

**SOME NEW CESÀRO SEQUENCE SPACES OF ORDER  $\alpha$** **ДЕЯКІ НОВІ ПРОСТОРИ ПОСЛІДОВНОСТЕЙ ЧЕЗАРО ПОРЯДКУ  $\alpha$** 

We introduce the spaces  $\ell_\infty(C_\alpha)$ ,  $f(C_\alpha)$ , and  $f_0(C_\alpha)$  of Cesàro bounded, Cesàro almost convergent, and Cesàro almost null sequences of order  $\alpha$ , respectively. Moreover, we establish some inclusion relations for these spaces and determine the  $\alpha$ -,  $\beta$ - and  $\gamma$ -duals of the spaces  $\ell_\infty(C_\alpha)$  and  $f(C_\alpha)$ . Finally, we characterize the classes of matrix transformations from the space  $f(C_\alpha)$  to any sequence space  $Y$  and from any sequence space  $Y$  to the space  $f(C_\alpha)$ .

Введено простори  $\ell_\infty(C_\alpha)$ ,  $f(C_\alpha)$  і  $f_0(C_\alpha)$  Чезаро обмежених, Чезаро майже збіжних і Чезаро майже нульових послідовностей порядку  $\alpha$  відповідно. Крім того, встановлено деякі співвідношення включення для цих просторів і визначено  $\alpha$ -,  $\beta$ - та  $\gamma$ -дуальні простори для  $\ell_\infty(C_\alpha)$  та  $f(C_\alpha)$ . Насамкінець охарактеризовано класи матричних перетворень з простору  $f(C_\alpha)$  в довільний простір послідовностей  $Y$  та з довільного простору послідовностей  $Y$  у простір  $f(C_\alpha)$ .

**1. Introduction.** We denote the space of all complex valued sequences by  $\omega$ . Any vector subspace of  $\omega$  is called a *sequence space*. We write  $\ell_\infty$ ,  $c$  and  $c_0$  for the spaces of all bounded, convergent and null sequences which are Banach spaces endowed with the sup-norm  $\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|$ , respectively, where  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Also by  $bs$ ,  $cs$  and  $\ell_1$ , we mean the spaces of all bounded, convergent and absolutely convergent series, as usual, which are Banach spaces endowed with the norms  $\|x\| = \sup_{k \in \mathbb{N}} \left| \sum_{j=0}^k x_j \right|$  and  $\|x\|_1 = \sum_k |x_k|$ , respectively. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to  $\infty$ .

The *multiplier space*  $S(X, Y)$  of the sequence spaces  $X$  and  $Y$  is defined by

$$S(X, Y) = \{z = (z_k) \in \omega : xz = (x_k z_k) \in Y \text{ for all } x \in X\}. \quad (1.1)$$

With the notation of (1.1), the  $\alpha$ -,  $\beta$ - and  $\gamma$ -duals  $X^\alpha$ ,  $X^\beta$  and  $X^\gamma$  of a sequence space  $X$  are defined by

$$X^\alpha = S(X, \ell_1), \quad X^\beta = S(X, cs) \quad \text{and} \quad X^\gamma = S(X, bs).$$

If a normed sequence space  $X$  contains a sequence  $(b_n)$  with the property that for every  $x \in X$  there is a unique sequence of scalars  $(\alpha_n)$  such that

$$\lim_{n \rightarrow \infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \dots + \alpha_n b_n)\| = 0,$$

then  $(b_n)$  is called a *Schauder basis* (or, briefly, *basis*) for  $X$ . The series  $\sum_k \alpha_k b_k$ , which has the sum  $x$ , is called the expansion of  $x$  with respect to  $(b_n)$ , and is written as  $x = \sum_k \alpha_k b_k$ .

