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SUFFICIENT CONDITIONS AND RADIUS PROBLEMS FOR THE SILVERMAN CLASS

ДОСТАТНІ УМОВИ ТА ЗАДАЧІ ПРО РАДІУС ДЛЯ КЛАСУ СІЛЬВЕРМАНА

For $0<\alpha\leq 1$ and $\lambda>0$, let

$$G_{\lambda,\alpha}=\left\{f\in\mathcal{A}\colon \left|\frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)}-(1-\alpha)\right|<\lambda,\ z\in\mathbb{D}\right\}. \tag{0.1}$$

The general form of the Silverman class introduced by Tuneski and Irmak [Int. J. Math. and Math. Sci., **2006**, Article ID 38089 (2006)]. Our differential inequality formulation lays out several sufficient conditions for this class. Further, we consider a class Ω given by

$$\Omega=\left\{f\in\mathcal{A}\colon |zf'(z)-f(z)|<\frac{1}{2},\ z\in\mathbb{D}\right\}. \tag{0.2}$$

For these two classes, we establish inclusion relations involving some well-known subclasses of $\mathcal{S}^*$ and compute radius estimates featuring various pairings of these classes.

Нехай для $0<\alpha\leq 1$ і $\lambda>0$

$$G_{\lambda,\alpha}=\left\{f\in\mathcal{A}\colon \left|\frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)}-(1-\alpha)\right|<\lambda,\ z\in\mathbb{D}\right\}. \tag{0.1}$$

Загальну форму класу Сільвермана ввели Тунескі та Ірмак [Int. J. Math. and Math. Sci., **2006**, Article ID 38089 (2006)]. Наше формулювання диференціальної нерівності містить кілька достатніх умов для цього класу. Крім того, розглянуто клас Ω, що задається формулою

$$\Omega=\left\{f\in\mathcal{A}\colon |zf'(z)-f(z)|<\frac{1}{2},\ z\in\mathbb{D}\right\}. \tag{0.2}$$

Для цих двох класів встановлено співвідношення включення, що містить деякі відомі підкласи $\mathcal{S}^*$, і обчислено оцінки радіусів для різних пар класів, що вивчаються.

1. Introduction. Let $\mathbb{D}:=\{z\colon |z|<1\}$ be the open unit disk and $\mathcal{H}$ be the class of all analytic functions defined on $\mathbb{D}$. In addition, let $\mathcal{A}_n$ be the class of all normalized analytic functions of the form $f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\ldots$ with $\mathcal{A}:=\mathcal{A}_1$. With the symbol $\mathcal{S}$, we denote the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\mathcal{S}^*$ and $\mathcal{C}$ denote the class of starlike and convex functions, respectively. For two analytic functions f and F, it is said that f is subordinate to F, denoted by $f\prec F$ if there exists a Schwarz function ω such that $f(z)=F(\omega(z))$. Let Φ_M denotes the Ma – Minda class, consisting of the functions ϕ satisfying the following properties: (i) ϕ is analytic and univalent; (ii) ϕ is symmetric with respect to real axis; (iii) ϕ has positive real part in $\mathbb{D}$; (iv) ϕ is starlike with respect to $\phi(0)=1$; (v) $\phi'(0)>0$. For $\phi\in\Phi_M$, Ma and Minda [7] introduced a general subclass of $\mathcal{S}^*$, defined as the class of all the functions $f\in\mathcal{A}$ such that $zf'(z)/f(z)\prec\phi(z)$, denoted by $\mathcal{S}^*(\phi)$. In later years, many authors came up with different subclasses of $\mathcal{S}^*$, which they defined by taking ϕ as a particular Ma – Minda function. Some of the classes which are used in

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the present work are listed as follows: the class $\mathcal{S}^*_L$, introduced by Sokół [17], where ϕ is taken as $\sqrt{1+z}$; the class $\mathcal{S}^*_e$ with $\phi(z)=e^z$ defined by Mendiratta et al. [9]; the class $\mathcal{S}^*_{RL}$ introduced by Mendiratta et al. [18]; the class $\mathcal{S}^*_C$ introduced by Sharma et al. [14], where $\phi(z)=1+4z/3+2z^2/3$; the class $\mathcal{S}^*_S$ introduced by Cho et al. [2], where $\phi(z)=1+\sin z$; the class $\mathcal{S}^*_{Cr}$ associated with the crescent $z+\sqrt{1+z^2}$, introduced by Sharma et al. [15]; the class $\mathcal{S}^*_{SG}$ introduced by Goel and Kumar [4] with $\phi(z)=2/(1+e^{-z})$; the class $\mathcal{S}^*_\wp$ introduced by Kumar and Gangania [5], where ϕ represents a cardioid given by $1+ze^z$; the class $\mathcal{S}^*_{Ne}$ introduced by Wani and Swaminathan [20], where ϕ is taken as $1+z-z^3/3$. In the year 1999, Silverman [16] introduced the following class:

$$G_b=\left\{f\in\mathcal{A}\colon \left|\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}-1\right|<b,\ z\in\mathbb{D}\right\},\quad b>0.$$

The author established conditions on b for which the class G_b is contained in the class of starlike functions and further in the class $\mathcal{S}^*(\alpha)$, the class of starlike functions of order α. In addition, the author estimated the largest radius for which every starlike function of order $1/2$ belongs to G_b. The proofs of these theorems are based on the properties of Schwarz function. Some of the properties are given as follows:

Lemma 1.1 (Schwarz–Pick lemma [6]). *Let* ω *be a function analytic on* $\mathbb{D}$ *such that* $|\omega(z)|\le 1$ *and* $\omega(0)=0$. *Then*, *for all* $z\in\mathbb{D}$,

$$|\omega'(z)|\le\frac{1-|\omega(z)|^2}{1-|z|^2}.$$

Lemma 1.2 [3]. *Let* $\omega:\mathbb{D}\to\mathbb{D}$ *be analytic*, *then*, *for all* $z\in\mathbb{D}$,

$$|\omega'(z)|\le\begin{cases}1, & |z|\le\sqrt{2}-1,\\ \dfrac{(1+r^2)^2}{4r(1-r^2)}, & |z|\ge\sqrt{2}-1.\end{cases}$$

