nonempty word. By using relations (7)–(11), we can prove that there exist words $w_{x_i} \in X_i^*$, $1 \leq i \leq n$, and $w_{u_i, u_{i+1}} \in \big\{z_{u_i, u_{i+1}}, u_i \in U_i, u_{i+1} \in U_{i+1}, 1 \leq i \leq n-1)\big\}$ such that $w = w_{x_n} w_{u_{n-1}, u_n} w_{x_{n-1}} \ldots w_{x_3} w_{u_2, u_3} w_{x_2} w_{u_1, u_2} w_{x_1}$ in M. Moreover, it can be noted that relations (12) and (13) can be used to prove that there exists a set $P(w_{u_i, u_{i+1}}) \subseteq U_i \times U_{i+1}$ such that $w_{u_i, u_{i+1}} = \prod_{(u_i, u_{i+1}) \in P(w_{u_i, u_{i+1}})} z_{u_i, u_{i+1}}$. Therefore, for any word $w \in Z^*$, we have

$$\overline{\theta}(w) = \theta(w) = \theta\big(w_{x_n} w_{u_{n-1}, u_n} w_{x_{n-1}} \ldots w_{x_3} w_{u_2, u_3} w_{x_2} w_{u_1, u_2} w_{x_1}\big)$$

$$= \theta(w_{x_n})\theta(w_{u_{n-1}, u_n})\theta(w_{x_{n-1}}) \ldots \theta(w_{x_3})\theta(w_{u_2, u_3})\theta(w_{x_2})\theta(w_{u_1, u_2})\theta(w_{x_1})$$

$$= \big(1_{U_1}, \emptyset, 1_{U_2}, \ldots, 1_{U_{n-1}}, \emptyset, w_{x_n}\big)\big(1_{U_1}, \emptyset, 1_{U_2}, \ldots, 1_{U_{n-1}}, P(w_{u_{n-1}, u_n}), 1_{U_n}\big) \ldots$$

$$\big(1_{U_1}, P(w_{u_1, u_2}), 1_{U_2}, \ldots, 1_{U_{n-1}}, \emptyset, 1_{U_n}\big)\big(w_{x_1}, \emptyset, 1_{U_2}, \ldots, 1_{U_{n-1}}, \emptyset, 1_{U_n}\big)$$

$$= \big(w_{x_1}, P(w_{u_1, u_2}), w_{x_2}, \ldots, w_{x_{n-1}}, P(w_{u_{n-1}, u_n}), w_{x_n}\big).$$

Now, if $\theta(w) = \theta(w')$ for some $w, w' \in Z^*$, then $w_{x_i} = w'_{x_i}$ in U_i and $P(w_{u_i, u_{i+1}}) = P(w'_{u_i, u_{i+1}})$. Relations (14) and (15) imply that $w_{x_i} = w'_{x_i}$ holds in M. So that $w = w'$ holds as well. Therefore, $\overline{\theta}$ is injective.

Theorem 2.2 is proved.

Note that the n-generalized Schützenberger-crossed product $U_1\diamondsuit_{\alpha_1}^{f_1} U_2\diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n$ contains constructions of n-generalized Schützenberger product $U_1\diamondsuit U_2\diamondsuit \ldots \diamondsuit U_n$, iterated crossed product $U_1\#_{\alpha_1}^{f_1} U_2\#_{\alpha_2}^{f_2} \ldots \#_{\alpha_{n-1}}^{f_{n-1}} U_n$, iterated semidirect product $U_1 \rtimes_{\alpha_1} U_2 \rtimes_{\alpha_2} \ldots \rtimes_{\alpha_{n-1}} U_n$, Schützenberger product $U_1\diamondsuit U_2$, crossed product $U_1\#_\alpha^f U_2$ and semidirect product $U_1 \rtimes_\alpha U_2$.