Let  $X$  and  $Y$  be any two sequence spaces, and  $A = (a_{nk})$  be an infinite matrix of complex numbers  $a_{nk}$ , where  $k, n \in \mathbb{N}$ . Then we say that  $A$  defines a *matrix transformation* from  $X$  into

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$Y$  and denote it by writing  $A: X \rightarrow Y$ , if, for every sequence  $x = (x_k) \in X$ , the sequence  $Ax = \{(Ax)_n\}$ , the  $A$ -transform of  $x$  is in  $Y$ , where

$$(Ax)_n = \sum_k a_{nk}x_k \quad \text{for each } n \in \mathbb{N}. \quad (1.2)$$

By  $(X: Y)$ , we denote the class of all matrices  $A$  such that  $A: X \rightarrow Y$ . Thus,  $A \in (X: Y)$  if and only if the series on the right-hand side of (1.2) converges for each  $n \in \mathbb{N}$  and every  $x \in X$ , and we have  $Ax \in Y$  for all  $x \in X$ . Also, we write  $A_n = (a_{nk})_{k \in \mathbb{N}}$  for the sequence in the  $n$ th row of  $A$ .

Banach [1] proved the existence of a functional  $L$  on the space  $\ell_\infty$  satisfying the following conditions for all  $x, y \in \ell_\infty$  and all scalars  $\lambda$  and  $\mu$ :

- (i)  $L(\lambda x + \mu y) = \lambda L(x) + \mu L(y)$ ,
- (ii)  $x_k \geq 0$  for all  $k \in \mathbb{N}$  implies  $L((x_k)_{k=0}^\infty) \geq 0$ ,
- (iii)  $L((x_{n+k})_{k=0}^\infty) = L((x_k)_{k=0}^\infty)$  for all  $n \in \mathbb{N}$ ,
- (iv)  $L(e) = 1$ , where  $e = (1, 1, 1, \dots)$ .

Lorentz [15] defined a *Banach limit* to be any functional on  $\ell_\infty$  satisfying the conditions in (i)–(iv). Then a sequence  $x = (x_k) \in \ell_\infty$  is said to be *almost convergent* to the generalized limit  $l$  if all Banach limits of  $x$  are coincide and are equal to  $l$  [15]. This is denoted by  $f - \lim x_k = l$ . The *shift operator*  $P$  is defined on  $\omega$  by  $P_n(x) = x_{n+1}$  for all  $n \in \mathbb{N}$ . Let  $P^i$  be the composition of  $P$  with itself  $i$  times and write, for a sequence  $x = (x_k)$ ,

$$t_{mn}(x) = \frac{1}{m+1} \sum_{i=0}^m P_n^i(x) \quad \text{for all } m, n \in \mathbb{N}.$$

Lorentz [15] proved that  $f - \lim x_k = l$  if and only if  $\lim_{m \rightarrow \infty} t_{mn}(x) = l$  uniformly in  $n$ . It is well-known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. For more detail on the Banach limit, the reader may refer to Çolak and Çakar [9], and Das [10]. Therefore, we define the spaces  $f_0$  and  $f$  of almost null and almost convergent sequences by

$$f_0 := \left\{ x = (x_k) \in \ell_\infty : \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{x_{n+k}}{m+1} = 0 \text{ uniformly in } n \right\},$$

$$f := \left\{ x = (x_k) \in \ell_\infty : \exists \alpha \in \mathbb{C} \text{ such that } \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{x_{n+k}}{m+1} = \alpha \text{ uniformly in } n \right\}.$$

One can easily see that the inclusions

$$c_0 \subset f_0, \quad c \subset f \quad \text{and} \quad f_0 \subset f$$

are strictly hold.

Let  $\alpha \in \mathbb{R}$  with  $\alpha > -1$  such that  $-\alpha \notin \mathbb{N}$ . The *Cesàro matrix of order  $\alpha$*  or, in short, the  $C_\alpha$ -matrix is defined by the matrix  $C_\alpha = (c_{nk}^{(\alpha)})$  which is given by

$$c_{nk}^{(\alpha)} = \begin{cases} \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}}, & 0 \leq k \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

for all  $k, n \in \mathbb{N}$ . Then we define the sequence  $y = (y_k)$  by the  $C_\alpha$ -transform of a sequence  $x = (x_k)$ , i.e.,

$$y_k = (C_\alpha x)_k = \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^k \binom{k-j+\alpha-1}{k-j} x_j \quad \text{for all } k \in \mathbb{N}. \quad (1.3)$$