Besides these inequalities, Dieudonné proved a number of others relating to derivatives of Schwarz function [3]. In 2006, the class G_b was generalized by Tuneski and Irmak [19] in the form given by (0.1). By taking $\alpha=1/2$, $G_{\lambda,\alpha}$ reduces to the class G_b with $b=2\lambda$. In 2017, Peng and Zhong [13] introduced a new subclass Ω of $\mathcal{A}$ given by (0.2). For this class, the authors proved that $f\in\Omega$ is equivalent to saying that

$$f(z)=z+\frac{1}{2}z\int_0^z\varphi(\zeta)d\zeta,\tag{1.1}$$

where φ is analytic in $\mathbb{D}$ and $|\varphi(z)|\le 1$, $z\in\mathbb{D}$. They also proved its inclusion in $\mathcal{S}^*$, estimated radius of convexity and discussed many other properties of Ω. In 2019, Peng and Obradović [12] estimated logarithmic and inverse coefficients, proved Robertson's $1/2$ conjecture and $1/2$ theorem and other results related to Hadamard product and coefficient multipliers. Later in this year, Swaminathan and Wani [18] defined a new class $\Omega_n=\big\{f\in\mathcal{A}_n\colon |zf'(z)-f(z)|<1/2,\ z\in\mathbb{D}\big\}$. They obtained sufficient conditions for Ω_n, proved inclusion properties of Ω and derived sharp radii estimates for different subclasses of $\mathcal{S}^*$. Motivated by their work, we consider similar problems for the class $G_{\lambda,\alpha}$. We allot double integral functions to this class by proving sufficient conditions and utilizing the conditions to construct such functions. Further by using the concept of subordination, we prove several inclusion relations between the class $G_{\lambda,\alpha}$, Ω and other well-known subclasses of $\mathcal{S}^*$ mentioned above. We also obtain some radius estimates for the functions belonging to the different forms of $\mathcal{S}^*(\phi)$ ensuring that they are contained in Ω as well as $G_{\lambda,\alpha}$.

2. Main results.

Theorem 2.1. *Let* $f \in \mathcal{A}_n$, $0 \le \alpha < 1$ *and* $\lambda > 0$. *If*

$$\left| zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z} \right) \right| < \delta, \tag{2.1}$$

where δ *is the smallest positive root of*

$$\begin{aligned}\phi(r) :=\ & (1+n)(2\alpha n - \lambda(n+1) - n)r^2 \\ & + n(1-\alpha+n)(2\lambda(n+1) + n + \alpha n^2)r - \lambda n^2(n+1-\alpha)^2,\end{aligned} \tag{2.2}$$

then $f \in G_{\lambda,\alpha}$.

Proof. From (2.1), we have

$$zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z} \right) \prec \delta z, \quad z \in \mathbb{D}.$$

Let $P(z) = f'(z) - f(z)/z$, then $P(0) = 0$ and

$$(1-\alpha)P(z) + zP'(z) = zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z} \right) \prec \delta z.$$

Now applying [10, Theorem 3.1b] for $h(z) = \delta z/(1-\alpha)$ and $\gamma = 1-\alpha$, we obtain

$$P(z) \prec \frac{\delta z}{n+1-\alpha},$$

which is equivalent to

$$f'(z) - \frac{f(z)}{z} \prec \frac{\delta z}{n+1-\alpha}. \tag{2.3}$$

Now let us suppose $p(z) = f(z)/z$, then from (2.3)

$$zp'(z) = f'(z) - \frac{f(z)}{z} \prec \frac{\delta z}{n+1-\alpha}. \tag{2.4}$$

Now by using [10, Lemma 8.2a], we get

$$p(z) = \frac{f(z)}{z} \prec 1 + \frac{\delta z}{n(n+1-\alpha)},$$

which further yields the inequality

$$1 - \frac{\delta}{n(n+1-\alpha)} < \left| \frac{f(z)}{z} \right| < 1 + \frac{\delta}{n(n+1-\alpha)}. \tag{2.5}$$

From (2.4), it is clear that

$$\left|f'(z)-\frac{f(z)}{z}\right|<\frac{\delta}{n+1-\alpha}, \tag{2.6}$$

which further implies

$$|f'(z)|>\left|\frac{f(z)}{z}\right|-\frac{\delta}{n+1-\alpha}. \tag{2.7}$$

From (2.5) and (2.7), we may conclude that

$$|f'(z)|>1-\frac{\delta(n+1)}{n(n+1-\alpha)}. \tag{2.8}$$

From (2.1), we have

$$\left|f'(z)\left(\frac{zf''(z)}{f'(z)}\right)-\alpha\left(f'(z)-\frac{f(z)}{z}\right)\right|<\delta. \tag{2.9}$$

Now from (2.8) and (2.9), we observe that

$$\left(1-\frac{\delta(n+1)}{n(n+1-\alpha)}\right)\left|\frac{zf''(z)}{f'(z)}\right|<|f'(z)|\left|\frac{zf''(z)}{f'(z)}\right|<\delta+\alpha\left|f'(z)-\frac{f(z)}{z}\right|. \tag{2.10}$$

Here, we may note that δ is the smaller of the two roots of $\phi(r)$, which is given by (2.2). So, we get

$$\delta=\frac{1}{2(n+1)(\lambda(n+1)+n-2\alpha n)}\Big(n(n+1-\alpha)(n+\alpha n^2+2\lambda(n+1)$$
$$-\sqrt{n^2+\alpha^2n^4+2\alpha n^3+8\alpha\lambda n+12\alpha\lambda n^2+4\alpha\lambda n^3})\Big).$$

Since

$$\left(n+\alpha n^2+2\lambda(n+1)-\sqrt{n^2+\alpha^2n^4+2\alpha n^3+8\alpha\lambda n+12\alpha\lambda n^2+4\alpha\lambda n^3}\right)$$
$$\times\left(n+\alpha n^2+2\lambda(n+1)+\sqrt{n^2+\alpha^2n^4+2\alpha n^3+8\alpha\lambda n+12\alpha\lambda n^2+4\alpha\lambda n^3}\right)$$
$$=4\lambda(n+1)(\lambda(n+1)+n-2\alpha n),$$

we have

$$\delta=\frac{2\lambda n(n+1-\alpha)}{n+\alpha n^2+2\lambda(n+1)+\sqrt{n^2+\alpha^2n^4+2\alpha n^3+8\alpha\lambda n+12\alpha\lambda n^2+4\alpha\lambda n^3}}$$
$$\le\frac{2\lambda n(n+1-\alpha)}{2\lambda(n+1)}.$$

Therefore,

$$1-\frac{\delta(n+1)}{n(n+1-\alpha)}>0$$

and, thus, (2.10) implies

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{\delta + \dfrac{\alpha\delta}{n+1-\alpha}}{1 - \dfrac{\delta(n+1)}{n(n+1-\alpha)}} = \frac{n(n+1)\delta}{n(n+1-\alpha) - \delta(n+1)}. \tag{2.11}$$