2.3. Regularity for $U_1\diamondsuit_{\alpha_1}^{f_1} U_2\diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n$. In this subsection, we obtained the regularity of the n-generalized Schützenberger-crossed product of monoids $U_1, U_2, \ldots, U_{n-1}$ and U_n. We recall that a monoid M is called *regular* if, for every $a \in M$, there exists $b \in M$ such that

$aba = a$ and $bab = b$ (or, equivalently, for the set of inverses of a in M, that is, $a^{-1} = \{b \in B : aba = a$ and $bab = b\}$, M is regular if and only if, for all $a \in M$, the set a^{-1} is not equal to the emptyset). We are referred to [8, 9, 13] for a detailed survey on regularity.

The following theorem we use the notation c and d instead of $\big(u_3 \triangleleft_{\alpha_2} \big(u_4 \triangleleft_{\alpha_3} \ldots \big(u_n \triangleleft_{\alpha_{n-1}} v_1\big) \ldots \big)\big)$ and $\big(v_3 \triangleleft_{\alpha_2} \big(v_4 \triangleleft_{\alpha_3} \ldots \big(v_n \triangleleft_{\alpha_{n-1}} u_1\big) \ldots \big)\big)$, respectively, to have more convenience expressions in multiplications.

Theorem 2.3. *Let $U_1, U_2, \ldots, U_{n-1}$ and U_n be any monoids. Then n-generalized Schützenberger-crossed product $U_1 \diamond_{\alpha_1}^{f_1} U_2 \diamond_{\alpha_2}^{f_2} \ldots \diamond_{\alpha_{n-1}}^{f_{n-1}} U_n$ is regular if and only if U_1 is regular monoid and U_i, $2 \le i \le n$, are groups.*

Proof. Let us suppose that $U_1 \diamond_{\alpha_1}^{f_1} U_2 \diamond_{\alpha_2}^{f_2} \ldots \diamond_{\alpha_{n-1}}^{f_{n-1}} U_n$ is regular. Thus, for

$$\big(u_1, \{(1_{U_1}, 1_{U_2})\}, 1_{U_2}, \{(1_{U_2}, 1_{U_3})\}, \ldots, 1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}\big) \in U_1 \diamond_{\alpha_1}^{f_1} U_2 \diamond_{\alpha_2}^{f_2} \ldots \diamond_{\alpha_{n-1}}^{f_{n-1}} U_n,$$

there exists $(v_1, Q_1, v_2, Q_2, \ldots, Q_{n-1}, v_n)$ such that

$$\big(u_1, \{(1_{U_1}, 1_{U_2})\}, 1_{U_2}, \{(1_{U_2}, 1_{U_3})\}, \ldots, 1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}\big)$$

$$= \big(u_1, \{(1_{U_1}, 1_{U_2})\}, 1_{U_2}, \{(1_{U_2}, 1_{U_3})\}, \ldots,$$

$$1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}\big)(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n)$$

$$\big(u_1, \{(1_{U_1}, 1_{U_2})\}, 1_{U_2}, \{(1_{U_2}, 1_{U_3})\}, \ldots, 1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}\big)$$

$$= \big(u_1 v_1, \{(1_{U_1}, 1_{U_2})\} v_2 \cup Q_1, v_2, \ldots, v_{n-1}, \{(1_{U_{n-1}}, 1_{U_n})\} v_n \cup Q_{n-1}, v_n\big)$$

$$\big(u_1, \{(1_{U_1}, 1_{U_2})\}, 1_{U_2}, \{(1_{U_2}, 1_{U_3})\}, \ldots, 1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}\big)$$

$$= \big(u_1 v_1 \big(v_2 \triangleleft_{\alpha_1} \big(v_3 \triangleleft_{\alpha_2} \ldots \big(v_n \triangleleft_{\alpha_{n-1}} u_1\big) \ldots \big)\big), \{(1_{U_1}, 1_{U_2})\} v_2 \cup Q_1 \cup \{(1_{U_1}, 1_{U_2})\}, v_2, \ldots, v_n\big)$$

and

$$(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n) = (v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n)$$

$$\big(u_1, \{(1_{U_1}, 1_{U_2})\}, 1_{U_2}, \{(1_{U_2}, 1_{U_3})\}, \ldots,$$

$$1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}\big)(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n)$$