Then the inverse  $C_\alpha^{-1} = (\tilde{c}_{nk}^{(\alpha)})$  of the  $C_\alpha$ -matrix is determined by

$$\tilde{c}_{nk}^{(\alpha)} = \begin{cases} \binom{n-k-\alpha-1}{n-k} \binom{k+\alpha}{k}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all  $k, n \in \mathbb{N}$  [21, p. 9, Problem 3] and [18, p. 35, Example 7.7]. Therefore, the  $\tilde{C}_\alpha$ -transform of the sequence  $y = (y_k^{(\alpha)})$  is given by

$$x_n = (\tilde{C}_\alpha y)_n = \sum_{k=0}^n \binom{n-k-\alpha-1}{n-k} \binom{k+\alpha}{k} y_k^{(\alpha)} \quad (1.4)$$

for all  $\alpha > -1$  and  $n \in \mathbb{N}$ . We should note here that the reader can refer to Malkowsky and Rakocevic [18, p. 28–44] for some details related to the Cesàro methods of order greater than  $-1$ . Also, each  $C_\alpha$ -matrix is a regular matrix when  $\alpha \geq 0$ . When  $-1 < \alpha < 0$ ,  $C_\alpha$ -matrix is not regular [8, 16].

If we choose  $\alpha \in \mathbb{N}$  and  $0 \leq k \leq n$ , then

$$\binom{n-k-\alpha-1}{n-k} = \frac{(n-k-\alpha-1)(n-k-\alpha-2)\dots(-\alpha)}{(n-k)!} = 0$$

for  $n-k \in \mathbb{R}_{\alpha+1} = \{\alpha+1, \alpha+2, \alpha+3, \dots\}$ . Hence,  $\binom{n-k-\alpha-1}{n-k} \neq 0$  for  $\max\{0, n-\alpha\}$ .

Therefore, the inverse matrix  $\tilde{C}_\alpha = (\tilde{c}_{nk}^{(\alpha)})_{n,k \in \mathbb{N}}$  of the matrix  $C_\alpha$  of order  $\alpha \in \mathbb{N}$  is obtained by (1.4) that

$$\tilde{c}_{nk}^{(\alpha)} = \begin{cases} \binom{n-k-\alpha-1}{n-k} \binom{k+\alpha}{k}, & \max\{0, n-\alpha\} \leq k \leq n, \\ 0, & k > n, \end{cases} \quad (1.5)$$

for all  $k, n \in \mathbb{N}$ .

For example, in the special case of  $\alpha = 1$ , one can derive from (1.4) and (1.5) that

$$x_n = \begin{cases} y_0^{(1)}, & n = 0, \\ -ny_{n-1}^{(1)} + (n+1)y_n^{(1)}, & n \geq 1. \end{cases}$$

Therefore, we conclude for the matrix  $\tilde{C}_1 = (\tilde{c}_{nk}^{(1)})_{n,k \in \mathbb{N}}$  with the convention that terms with a negative index are equal to zero, as follows:

$$\tilde{c}_{nk}^{(1)} = \begin{cases} -n, & k = n - 1, \\ n + 1, & k = n, \\ 0, & 0 \leq k < n - 1 \text{ or } k > n, \end{cases}$$

for all  $k, n \in \mathbb{N}$ .

**2. Some new Cesàro sequence spaces of order  $\alpha$ .** We introduce the spaces  $\ell_\infty(C_\alpha)$ ,  $f(C_\alpha)$ , and  $f_0(C_\alpha)$  as the sets of all Cesàro bounded, Cesàro almost convergent, and Cesàro almost null sequences of order  $\alpha$ , respectively.