Now let us consider the following inequality:

$$\begin{aligned}
&\left(1 - \frac{\delta(n+1)}{n(n+1-\alpha)}\right)\left|\alpha\frac{f(z)f''(z)}{(f'(z))^2} - (1-\alpha) + (1-\alpha)\frac{f(z)}{zf'(z)}\right| \\
&< |f'(z)|\left|\alpha\frac{f(z)f''(z)}{(f'(z))^2} - (1-\alpha) + (1-\alpha)\frac{f(z)}{zf'(z)}\right| \\
&= \left|\alpha\frac{f(z)f''(z)}{f'(z)} - (1-\alpha)\left(f'(z) - \frac{f(z)}{z}\right)\right| \\
&< \alpha\left|\frac{f(z)}{z}\right|\left|\frac{zf''(z)}{f'(z)}\right| + (1-\alpha)\left|f'(z) - \frac{f(z)}{z}\right|.
\end{aligned}$$

By using (2.5), (2.6) and (2.11) in the above inequality, we get

$$\begin{aligned}
&\left(1 - \frac{\delta(n+1)}{n(n+1-\alpha)}\right)\left|\alpha\frac{f(z)f''(z)}{(f'(z))^2} - (1-\alpha) + (1-\alpha)\frac{f(z)}{zf'(z)}\right| \\
&< \alpha\left(1 + \frac{\delta}{n(n+1-\alpha)}\right)\left(\frac{n(n+1)\delta}{n(n+1-\alpha) - \delta(n+1)}\right) + (1-\alpha)\left(\frac{\delta}{n+1-\alpha}\right) =: \tau,
\end{aligned}$$

which implies

$$\left|\alpha\frac{f(z)f''(z)}{(f'(z))^2} - (1-\alpha) + (1-\alpha)\frac{f(z)}{zf'(z)}\right| < \left(\frac{n(n+1-\alpha)}{n(n+1-\alpha) - \delta(n+1)}\right)\tau = \lambda.$$

Thus, we have

$$\left|\frac{1 - \alpha + \alpha zf''(z)/f'(z)}{zf'(z)/f(z)} - (1-\alpha)\right| < \lambda.$$

Theorem 2.1 is proved.

***Corollary* 2.1.** *Let* $0 \le \alpha < 1$, $\lambda > 0$ *and* $g \in \mathcal{H}$. *If* $|g(z)| < \delta$, *where* δ *is the smallest positive root of*

$$\phi(r) := (1+n)(2\alpha n - \lambda(n+1) - n)r^2 + n(1-\alpha+n)(2\lambda(n+1) + n + \alpha n^2)r - \lambda n^2(n+1-\alpha)^2,$$

then

$$f(z) = z + z^{n+1}\int_0^1\int_0^1 g(rsz)r^{n-\alpha}s^{n-1}\,dr\,ds$$

is in $G_{\lambda,\alpha}$.

Proof. Suppose that $f(z)$ satisfies the differential equation

$$zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right) = z^n g(z). \tag{2.12}$$

Let

$$H(z) = f'(z) - \frac{f(z)}{z},$$

then, from (2.12), we have

$$(1-\alpha)H(z) + zH'(z) = z^n g(z).$$

Now applying [10, Theorem 3.1b], we obtain the solution of the above differential equation, given by

$$H(z) = \frac{1}{z^{1-\alpha}} \int\limits_0^z g(t)t^{n-\alpha}dt.$$

Now if we substitute $t = rz$ in the above equation, then

$$H(z) = z^n \int\limits_0^1 g(rz)r^{n-\alpha}dr.$$

Taking $h(z) = f(z)/z$, we have

$$zh'(z) = f'(z) - \frac{f(z)}{z} = H(z).$$

Now by using [10, Lemma 8.2a], we obtain

$$h(z) = 1 + \int\limits_0^z \frac{H(t)}{t}dt.$$

Substituting $t = sz$, we get

$$\begin{aligned} h(z) &= 1 + \int\limits_0^1 \frac{H(sz)}{s}ds \\ &= 1 + \int\limits_0^1 \left(\frac{(sz)^n}{s} \int\limits_0^1 g(rsz)r^{n-\alpha}dr\right)ds \\ &= 1 + z^n \int\limits_0^1 \int\limits_0^1 g(rsz)r^{n-\alpha}s^{n-1}drds. \end{aligned}$$

Thus,

$$f(z) = z + z^{n+1} \int\limits_0^1 \int\limits_0^1 g(rsz)r^{n-\alpha}s^{n-1}drds.$$

Now using Theorem 2.1 along with the fact that $|g(z)| < \delta$, we have that $f \in G_{\lambda,\alpha}$.

***Corollary* 2.2.** *Let $f \in \mathcal{A}$ satisfies*

$$\left|zf''(z) - \frac{1}{2}\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{3}{8}\big(5 - \sqrt{21}\,\big), \tag{2.13}$$

then $z(zf'(z)/f(z))$ is univalent in $\mathbb{D}$.

Proof. If we take $n = 1$, $\alpha = 1/2$ and $\delta = 3\big(5 - \sqrt{21}\,\big)/8$ in Theorem 2.1, then (2.13) implies that $f \in G_{\frac{1}{4},\frac{1}{2}}$. We know that $G_{\frac{1}{4},\frac{1}{2}} = G_{\frac{1}{2}}$, and, thus, by using [11, Theorem 2], the result follows.

Theorem 2.2. *Let $f \in \mathcal{A}_n$, $0 \le \alpha < 1$ and $\lambda > 0$. If*

$$\big|zf''(z) - \alpha(f'(z) - 1)\big| < \frac{\delta(n+1)(n-\alpha)}{\alpha + (n+1)(n-\alpha)}, \quad z \in \mathbb{D}, \tag{2.14}$$

where δ is the smallest positive root of

$$\phi(r) := (1+n)(2\alpha n - \lambda(n+1) - n)r^2 + n(1-\alpha+n)(2\lambda(n+1) + n + \alpha n^2)r - \lambda n^2(n+1-\alpha)^2,$$

then $f \in G_{\lambda,\alpha}$.

Proof. From (2.14), we obtain, for $z \in \mathbb{D}$,

$$zf''(z) - \alpha(f'(z) - 1) \prec \frac{\delta(n+1)(n-\alpha)z}{\alpha + (n+1)(n-\alpha)}.$$

Let $P(z) = f'(z) - (1+\alpha)f(z)/z$, then

$$P(z) + zP'(z) = zf''(z) - \alpha f'(z) \prec \frac{\delta(n+1)(n-\alpha)z}{\alpha + (n+1)(n-\alpha)} - \alpha.$$

By using Lemma [10, Theorem 3.1b], we get

$$P(z) \prec \frac{\delta(n-\alpha)z}{\alpha + (n+1)(n-\alpha)} - \alpha,$$

which further implies

$$f'(z) - (1+\alpha)\frac{f(z)}{z} \prec \frac{\delta(n-\alpha)z}{\alpha + (n+1)(n-\alpha)} - \alpha.$$

Now let us take

$$p(z) = \frac{f(z)}{z} - 1 \qquad \text{and} \qquad q(z) = \frac{\delta z}{\alpha + (n+1)(n-\alpha)}.$$