$$= \big(v_1 \big(v_2 \triangleleft_{\alpha_1} \big(v_3 \triangleleft_{\alpha_2} \ldots \big(v_n \triangleleft_{\alpha_{n-1}} u_1\big) \ldots \big)\big), Q_1 \cup \{(1_{U_1}, 1_{U_2})\}, v_2, \ldots,$$

$$Q_{n-1} \cup \{(1_{U_{n-1}}, 1_{U_n})\}, v_n\big)(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n)$$

$$= \Big(v_1 \big(v_2 \triangleleft_{\alpha_1} \big(v_3 \triangleleft_{\alpha_2} \ldots \big(v_n \triangleleft_{\alpha_{n-1}} u_1\big) \ldots \big)\big)\big(v_2 \triangleleft_{\alpha_1} \big(v_3 \triangleleft_{\alpha_2} \ldots \big(v_n \triangleleft_{\alpha_{n-1}} v_1\big) \ldots \big)\big)$$

$$f_1(f_2(f_3(\ldots (f_{n-1}(v_n, v_n), v_{n-1}), \ldots), v_3), v_2),$$

$$(Q_1 \cup \{(1_{U_1}, 1_{U_2})\}) v_2 \cup Q_1, b_2^2, \ldots, (Q_{n-1} \cup \{(1_{U_{n-1}}, 1_{U_n})\}) v_n \cup Q_{n-1}, b_n^2\Big).$$

Therefore, we obtain that $u_i = 1_{U_i}$, $2 \leq i \leq n$. This gives that $v_1 u_1 v_1 = u_1$ and $u_1 v_1 u_1 = v_1$. Hence, U_1 is regular.

By using the similar argument, for

$$\left(1_{U_1}, \{(1_{U_1}, 1_{U_2})\}, u_2, \{(1_{U_2}, 1_{U_3})\}, \ldots, 1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}\right) \in U_1 \diamond_{\alpha_1}^{f_1} U_2 \diamond_{\alpha_2}^{f_2} \ldots \diamond_{\alpha_{n-1}}^{f_{n-1}} U_n,$$

there exists $(v_1, Q_1, v_2, Q_2, \ldots, Q_{n-1}, v_n)$ such that

$$(1_{U_1}, \{(1_{U_1}, 1_{U_2})\}, u_2, \{(1_{U_2}, 1_{U_3})\}, \ldots, 1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n})$$

$$= \Big(1_{U_1}, \{(1_{U_1}, 1_{U_2})\}, u_2, \{(1_{U_2}, 1_{U_3})\}, \ldots,$$

$$1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}\Big)(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n)$$

$$\left(1_{U_1}, \{(1_{U_1}, 1_{U_2})\}, u_2, \{(1_{U_2}, 1_{U_3})\}, \ldots, 1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}\right)$$

$$= \left(u_2 \triangleleft_{\alpha_1} v_1, \{(1_{U_1}, 1_{U_2})\}v_2 \cup Q_1, u_2 v_2, \ldots, v_{n-1}, \{(1_{U_{n-1}}, 1_{U_n})\}v_n \cup Q_{n-1}, v_n\right)$$

$$\left(1_{U_1}, \{(1_{U_1}, 1_{U_2})\}, u_2, \{(1_{U_2}, 1_{U_3})\}, \ldots, 1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}\right)$$

$$= \Big(u_2 \triangleleft_{\alpha_1} v_1, \{(1_{U_1}, 1_{U_2})\}v_2 u_2 \cup Q_1 u_2 \cup \{(1_{U_1}, 1_{U_2})\}, u_2 v_2 u_2, \ldots,$$

$$\{(1_{U_{n-1}}, 1_{U_n})\}v_n u_n \cup Q_{n-1} u_n \cup \{(1_{U_{n-1}}, 1_{U_n})\}, v_n\Big)$$

and

$$(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n) = (v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n)$$