Now, we introduce the sequence spaces  $\ell_\infty(C_\alpha)$ ,  $f(C_\alpha)$ , and  $f_0(C_\alpha)$  by

$$\ell_\infty(C_\alpha) := \left\{ x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^k \binom{k-j+\alpha-1}{k-j} x_j \right| < \infty \right\},$$

$$f(C_\alpha) := \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{m \rightarrow \infty} \frac{1}{m+1} \right. \\ \left. \times \sum_{k=0}^m \sum_{j=0}^{n+k} \frac{\binom{n+k-j+\alpha-1}{n+k-j}}{\binom{n+k+\alpha}{n+k}} x_j = l \text{ uniformly in } n \right\},$$

$$f_0(C_\alpha) := \left\{ x = (x_k) \in \omega : \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^{n+k} \frac{\binom{n+k-j+\alpha-1}{n+k-j}}{\binom{n+k+\alpha}{n+k}} x_j = 0 \text{ uniformly in } n \right\}.$$

Let  $X$  be any sequence space. Then the domain  $X_A$  of an infinite matrix  $A$  in  $X$  is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}. \quad (2.1)$$

Using the notation in (2.1), one can redefine the sequence spaces  $\ell_\infty(C_\alpha)$ ,  $f(C_\alpha)$ , and  $f_0(C_\alpha)$  by  $\ell_\infty(C_\alpha) = \{\ell_\infty\}_{C_\alpha}$ ,  $f(C_\alpha) = \{f\}_{C_\alpha}$ , and  $f_0(C_\alpha) = \{f_0\}_{C_\alpha}$ , respectively.

The reader can refer to the monographs [3] and [19] for the background on the normed and paranormed sequence spaces, and summability theory and related topics.

**Theorem 2.1.** *The sequence spaces  $f_0(C_\alpha)$  and  $f(C_\alpha)$  are BK-spaces endowed with the norm given by*

$$\|x\|_{f(C_\alpha)} = \|C_\alpha x\|_f = \sup_{m,n \in \mathbb{N}} |t_{mn}(C_\alpha x)|,$$

where

$$\begin{aligned} t_{mn}(C_\alpha x) &= \frac{1}{m+1} \sum_{k=0}^m (C_\alpha x)_{n+k} \\ &= \frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^{n+k} \frac{\binom{n+k-j+\alpha-1}{n+k-j}}{\binom{n+k+\alpha}{n+k}} x_j \quad \text{for all } m, n \in \mathbb{N}. \end{aligned}$$

**Proof.** Since  $f_0$  and  $f$  endowed with the norm  $\|\cdot\|_\infty$  are  $BK$ -spaces [7, Example 7.3.2 (b)] and  $C_\alpha$  is a triangle matrix, Theorem 4.3.2 of [25, p. 61] gives the fact that  $f_0(C_\alpha)$  and  $f(C_\alpha)$  are  $BK$ -spaces endowed with the norm  $\|\cdot\|_{f(C_\alpha)}$ .

**Remark 2.1.** It is immediate that the absolute property does not hold on the spaces  $f_0(C_\alpha)$  and  $f(C_\alpha)$ , i.e.,  $\|x\|_{f(C_\alpha)} \neq \| |x| \|_{f(C_\alpha)}$  for at least one sequence  $x$  in each of these spaces, where  $|x| = (|x_k|)$ . Thus, the spaces  $f_0(C_\alpha)$  and  $f(C_\alpha)$  are  $BK$ -spaces of nonabsolute type.

**Theorem 2.2.** The sequence spaces  $f_0(C_\alpha)$  and  $f(C_\alpha)$  are norm isomorphic to the spaces  $f_0$  and  $f$ , respectively, i.e.,  $f_0(C_\alpha) \cong f_0$  and  $f(C_\alpha) \cong f$ .