It is easy to observe that $q(0) = 0$, $q'(0) \neq 0$ and $\operatorname{Re}\left(1 + \dfrac{zq''(z)}{q'(z)}\right) = 1 > \dfrac{\alpha}{n}$. Next, we observe

$$\begin{aligned} zp'(z) - \alpha p(z) &= f'(z) - (1+\alpha)\frac{f(z)}{z} + \alpha \\ &\prec \frac{\delta(n-\alpha)z}{\alpha + (n+1)(n-\alpha)} = nzq'(z) - \alpha q(z). \end{aligned}$$

Then, by using Lemma [10, Lemma 8.2a], we have

$$\frac{f(z)}{z} - 1 = p(z) \prec q(z) = \frac{\delta z}{\alpha + (n+1)(n-\alpha)},$$

which implies

$$\left|\frac{f(z)}{z} - 1\right| < \frac{\delta}{\alpha + (n+1)(n-\alpha)}. \tag{2.15}$$

Finally, from (2.14) and (2.15), we obtain

$$\left|zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| \le \left|zf''(z) - \alpha(f'(z) - 1)\right| + \alpha\left|\frac{f(z)}{z} - 1\right|$$
$$< \frac{\delta(n+1)(n-\alpha)}{\alpha + (n+1)(n-\alpha)} + \frac{\alpha\delta}{\alpha + (n+1)(n-\alpha)} = \delta.$$

Applying Theorem 2.1, we get the result.

***Corollary* 2.3.** *Let* $0 \le \alpha < 1$, $\lambda > 0$ *and* $g \in \mathcal{H}$. *If*

$$|g(z)| < \frac{\delta(n+1)(n-\alpha)}{\alpha + (n+1)(n-\alpha)}, \quad z \in \mathbb{D},$$

where δ *is the smallest positive root of*

$$\phi(r) := (1+n)(2\alpha n - \lambda(n+1) - n)r^2 + n(1-\alpha+n)(2\lambda(n+1) + n + \alpha n^2)r - \lambda n^2(n+1-\alpha)^2,$$

then

$$f(z) = z + z^{n+1} \int_0^1 \int_0^1 g(rsz) r^{n-1-\alpha} s^n \, dr ds$$

is in $G_{\lambda,\alpha}$.

Proof. Suppose that $f \in \mathcal{A}_n$ satisfies

$$zf''(z) - \alpha(f'(z) - 1) = z^n g(z).$$

Taking $H(z) = f'(z) - 1$, we reduce the above equation to

$$zH'(z) - \alpha H(z) = z^n g(z).$$

By using [10, Theorem 3.1b], we have the solution of the above differential equation as follows:

$$H(z) = z^\alpha \int_0^z g(t) t^{n-\alpha-1} dt.$$

Taking $t = rz$, we reduce it to

$$H(z) = z^n \int_0^1 g(rz) r^{n-\alpha-1} dr$$

and, thus,

$$f(z)=z+z^{n+1}\int\limits_0^1\int\limits_0^1 g(rsz)r^{n-1-\alpha}s^n drds.$$

By Theorem 2.2, the result follows.

Theorem 2.3. *Let* $f\in G_{\lambda,\alpha}$ $(\lambda>0,\ 1/3<\alpha\le 1)$. *Then* $zf'(z)/f(z)\prec 1/(1\pm cz)$, *where* $c=\lambda/(3\alpha-1)$, *and the result is sharp.*

Proof. Let $p(z)=zf'(z)/f(z)=1/(1+c\omega(z))$. Then

$$\begin{aligned}\left|\frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)}-(1-\alpha)\right| &= \left|\frac{1-2\alpha}{p(z)}+\frac{\alpha zp'(z)}{p^2(z)}+2\alpha-1\right|\\ &= \big|(1-2\alpha)c\omega(z)-\alpha cz\omega'(z)\big|.\end{aligned}$$

Now we show that $|\omega(z)|<1$ for $z\in\mathbb{D}$. Suppose that on contrary there exists a point $z_0\in\mathbb{D}$ such that $|\omega(z_0)|=1$ and $z_0\omega'(z_0)=k\omega(z_0)$, $k\ge 1$. Then

$$\begin{aligned}\left|\frac{1-2\alpha}{p(z_0)}+\frac{\alpha zp'(z_0)}{p^2(z_0)}+2\alpha-1\right| &= |\omega(z_0)c(1-\alpha(k+2))|\\ &= \left|\frac{\lambda}{1-3\alpha}(1-\alpha(k+2))\right|>\lambda,\end{aligned}$$

which is a contradiction to the assumption that $f\in G_{\lambda,\alpha}$. For the function $f(z)=z/(1\pm cz)$, we obtain that $zf'(z)/f(z)=1/(1\pm cz)$ and

$$\left|\frac{1-2\alpha}{p(z)}+\frac{\alpha zp'(z)}{p^2(z)}+2\alpha-1\right|=\lambda.$$

***Remark* 2.1.** For $\alpha=1/2$, $G_{\lambda,\alpha}$ reduces to the class G_b defined by Silverman and the result above reduces to [11, Theorem 1] with $b=2\lambda$.

***Remark* 2.2.** For $\alpha=1$, $G_{\lambda,\alpha}$ reduces to the class $G_{\lambda,1}$ defined by Tuneski and the result above reduces to [1, Theorem 3.1] with $h(z)=\lambda z$.

Theorem 2.4. *Let* $\lambda>0$ *and* $1/3<\alpha<1$ *such that* $\lambda<\big(2-\sqrt{3}\big)(3\alpha-1)$. *Then* $G_{\lambda,\alpha}\subset\Omega$.

Proof. Let $f\in G_{\lambda,\alpha}$, then Theorem 2.3 implies that

$$\frac{zf'(z)}{f(z)}\prec\frac{1}{1+cz}=:\phi_0(z),\quad \text{where}\quad c=\frac{\lambda}{3\alpha-1}.$$

By the structural formula, we know that $f\in\mathcal{S}^*(\phi_0)$ if and only if there exists a function $\phi(z)\prec\phi_0(z)$ such that

$$f(z)=z\exp\int\limits_0^z\frac{\phi(t)-1}{t}dt.$$

Taking $\phi(z)=\phi_0(z)$, we obtain the extremal function for the class $\mathcal{S}^*(\phi_0)$, given by $f_0(z)=z/(1+cz)$. Then, by the growth theorem, we have $|f(z)|\le f_0(r)$ on $|z|=r$. Hence, *Table* 1. Radii of the smallest disk with centered at 1, inscribed in $\mathcal{S}^*(\phi)$