$$\Big(1_{U_1}, \{(1_{U_1}, 1_{U_2})\}, u_2, \{(1_{U_2}, 1_{U_3})\}, \ldots,$$

$$1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}\Big)(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n)$$

$$= \Big(v_1, Q_1 u_2 \cup \{(1_{U_1}, 1_{U_2})\}, v_2 u_2, \ldots,$$

$$Q_{n-1} u_n \cup \{(1_{U_{n-1}}, 1_{U_n})\}, v_n\Big)(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n)$$

$$= \Big(v_1\big(v_2 u_2 \triangleleft_{\alpha_1} (v_3 \triangleleft_{\alpha_2} \ldots (v_n \triangleleft_{\alpha_{n-1}} v_1) \ldots)\big)$$

$$f_1(f_2(f_3(\ldots (f_{n-1}(v_n, v_n), v_{n-1}), \ldots), v_3), v_2),$$

$$Q_1 u_2 v_2 \cup \{(1_{U_1}, 1_{U_2})\}v_2 \cup Q_1, v_2 u_2 v_2, \ldots, Q_{n-1} u_n \cup \{(1_{U_{n-1}}, 1_{U_n})\}v_n u_n \cup Q_{n-1}, b_n^2\Big).$$

Here, since

$$\{(1_{U_1}, 1_{U_2})\} = \{(1_{U_1}, 1_{U_2})\}v_2 u_2 \cup Q_1 u_2 \cup \{(1_{U_1}, 1_{U_2})\},$$

$$Q_1 = Q_1 u_2 v_2 \cup \{(1_{U_1}, 1_{U_2})\}v_2 \cup Q_1,$$

$$Q_i = Q_i u_{i+1} v_{i+1} \cup \{(1_{U_i}, 1_{U_{i+1}})\} v_{i+1} \cup Q_i, \quad 2 \le i \le n-1,$$

and, in particular, $(1_{U_1}, v_2 u_2) = (1_{U_1}, 1_{U_2})$ and $Q_1 u_2 v_2 = Q_1$, we get $v_2 u_2 = u_2 v_2 = 1_{U_2}$. This says that U_2 is a group.

By applying the operations given above for the elements

$$\left.\begin{array}{l} (1_{U_1}, \{(1_{U_1}, 1_{U_2})\}, 1_{U_2}, \{(1_{U_2}, 1_{U_3})\}, u_3, \ldots, 1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}), \\ \dotfill \\ (1_{U_1}, \{(1_{U_1}, 1_{U_2})\}, 1_{U_2}, \{(1_{U_2}, 1_{U_3})\}, \ldots, u_{n-1}, \{(1_{U_{n-1}}, 1_{U_n})\}, 1_{U_n}), \\ (1_{U_1}, \{(1_{U_1}, 1_{U_2})\}, 1_{U_2}, \{(1_{U_2}, 1_{U_3})\}, \ldots, 1_{U_{n-1}}, \{(1_{U_{n-1}}, 1_{U_n})\}, u_n) \end{array}\right\}$$

$$\in U_1 \diamondsuit_{\alpha_1}^{f_1} U_2 \diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n$$

we obtain

$$\{(1_{U_i}, 1_{U_{i+1}})\} = \{(1_{U_i}, 1_{U_{i+1}})\} v_{i+1} u_{i+1} \cup Q_i u_{i+1} \cup \{(1_{U_i}, 1_{U_{i+1}})\}, \quad 3 \le i \le n-1,$$

and

$$Q_i = Q_i u_{i+1} v_{i+1} \cup \{(1_{U_i}, 1_{U_{i+1}})\} v_{i+1} \cup Q_i, \quad 2 \le i \le n-1.$$

Thus, $(1_{U_i}, v_{i+1} u_{i+1}) = (1_{U_i}, 1_{U_{i+1}})$, $Q_i u_{i+1} v_{i+1} = Q_i$, and we get $v_{i+1} u_{i+1} = u_{i+1} v_{i+1} = 1_{U_{i+1}}$. This says that U_i, $3 \le i \le n$ are groups.