**Proof.** Since it can be easily shown in a similar way that the spaces  $f_0(C_\alpha)$  and  $f_0$  are linearly norm isomorphic, we consider only the case  $f(C_\alpha) \cong f$ . To prove this, we should show the existence of a linear bijection between the spaces  $f(C_\alpha)$  and  $f$  which preserves the norm. Consider the transformation  $T$  defined, with the notation of (1.3),

$$\begin{aligned} T : f(C_\alpha) &\longrightarrow f \\ x &\longmapsto y = Tx = C_\alpha x. \end{aligned} \quad (2.2)$$

Since the transformation  $T$  is represented by the matrix  $C_\alpha$ , the linearity of  $T$  is clear. Further, it is trivial that  $x = \theta$  whenever  $Tx = \theta$  and, hence,  $T$  is injective.

Let us take any  $y = (y_k) \in f$  and define the sequence  $x = (x_k)$  by

$$x_k = \sum_{j=0}^k \binom{k-j-\alpha-1}{k-j} \binom{j+\alpha}{j} y_j \quad (2.3)$$

for all  $k \in \mathbb{N}$ , where  $\max\{0, k-\alpha\} \leq j$ . Then it is immediate that

$$\sum_{j=0}^{n+k} \frac{\binom{n+k-j+\alpha-1}{n+k-j}}{\binom{n+k+\alpha}{n+k}} x_j = \sum_{j=0}^{n+k} \frac{\binom{n+k-j+\alpha-1}{n+k-j}}{\binom{n+k+\alpha}{n+k}} \sum_{i=0}^j \binom{j-i-\alpha-1}{j-i} \binom{i+\alpha}{i} y_i = y_{n+k},$$

which gives that

$$\frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^{n+k} \frac{\binom{n+k-j+\alpha-1}{n+k-j}}{\binom{n+k+\alpha}{n+k}} x_j = \frac{1}{m+1} \sum_{k=0}^m y_{n+k}.$$

Therefore, we have

$$\lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m (C_\alpha x)_{n+k} = \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m y_{n+k} = l \quad \text{uniformly in } n.$$

This means that  $x \in f(C_\alpha)$  and, hence,  $T$  is surjective. Thus, one can easily see from (2.2) that  $\|x\|_{f(C_\alpha)} = \|Tx\|_f$ , i.e.,  $T$  is a norm preserving transformation.

The theorem is proved.

**Theorem 2.3.** *The sequence space  $\ell_\infty(C_\alpha)$  is a Banach space endowed with the norm*

$$\|x\|_{\ell_\infty(C_\alpha)} = \|C_\alpha x\|_\infty = \sup_{k \in \mathbb{N}} \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^k \binom{k-j+\alpha-1}{k-j} x_j \right|,$$

and the space  $\ell_\infty(C_\alpha)$  is linearly norm isomorphic to the space  $\ell_\infty$ .

**Proof.** Since this is a routine verification, we omit details.

It is known from Corollary 3.3 of Başar and Kirişçi [6] that the Banach space  $f$  has no Schauder basis. It is also known from Theorem 2.3 of Jarrah and Malkowsky [12] that the domain  $X_A$  of a matrix  $A$  in a normed sequence space  $X$  has a basis if and only if  $X$  has a basis whenever  $A = (a_{nk})$  is a triangle. Combining these two facts one can immediately conclude that both the space  $f(C_\alpha)$  and the space  $f_0(C_\alpha)$  have no Schauder basis.

**Definition 2.1** (Petersen [20]). *A regular matrix  $A = (a_{nk})$  is strongly regular if and only if it is translatable, i.e.,*

$$\lim_{n \rightarrow \infty} \sum_k |a_{nk} - a_{n,k+1}| = 0.$$

**Theorem 2.4.** *Let  $\alpha \geq 0$ . Then the Cesàro matrix of order  $\alpha$  is strongly regular.*

**Proof.** Let  $\alpha \geq 0$ . From the definition of the Cesàro matrix  $C_\alpha = (c_{nk}^{(\alpha)})$  of order  $\alpha$ , we have

$$\frac{c_{nk}^{(\alpha)}}{c_{n,k+1}^{(\alpha)}} = \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n-k+\alpha-2}{n-k-1}} = \frac{n-k+\alpha-1}{n-k} = 1 + \frac{\alpha-1}{n-k} \geq 1.$$