ϕ	$\mathcal{S}^*(\phi)$	r_1	Reference	ϕ	$\mathcal{S}^*(\phi)$	r_1	Reference
$\dfrac{2}{1+e^{-z}}$	$\mathcal{S}^*_{SG}$	$\dfrac{e-1}{e+1}$	[4]	$1+z-\dfrac{z^3}{3}$	$\mathcal{S}^*_{Ne}$	$\dfrac{2}{3}$	[20]
e^z	$\mathcal{S}^*_e$	$1-\dfrac{1}{e}$	[9]	$1+\dfrac{4}{3}z+\dfrac{2}{3}z^2$	$\mathcal{S}^*_C$	$\dfrac{2}{3}$	[14]
$1+\sin z$	$\mathcal{S}^*_S$	$\sin 1$	[2]	$z+\sqrt{1+z^2}$	$\mathcal{S}^*_{Cr}$	$2-\sqrt{2}$	[15]
$\sqrt{1+z}$	$\mathcal{S}^*_L$	$\sqrt{2}-1$	[17]	$1+ze^z$	$\mathcal{S}^*_{\wp}$	$\dfrac{1}{e}$	[5]

$$|zf'(z)-f(z)| = |f(z)|\left|\frac{zf'(z)}{f(z)}-1\right| \le |f_0(1)|\left|\frac{-cz}{1+cz}\right| \le \frac{c}{(1-c)^2}.$$

We get $c=\lambda/(3\alpha-1)<2-\sqrt{3}$, which further implies that

$$|zf'(z)-f(z)| \le \frac{c}{(1-c)^2} < \frac{1}{2}.$$

Theorem 2.4 is proved.

Lemma 2.1. *Let* $\lambda>0,\ 1/3<\alpha<1$ *and* $\phi\in\Phi_M$ *with* $\phi(\mathbb{D})=\Delta$. *Then* $G_{\lambda,\alpha}\subset\mathcal{S}^*(\phi)$, *whenever* $(1+r_1)\lambda<(3\alpha-1)r_1$, *where* r_1 *is the radius of the largest disk contained in* Δ *and centered at* 1.

Proof. Let $f\in G_{\lambda,\alpha}$. Then, from the proof of Theorem 2.4, we obtain

$$\left|\frac{zf'(z)}{f(z)}-1\right| < \frac{c}{1-c} \quad \text{with} \quad c=\frac{\lambda}{3\alpha-1}.$$

Since $(1+r_1)\lambda<(3\alpha-1)r_1$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| < \frac{c}{1-c} = \frac{\lambda}{3\alpha-\lambda-1} < r_1.$$

Therefore, $zf'(z)/f(z)$ lies in Δ and, hence, $f\in\mathcal{S}^*(\phi)$.

Theorem 2.5. *The class* $G_{\lambda,\alpha}$ $(\lambda>0,\ 1/3<\alpha<1)$ *satisfies the following inclusion relations*:

(i) $G_{\lambda,\alpha}\subset\mathcal{S}^*_{SG}$, *whenever* $2\lambda e<(e-1)(3\alpha-1)$;

(ii) $G_{\lambda,\alpha}\subset\mathcal{S}^*_e$, *whenever* $(2e-1)\lambda<(e-1)(3\alpha-1)$;

(iii) $G_{\lambda,\alpha}\subset\mathcal{S}^*_S$, *whenever* $(1+\sin(1))\lambda e<(1+\sin(1))(3\alpha-1)$;

(iv) $G_{\lambda,\alpha}\subset\mathcal{S}^*_L$, *whenever* $\sqrt{2}\lambda<\big(\sqrt{2}-1\big)(3\alpha-1)$;

(v) $G_{\lambda,\alpha}\subset\mathcal{S}^*_{Ne}$, *whenever* $5\lambda<2(3\alpha-1)$;

(vi) $G_{\lambda,\alpha}\subset\mathcal{S}^*_C$, *whenever* $5\lambda<2(3\alpha-1)$;

(vii) $G_{\lambda,\alpha}\subset\mathcal{S}^*_{Cr}$, *whenever* $(3-\sqrt{2})\lambda<(2-\sqrt{2})(3\alpha-1)$;

(viii) $G_{\lambda,\alpha}\subset\mathcal{S}^*_P$, *whenever* $(e+1)\lambda<(3\alpha-1)$.

Proof. For different choices of ϕ with respective values of r_1 (refer to Table 1), we apply Lemma 2.1 and the result follows directly.

Theorem 2.6. *If* $f \in \Omega$, *then* $f \in G_{\frac{1}{2},\frac{1}{2}}$ *in the disc* $|z| < r_0$, *where* $r_0 \approx 0.430496$ *is the smallest positive root of*

$$\begin{aligned}55r^{12} - 28r^{11} - 854r^{10} + 148r^9 + 2969r^8 - 212r^7 \\ - 4286r^6 + 28r^5 + 2875r^4 + 96r^3 - 888r^2 - 32r + 96 = 0.\end{aligned}$$

Proof. Let $f \in \Omega$, then f can be written in the from (1.1). Now let $\omega(z) = \int_0^z \varphi(\zeta)d\zeta$, then clearly $\omega(z)$ and $\omega'(z)$ are analytic in $\mathbb{D}$ and we can write f as

$$f(z) = z + \frac{1}{2}z\omega(z). \tag{2.16}$$

Now by using the properties of φ, we have

$$|\omega(z)| = \left|\int\limits_0^z \varphi(\zeta)d\zeta\right| \le \int\limits_0^z |\varphi(\zeta)|d\zeta \le |z|$$

and

$$|\omega'(z)| = |\varphi(z)| \le 1.$$

By using Schwarz – Pick lemma, we get, for $z \in \mathbb{D}$,

$$|\omega''(z)| \le \frac{1 - |\omega'(z)|^2}{1 - |z|^2}. \tag{2.17}$$

Dieudonné [3] proved certain results which yield the inequalities

$$|\omega'(z)| \ge \frac{(|\omega(z)| - r^2)(1 + |\omega(z)|)}{r(1 - r^2)} \tag{2.18}$$

and

$$|z\omega'(z) - \omega(z)| \le \frac{r^2 - |\omega(z)|^2}{1 - r^2} \tag{2.19}$$

on $|z| = r$, where $|\omega(z)| \le r$. In view of (2.16), we obtain

$$\begin{aligned}\left|\frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} - 1\right| &= \left|\frac{z\big(z(\omega(z) + 2)\omega''(z) - z\omega'(z)^2 + (\omega(z) + 2)\omega'(z)\big)}{(z\omega'(z) + \omega(z) + 2)^2}\right| \\ &\le \frac{r((2 + |z\omega'(z) - \omega(z)|)|\omega'(z)|) + r|\omega''(z)|(2 + |\omega(z)|)}{(2(1 - |\omega(z)|) - |z\omega'(z) - \omega(z)|)^2}.\end{aligned}$$