Conversely, suppose that U_1 is regular monoid and U_i, $2 \le i \le n$, are groups. Since U_i, $2 \le i \le n$, are groups, there exist u_i, $v_i \in U_i$ such that $u_i v_i = v_i u_i = 1_{U_i}$, $2 \le i \le n$. Also, since U_1 is regular, we can take $c = u_2 \triangleleft_{\alpha_1} k$ for some $f(u_2, v_2) k \in a_1^{-1}$ such that $k \in (u_1 f(u_2, v_2))^{-1}$. Now, let us consider the following:

$$u_1 \big(u_2 \triangleleft_{\alpha_1} \big(u_3 \ldots \triangleleft_{\alpha_{n-2}} \big(u_n \triangleleft_{\alpha_{n-1}} v_1\big) \ldots\big)\big) f_1(u_2, v_2) \big(u_2 v_2 \triangleleft_{\alpha_1} \big(\ldots \big(u_n v_n \triangleleft_{\alpha_{n-1}} u_1\big) \ldots\big)\big)$$

$$f_1(u_2 v_2, u_2) = u_1(u_2 \triangleleft_{\alpha_1} c) f_1(u_2, v_2) \big(u_2 v_2 \triangleleft_{\alpha_1} \big(\ldots \triangleleft_{\alpha_{n-2}} \big(u_n v_n \triangleleft_{\alpha_{n-1}} u_1\big) \ldots\big)\big)$$

$$f_1(u_2 v_2, u_2) = u_1(u_2 \triangleleft_{\alpha_1} c) f_1(u_2, v_2) u_1 = u_1(u_2 \triangleleft_{\alpha_1} (v_2 \triangleleft_{\alpha_1} k)) f_1(u_2, v_2) u_1$$

$$= u_1 f_1(u_2, v_2)(u_2 v_2 \triangleleft_{\alpha_1} k) u_1 = u_1 f_1(u_2, v_2) k u_1 = u_1.$$

Also, we can take $d = v_2 \triangleleft_{\alpha_1} l$ for some $f(v_2, u_2) l \in a_1^{-1}$ such that $l \in (v_1 f(v_2, u_2))^{-1}$. Then

$$v_1 \big(v_2 \triangleleft_{\alpha_1} \big(v_3 \ldots \triangleleft_{\alpha_{n-2}} \big(v_n \triangleleft_{\alpha_{n-1}} u_1\big) \ldots\big)\big) f_1(v_2, u_2) \big(v_2 v_2 \triangleleft_{\alpha_1} \big(\ldots \big(v_n u_n \triangleleft_{\alpha_{n-1}} v_1\big) \ldots\big)\big)$$

$$f_1(v_2 u_2, v_2) = v_1(v_2 \triangleleft_{\alpha_1} d) f_1(v_2, u_2) \big(v_2 u_2 \triangleleft_{\alpha_1} \big(\ldots \triangleleft_{\alpha_{n-2}} \big(v_n u_n \triangleleft_{\alpha_{n-1}} v_1\big) \ldots\big)\big)$$

$$f_1(v_2 u_2, v_2) = v_1(v_2 \triangleleft_{\alpha_1} d) f_1(v_2, u_2) v_1 = v_1(v_2 \triangleleft_{\alpha_1} (u_2 \triangleleft_{\alpha_1} l)) f_1(v_2, u_2) v_1$$

$$= v_1 f_1(v_2, u_2)(v_2 u_2 \triangleleft_{\alpha_1} l) v_1 = v_1 f_1(v_2, u_2) l v_1 = v_1.$$

Additionally, by choosing, $Q_i = P_i v_{i+1} \subseteq U_i \times U_{i+1}$, $1 \le i \le n-1$, where $P_i \subseteq U_i \times U_{i+1}$, we get

$$(P_i v_{i+1} \cup Q_i) u_{i+1} \cup P_i = P_i v_{i+1} u_{i+1} \cup Q_i u_{i+1} \cup P_i$$

$$= P_i v_{i+1} u_{i+1} \cup P_i v_{i+1} u_{i+1} \cup P_i = P_i \cup P_i \cup P_i = P_i$$

and

$$(Q_i u_{i+1} \cup P_i)v_{i+1} \cup Q_i = Q_i u_{i+1} v_{i+1} \cup P_i v_{i+1} \cup Q_i = P_i v_{i+1} u_{i+1} v_{i+1} \cup P_i v_{i+1} \cup P_i v_{i+1}$$
$$= P_i v_{i+1} \cup P_i v_{i+1} \cup P_i v_{i+1} = P_i v_{i+1} = Q_i.$$