By using inequalities (2.17), (2.18), and (2.19), we have

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}-1\right| \le \frac{r}{\left(2(1-|\omega(z)|)-\left(\frac{r^2-|\omega(z)|^2}{1-r^2}\right)\right)^2}\left[\left(2+\left(\frac{r^2-|\omega(z)|^2}{1-r^2}\right)\right)\right.$$

$$\left.\times\left(\frac{1-|\omega(z)|^2}{1-r^2}\right)+r\left(\frac{1-\left(\frac{(|\omega(z)|-r^2)(1+|\omega(z)|)}{r(1-r^2)}\right)^2}{1-r^2}\right)(2+|\omega(z)|)\right].$$

Writing $|\omega(z)|=\omega$, we get

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}-1\right| \le \frac{1}{(1-r^2)(2r^2\omega-3r^2+\omega^2-2\omega+2)^2}\Big(r^6\omega+2r^6-r^5\omega^2$$

$$+r^5-r^4\omega^3-4r^4\omega^2-7r^4\omega-6r^4-r^3\omega^4+4r^3\omega^2$$

$$-3r^3+2r^2\omega^4+8r^2\omega^3+10r^2\omega^2+5r^2\omega+2r^2$$

$$+r\omega^4-3r\omega^2+2r-\omega^5-4\omega^4-5\omega^3-2\omega^2\Big).$$

For f to be in $G_{\frac{1}{2},\frac{1}{2}}$, it suffices to show that

$$\frac{1}{(1-r^2)(2r^2\omega-3r^2+\omega^2-2\omega+2)^2}\Big(r^6\omega+2r^6-r^5\omega^2+r^5$$

$$-r^4\omega^3-4r^4\omega^2-7r^4\omega-6r^4-r^3\omega^4+4r^3\omega^2-3r^3+2r^2\omega^4$$

$$+8r^2\omega^3+10r^2\omega^2+5r^2\omega+2r^2+r\omega^4-3r\omega^2+2r-\omega^5-4\omega^4-5\omega^3-2\omega^2\Big)<1,$$

which is equivalent to

$$\Phi(\omega,r):=\omega^5+\big(r^3-3r^2-r+5\big)\omega^4+\big(1-3r^4\big)\omega^3+(-4r^6+r^5$$

$$+22r^4-4r^3-32r^2+3r+10)\omega^2+\big(11r^6-25r^4+23r^2-8\big)\omega$$

$$-11r^6-r^5+27r^4+3r^3-18r^2-2r+4>0.$$

We may note that $\omega=|\omega(z)|\le|z|=r$, so we have $0\le\omega\le r$. Let us write

$$A=-4r^6+r^5+22r^4-4r^3-32r^2+3r+10,$$

$$B=11r^6-25r^4+23r^2-8,$$

$$C=-11r^6-r^5+27r^4+3r^3-18r^2-2r+4,$$

then $B^2-4AC<0$, whenever $r<r_1\approx 0.430496$. Also $A>0$, whenever $r<r_2\approx 0.565244$. Thus,

$$(-4r^6+r^5+22r^4-4r^3-32r^2+3r+10)\omega^2$$
$$+\big(11r^6-25r^4+23r^2-8\big)\omega-11r^6-r^5+27r^4+3r^3-18r^2-2r+4>0,$$

whenever $r<\min\{r_1,r_2\}=r_1$. Next we observe that coefficients of ω^5 and ω^4 are always positive and coefficient of ω^3 is positive for the range $0\le r<r_3=(1/3)^{1/4}\approx 0.759836$. It can be easily concluded that

$$\Phi(\omega,r)>0 \quad \text{whenever} \quad r<r_0=\min\{r_1,r_2,r_3\}=r_1.$$

Theorem 2.6 is proved.

Theorem 2.7. *If* $f\in\mathcal{S}^*_e$, *then* $f\in\Omega$ *in the disc* $|z|<r_0$, *where* $r_0\approx 0.476813$ *is the smallest positive root of* $2(e^r-1)f_0(r)-1=0$, *where*

$$f_0(z)=z\exp\left(\int\limits_0^z\frac{e^t-1}{t}dt\right)=z+z^2+\frac{3z^3}{4}+\frac{17z^4}{36}+\frac{19z^5}{72}+\ldots. \tag{2.20}$$

Moreover, this estimate is sharp.

Proof. Let $f\in\mathcal{S}^*_e$. Then $zf'(z)/f(z)\prec e^z$, which further implies that

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le\max_{|z|=r}|e^{re^{i\theta}}-1|=e^r-1.$$

We apply [9, Theorem 2.7] on f and obtain that $|f(z)|\le f_0(r)$ $(|z|=r)$, where f_0 is given by (2.20). So,

$$\big|zf'(z)-f(z)\big|\le|f(z)|\left|\frac{zf'(z)}{f(z)}-1\right|\le f_0(r)(e^r-1) \quad \text{on} \quad |z|=r.$$

Taking $|z|<r_0$, we have $|zf'(z)-f(z)|<1/2$ and, for the function f_0, the inequality holds only in the disk $|z|<r_0$, therefore, result is sharp.

Theorem 2.8. *If* $f\in\mathcal{S}^*_{Cr}$, *then* $f\in\Omega$ *in the disc* $|z|<r_0$, *where* $r_0\approx 0.485894$ *is the smallest positive root of* $2(r+\sqrt{1+r^2}-1)f_0(r)-1=0$, *where*

$$f_0(z)=z\exp\left(\int\limits_0^z\frac{t+\sqrt{1+t^2}-1}{t}dt\right)=z+z^2+\frac{3z^3}{4}+\frac{5z^4}{12}+\frac{z^5}{6}+\ldots. \tag{2.21}$$

This result is sharp.

Proof. Let $f\in\mathcal{S}^*_{Cr}$. Then $zf'(z)/f(z)\prec z+\sqrt{1+z^2}$, which is sufficient to say that

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le\max_{|z|=r}\left|e^{i\theta}+\sqrt{1+r^2e^{2i\theta}}-1\right|=r+\sqrt{1+r^2}-1.$$

By using [15, Theorem 1], we obtain $|f(z)|\le f_0(r)$ $(|z|=r)$, where f_0 is given by (2.21). So on $|z|=r$, we have

$$\big|zf'(z)-f(z)\big|\le|f(z)|\left|\frac{zf'(z)}{f(z)}-1\right|\le f_0(r)\Big(r+\sqrt{1+r^2}-1\Big),$$

which is less that 1/2, provided $r<r_0$. For the function f_0, the inequality holds only in the disk $|z|<r_0$, therefore, result is sharp.

Theorem 2.9. *If* $f \in \mathcal{S}^*_{SG}$, *then* $f \in \Omega$ *in the disc* $|z| < r_0$, *where* $r_0 \approx 0.799269$ *is the smallest positive root of* $2\tan(r/2)f_0(r) - 1 = 0$, *where*

$$f_0(z) = z\exp\left(\int\limits_0^z \frac{e^t - 1}{t(e^t + 1)}dt\right) = z + \frac{z^2}{2} + \frac{z^3}{8} + \frac{z^4}{144} - \frac{5z^5}{1152} + \ldots . \tag{2.22}$$

Proof. Let $f \in \mathcal{S}^*_{SG}$, we get $zf'(z)/f(z) \prec 2/(1 + e^{-z})$. Therefore,

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \max_{|z|=r}\left|\frac{e^{re^{i\theta}} - 1}{e^{re^{i\theta}} - 1}\right| = \tan(r/2).$$

Applying [4, Theorem 1.1], we obtain $|f(z)| \leq f_0(r)$ $(|z| = r)$, where f_0 is given by (2.22). So on $|z| = r$

$$|zf'(z) - f(z)| \leq |f(z)|\left|\frac{zf'(z)}{f(z)} - 1\right| \leq f_0(r)\tan(r/2) < 1/2,$$

whenever $r < r_0$.

Theorem 2.9 is proved.

Theorem 2.10. *If* $f \in \mathcal{S}^*_S$, *then* $f \in \Omega$ *in the disc* $|z| < r_0$, *where* $r_0 \approx 0.531721$ *is the smallest positive root of* $2\sinh 1 f_0(r) - 1 = 0$, *where*

$$f_0(z) = z\exp\left(\int\limits_0^z \frac{\sin t}{t}dt\right) = z + \frac{z^2}{2} + \frac{z^3}{8} + \frac{z^4}{144} - \frac{5z^5}{1152} + \ldots . \tag{2.23}$$

Proof. Let $f \in \mathcal{S}^*_S$. Then $zf'(z)/f(z) \prec 1 + \sin z$, which further implies that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \max_{|z|=r}|\sin re^{i\theta}| = \sinh r.$$

Now by using the growth theorem given for $\mathcal{S}^*_S$ in [2], we have $|f(z)| \leq f_0(r)$ $(|z| = r)$, where f_0 is given by (2.23). Therefore,

$$|zf'(z) - f(z)| \leq |f(z)|\left|\frac{zf'(z)}{f(z)} - 1\right| \leq f_0(r)\sinh r \quad \text{on} \quad |z| = r.$$

Hence, $|zf'(z) - f(z)| \leq \sinh 1 f_0(r) < 1/2$, provided $r < r_0$.

Theorem 2.10 is proved.

Theorem 2.11. *If* $f \in \mathcal{S}^*_\wp$, *then* $f \in \Omega$ *in the disc* $|z| < r_0$, *where* $r_0 \approx 0.43384$ *is the smallest positive root of* $2r^2e^{e^r+r-1} - 1 = 0$. *This result is sharp.*

Proof. Let $f \in \mathcal{S}^*_\wp$, then we obtain $zf'(z)/f(z) \prec 1 + ze^z$, which further implies

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \max_{|z|=r}|ze^z| = \max_{0\leq\theta<2\pi}|re^{i\theta}e^{re^{i\theta}}| = re^r.$$

Now by using [9, Theorem 2.2(ii)], we get $|f(z)| \leq re^{e^r-1}$ on $|z| = r$. Finally, we have

$$|zf'(z) - f(z)| = |f(z)|\left|\frac{zf'(z)}{f(z)} - 1\right| \leq re^{e^r-1}(re^r) \quad \text{on} \quad |z| = r.$$

Since $r < r_0$, $|zf'(z) - f(z)| < 1/2$ and, thus, $f \in \Omega$. The result is sharp as for the function $f_0(z) = ze^{e^z-1}$, the inequality holds only in the disk $|z| < r_0$.

Theorem 2.12. *If* $f \in \mathcal{S}^*_{RL}$, *then* $f \in \Omega$ *in the disc* $|z| < r_0$, *where* $r_0 \approx 0.768$ *is the smallest positive root of* $2(\phi_0(-r) - 1)f_0(r) - 1 = 0$, *where*

$$\phi_0(z) = \sqrt{2} - \big(\sqrt{2} - 1\big)\sqrt{\frac{1 - z}{1 + 2\big(\sqrt{2} - 1\big)z}} \tag{2.24}$$

and

$$f_0(z) = z\left(\frac{\sqrt{1 - z} + \sqrt{1 + 2\big(\sqrt{2} - 1\big)z}}{2}\right)^{2\left(\sqrt{2}-1\right)} \exp\left(p_0(z)\right) \tag{2.25}$$

with

$$p_0(z) = \sqrt{2\big(\sqrt{2} - 1\big)}\tan^{-1}\left(\sqrt{2\big(\sqrt{2} - 1\big)}\left(\frac{\sqrt{1 + 2\big(\sqrt{2} - 1\big)z} - \sqrt{1 - z}}{\sqrt{1 + 2\big(\sqrt{2} - 1\big)z} + 2\big(\sqrt{2} - 1\big)\sqrt{1 - z}}\right)\right).$$

Proof. Let $f \in \mathcal{S}^*_{RL}$, then $zf'(z)/f(z) \prec \phi_0(z)$ (given by (2.24)). So on $|z| = r$

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \max_{0\le\theta<2\pi} |\phi_0(re^{i\theta}) - 1| = \sqrt{2} - \big(\sqrt{2} - 1\big)\sqrt{\frac{1 + r}{1 - 2\big(\sqrt{2} - 1\big)r}} - 1.$$

By using [8, Theorem 2.2(ii)], we get $|f(z)| \le |f_0(r)|$ on $|z| = r$, where f_0 is given by (2.25). Therefore,

$$\big|zf'(z) - f(z)\big| = |f(z)|\left|\frac{zf'(z)}{f(z)} - 1\right| \le (\phi_0(-r) - 1)f_0(r) < \frac{1}{2},$$

provided $r < r_0$.

Theorem 2.12 is proved.

Theorem 2.13. *If* $f \in \mathcal{S}^*_L$, *then* $f \in \Omega$ *in the disc* $|z| < r_0$, *where* $r_0 \approx 0.734453$ *is the positive root of*

$$8r\big(1 - \sqrt{1 - r}\big)\exp\big(2\sqrt{1 + r} - 2\big) - \big(1 + \sqrt{1 + r}\big)^2 = 0. \tag{2.26}$$

Proof. Let $f \in \mathcal{S}^*_L$, which implies that $zf'(z)/f(z) \prec \sqrt{1 + z}$. Thus, on $|z| = r$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \max_{|z|=r}\big|\sqrt{1 + z} - 1\big| = \max_{0\le\theta<2\pi}\left|\sqrt{1 + re^{i\theta}} - 1\right| = 1 - \sqrt{1 - r}.$$

Applying the growth theorem on f, we obtain

$$|f(z)| \le \frac{4r\exp\big(2\sqrt{1 + r} - 2\big)}{\big(1 + \sqrt{1 + r}\big)^2} \quad \text{on} \quad |z| = r.$$

We observe that

$$|zf'(z) - f(z)| = |f(z)|\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{4r\big(1 - \sqrt{1 - r}\,\big)\exp\big(2\sqrt{1 + r} - 2\big)}{\big(1 + \sqrt{1 + r}\,\big)^2},$$

which is less that $1/2$, provided $r < r_0$. Therefore, $f \in \Omega$.