Consequently, for every $(u_1, P_1, u_2, \ldots, P_{n-1}, u_n) \in U_1 \diamondsuit_{\alpha_1}^{f_1} U_2 \diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n$, there exists

$$(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n) \in U_1 \diamondsuit_{\alpha_1}^{f_1} U_2 \diamondsuit_{\alpha_2}^{f_2} \ldots \diamondsuit_{\alpha_{n-1}}^{f_{n-1}} U_n$$

such that

$$(u_1, P_1, u_2, \ldots, P_{n-1}, u_n)(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n)(u_1, P_1, u_2, \ldots, P_{n-1}, u_n)$$

$$= \Big(u_1 \big(u_2 \triangleleft_{\alpha_1} (\ldots \triangleleft_{\alpha_{n-2}} (u_n \triangleleft_{\alpha_{n-1}} v_1) \ldots) \big) \big) f_1 (f_2(\ldots (f_{n-1}(u_n, v_n), v_{n-1}), \ldots), v_2)$$

$$\big(u_2 v_2 \triangleleft_{\alpha_1} (\ldots \triangleleft_{\alpha_{n-2}} (u_n v_n \triangleleft_{\alpha_{n-1}} u_1) \ldots) \big) f_1 (f_2(\ldots (f_{n-1}(u_n v_n, u_n)), \ldots), u_2),$$

$$(P_1 v_2 \cup Q_1) u_2 \cup P_1, u_2 v_2 u_2, (P_2 v_3 \cup Q_2) u_3 \cup P_2, \ldots,$$

$$(P_{n-1} v_n \cup Q_{n-1}) u_n \cup P_{n-1}, u_n v_n u_n \Big)$$

$$= \Big(u_1 \big(u_2 \triangleleft_{\alpha_1} (\ldots (u_n \triangleleft_{\alpha_{n-1}} v_1) \ldots) \big) \big)$$

$$f_1(u_2, v_2) \big(u_2 v_2 \triangleleft_{\alpha_1} (\ldots (u_n v_n \triangleleft_{\alpha_{n-1}} u_1) \ldots) \big) f_1(u_2 v_2, u_2),$$

$$(P_1 v_2 \cup Q_1) u_2 \cup P_1, u_2 v_2 u_2, (P_2 v_3 \cup Q_2) u_3 \cup P_2, \ldots,$$

$$(P_{n-1} v_n \cup Q_{n-1}) u_n \cup P_{n-1}, u_n v_n u_n \Big)$$

$$= (u_1, P_1, u_2, \ldots, P_{n-1}, u_n)$$

and

$$(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n)(u_1, P_1, u_2, \ldots, P_{n-1}, u_n)(v_1, Q_1, v_2, \ldots, Q_{n-1}, v_n)$$

$$= \Big(v_1 \big(v_2 \triangleleft_{\alpha_1} (\ldots (v_n \triangleleft_{\alpha_{n-1}} u_1) \ldots) \big) \big) f_1 (f_2(\ldots (f_{n-1}(v_n, u_n), u_{n-1}), \ldots), u_2)$$

$$\big(v_2 u_2 \triangleleft_{\alpha_1} (\ldots \triangleleft_{\alpha_{n-2}} (v_n u_n \triangleleft_{\alpha_{n-1}} v_1) \ldots) \big) f_1 (f_2(\ldots (f_{n-1}(v_n u_n, bn)), \ldots), v_2),$$

$$(Q_1 u_2 \cup P_1) v_2 \cup Q_1, v_2 u_2 v_2, (Q_2 u_3 \cup P_2) v_3 \cup Q_2, \ldots,$$