Theorem 2.14. *If* $f \in \mathcal{S}^*_{Ne}$, *then* $f \in \Omega$ *in the disc* $|z| < r_0$, *where* $r_0 \approx 0.524752$ *is the positive root of*

$$2r\left(r+\frac{r^3}{3}\right)\exp\left(r-\frac{r^3}{9}\right)=0.$$

Proof. If $f \in \mathcal{S}^*_{Ne}$, then $zf'(z)/f(z) \prec 1+z-z^3/3$. We know that on $|z|=r$

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \max_{|z|=r}\left|z-\frac{z^3}{3}\right| = \max_{0\le\theta<2\pi}\left|re^{i\theta}-\frac{r^2e^{3i\theta}}{3}\right| = r+\frac{r^3}{3}.$$

The growth theorem for the class $\mathcal{S}^*_{Ne}$ implies that for any $f \in \mathcal{S}^*_{Ne}$, $|f(z)| \le |f_{Ne}(r)|$ on $|z|=r$, where

$$f_{Ne}(z) = z\exp\left(z-\frac{z^3}{9}\right).$$

By using the above inequalities, we get

$$\left|zf'(z)-f(z)\right| = |f(z)|\left|\frac{zf'(z)}{f(z)}-1\right| \le r\left(r+\frac{r^3}{3}\right)\exp\left(r-\frac{r^3}{9}\right).$$

Since $r < r_0$, we have $|zf'(z)-f(z)| < 1/2$ and, thus, $f \in \Omega$.

Theorem 2.15. *If* $f \in \mathcal{S}^*_C$, *then* $f \in \Omega$ *in the disc* $|z| < r_0$, *where* $r_0 \approx 0.411914$ *is the positive root of*

$$2re^{\frac{r^2}{3}+\frac{4r}{3}}\left(\frac{2r^2}{3}+\frac{4r}{3}\right)-1=0.$$

Proof. Let $f \in \mathcal{S}^*_C$. Then we have $zf'(z)/f(z) \prec 1+4z/3+2z^2/3$, which gives

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \left|\frac{4z}{3}+\frac{2z^2}{3}\right|.$$

Taking $z=re^{i\theta}$ $(0\le\theta<2\pi)$ it becomes

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \left|\frac{4re^{i\theta}}{3}+\frac{2r^2e^{2i\theta}}{3}\right| = \frac{2}{3}\sqrt{4r^2+r^2+4r^3\cos\theta} \le \frac{4r}{3}+\frac{2r^2}{3}.$$

Now by using the growth theorem for $\mathcal{S}^*_C$ (see [14]), we obtain $|f(z)| \le |f_0(r)|$ on $|z|=r$, where

$$f_0(z) = z\exp\left(\frac{4z}{3}+\frac{z^2}{3}\right).$$

We observe that

$$\left|zf'(z)-f(z)\right| = |f(z)|\left|\frac{zf'(z)}{f(z)}-1\right| \le re^{\frac{r^2}{3}+\frac{4r}{3}}\left(\frac{2r^2}{3}+\frac{4r}{3}\right) < \frac{1}{2},$$

provided $r < r_0$. We may note that for $f(z)=f_0(z)$ the inequality $|zf'(z)-f(z)| < 1/2$ holds only in the disk $|z| < r_0$ and, thus, the result is sharp.

Theorem 2.16. *Let* $f \in \mathcal{S}^*(\phi_i)$, $i = 1, 2, 3$, *then* $f \in G_{\frac{1}{2},\frac{1}{2}}$ *in the disk* $|z| < r_i$, $i = 1, 2, 3$, *for the following cases*:

(i) $\phi_1(z) = e^z$ *and* $r_1 \approx 0.537561$ *is the smallest positive root of* $e^r(1 + r^2)^2 - 4(1 - r^2) = 0$;

(ii) $\phi_2(z) = \sqrt{1 + z}$ *and* $r_2 \approx 0.429874$ *is the smallest positive root of* $(1 + r^2)^2 - 4(1 - r)^{3/2}(1 - r^2) = 0$;

(iii) $\phi_3(z) = 2/(1 + e^{-z})$ *and* $r_3 \approx 0.683447$ *is the smallest positive root of* $e^r(1 + r^2)^2 - 8(1 - r^2) = 0$.

Proof. Let $f \in \mathcal{S}^*(\phi)$, then we have $zf'(z)/f(z) \prec \phi(z)$. Thus, there exists a Schwarz function ω with $\omega(0) = 0$ and $|\omega(z)| \le |z|$ such that

$$\frac{zf'(z)}{f(z)} = \phi(\omega(z)),$$

which further implies

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 = z\omega'(z)\frac{\phi'(\omega(z))}{\phi^2(\omega(z))}.$$

For f to be in G_1, it is sufficient to show that $|z\omega'(z)\phi'(\omega(z))/\phi^2(\omega(z))| < 1$.

(i) Let $\phi(z) = e^z$, then, by using Lemma 1.2, we obtain

$$\left| z\omega'(z)\frac{\phi'(\omega(z))}{\phi^2(\omega(z))} \right| = \left| \frac{z\omega'(z)}{e^{\omega(z)}} \right| \le \frac{e^r(1 + r^2)^2}{4(1 - r^2)},$$

which is less than 1 provided $r < r_1$.

(ii) Let $\phi(z) = \sqrt{1 + z}$. By Lemma 1.2 we get, for $r < r_2$,

$$\left| z\omega'(z)\frac{\phi'(\omega(z))}{\phi^2(\omega(z))} \right| = \left| \frac{z\omega'(z)}{(1 + \omega(z))^{3/2}} \right| \le \frac{(1 + r^2)^2}{4(1 - r)^{3/2}(1 - r^2)} < 1.$$

(iii) Let $\phi(z) = 2/(1 + e^{-z})$, then, by using Lemma 1.2, we have

$$\left| z\omega'(z)\frac{\phi'(\omega(z))}{\phi^2(\omega(z))} \right| = \left| \frac{z\omega'(z)}{2e^{\omega(z)}} \right| \le \frac{e^r(1 + r^2)^2}{8(1 - r^2)},$$

which is less than 1 whenever $r < r_3$.

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