$$(Q_{n-1} u_n \cup P_{n-1}) v_n \cup Q_{n-1}, v_n u_n v_n \Big)$$

$$= \Big(v_1 \big(v_2 \triangleleft_{\alpha_1} (\ldots (v_n \triangleleft_{\alpha_{n-1}} u_1) \ldots) \big) \big)$$

$$f_1(v_2, u_2) \big(v_2 u_2 \triangleleft_{\alpha_1} (\ldots \triangleleft_{\alpha_{n-2}} (v_n u_n \triangleleft_{\alpha_{n-1}} v_1) \ldots) \big) f_1(v_2 u_2, v_2),$$

$$(Q_1 u_2 \cup P_1) v_2 \cup Q_1, v_2 u_2 v_2, (Q_2 u_3 \cup P_2) v_3 \cup Q_2, \ldots,$$

$$\big(Q_{n-1}u_n \cup P_{n-1})v_n \cup Q_{n-1}, v_n u_n v_n\big) = (v_1, Q_1, v_2, \dots, Q_{n-1}, v_n).$$

Theorem 2.3 is proved.

References

1. A. L. Agore, G. Militaru, *Crossed product of groups, applications*, Arab. J. Sci. Eng., **33**, 1–17 (2008).
2. A. L. Agore, D. Fratila, *Crossed product of cyclic groups*, Czech. Math. J., **60**, 889–901 (2010) .
3. F. Ateş, *Some new monoid and group constructions under semidirect product*, Ars Combin., **91**, 203–218 (2009).
4. F. Ateş, A. S. Çevik, *Knit products of some groups and their applications*, Rend. Semin. Mat. Univ. Padova, **2**, 1–12 (2009).
5. E. K. Çetinalp, E. G. Karpuz, F. Ateş, A. S. Çevik, *Two-sided crossed product of groups*, Filomat, **30**, № 4, 1005–1012 (2016).
6. E. K. Çetinalp, E. G. Karpuz, *Iterated crossed product of cyclic groups*, Bull. Iran. Math. Soc., **44**, № 6, 1493–1508 (2018).
7. E. K. Çetinalp, E. G. Karpuz, A. S. Çevik, *Complete rewriting system for Schützenberger product of n groups*, Asian-Eur. J. Math., **12** № 1 (2019).
8. E. K. Çetinalp, *Regularity of n-generalized Schützenberger product of monoids*, J. Balkesir University Institute of Science and Technology, **24**, № 1 (2022).
9. E. K. Çetinalp, *Regularity of iterated crossed product of monoids*, Bull. Int. Math. Virtual Inst., **12**, № 1, 151–158 (2022).
10. G. M. S. Gomes, H. Sezinando, J. E. Pin, *Presentations of the Schützenberger product of n groups*, Commun. Algebra, **34**, № 4, 1213–1235 (2006).
11. J. M. Howie, N. Ruškuc, *Constructions and presentations for monoids*, Commun. Algebra, **22**, 6209–6224 (1994).
12. A. Emin, F. Ateş, S. İkikardeş, İ. N. Cangül, *A new monoid construction under crossed products*, J. Inequal. and Appl., **244** (2013).
13. E. G. Karpuz, F. Ateş, A. S. Çevik, *Regular and π-inverse monoids under Schützenberger products*, Algebras, Groups and Geometries, **27**, 455–471 (2010).
14. E. G. Karpuz, E. K. Çetinalp, *Some remarks on the Schützenberger product of n monoids*, Ric. Mat. (2022); doi/10.1007/s11587-022-00743-z.
15. M. P. Schützenberger, *On finite monoids having only trivial subgroups*, Inform. and Control, **8**, 190–194 (1965).
16. H. Straubing, *A generalization of the Schützenberger product of finite monoids*, Theoret. Comput. Sci., **13**, 137–150 (1981).
17. M. A. Rudkovskii, *Twisted product of Lie groups*, Sib. Math. J., **38**, 1120–1129 (1997).