Case 1. For all  $n \leq m$ ,

$$\begin{aligned} \sum_{k=0}^m |c_{nk}^{(\alpha)} - c_{n,k+1}^{(\alpha)}| &= \sum_{k=0}^m (c_{nk}^{(\alpha)} - c_{n,k+1}^{(\alpha)}) \\ &= (c_{n0}^{(\alpha)} - c_{n1}^{(\alpha)}) + (c_{n1}^{(\alpha)} - c_{n2}^{(\alpha)}) + \dots + (c_{n,m-1}^{(\alpha)} - c_{nm}^{(\alpha)}) + (c_{nm}^{(\alpha)} - c_{n,m+1}^{(\alpha)}) \\ &= c_{n0}^{(\alpha)} = \frac{\alpha}{n+\alpha}. \end{aligned} \tag{2.4}$$

Taking limit in (2.4) as  $m \rightarrow \infty$ , we obtain

$$\lim_{m \rightarrow \infty} \sum_{k=0}^m |c_{nk}^{(\alpha)} - c_{n,k+1}^{(\alpha)}| = \sum_{k=0}^{\infty} |c_{nk}^{(\alpha)} - c_{n,k+1}^{(\alpha)}| = \lim_{m \rightarrow \infty} \frac{\alpha}{n+\alpha} = \frac{\alpha}{n+\alpha}.$$

Therefore, we get

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} |c_{nk}^{(\alpha)} - c_{n,k+1}^{(\alpha)}| = \lim_{n \rightarrow \infty} \frac{\alpha}{n + \alpha} = 0,$$

as desired.

*Case 2.* For all  $n > m$ ,

$$\begin{aligned} \sum_{k=0}^m |c_{nk}^{(\alpha)} - c_{n,k+1}^{(\alpha)}| &= \sum_{k=0}^m (c_{nk}^{(\alpha)} - c_{n,k+1}^{(\alpha)}) \\ &= (c_{n0}^{(\alpha)} - c_{n1}^{(\alpha)}) + (c_{n1}^{(\alpha)} - c_{n2}^{(\alpha)}) + \dots + (c_{n,m-1}^{(\alpha)} - c_{nm}^{(\alpha)}) + (c_{nm}^{(\alpha)} - c_{n,m+1}^{(\alpha)}) \\ &= c_{n0}^{(\alpha)} - c_{n,m+1}^{(\alpha)} = \frac{\binom{n + \alpha - 1}{n}}{\binom{n + \alpha}{\alpha}} - \frac{\binom{n - (m + 1) + \alpha - 1}{n - (m + 1)}}{\binom{n + \alpha}{\alpha}} \\ &< \frac{\binom{n + \alpha - 1}{n}}{\binom{n + \alpha}{\alpha}} = \frac{\alpha}{n + \alpha}. \end{aligned}$$

We conclude using similar calculations as in Case 1 with Sandwich theorem.

The theorem is proved.

**Definition 2.2.** An infinite matrix  $A$  is said to be almost strongly regular if  $A$  transforms all almost convergent sequences into an almost convergent sequence with the same limit [11].

**Theorem 2.5.** Let  $\alpha \geq 0$ . Then the inclusions  $f_0 \subset f_0(C_\alpha)$  and  $f \subset f(C_\alpha)$  strictly hold.

**Proof.** Let  $\alpha \geq 0$ . We take a sequence  $x = (x_k)$  in the space  $f$ . The matrix  $C_\alpha = (c_{nk}^{(\alpha)})$  is an almost strongly regular matrix by Part (i) of Example 4.5.5 in [19]. So,  $C_\alpha x \in f$  which implies that  $x \in f(C_\alpha)$ . Therefore, the inclusion  $f \subset f(C_\alpha)$  holds.

Moreover, consider the sequence  $x = (x_k)$  defined by

$$x_k := \sum_{i=0}^k (-1)^i \binom{k - i - \alpha - 1}{k - i} \binom{i + \alpha}{i} \quad (2.5)$$

for all  $k \in \mathbb{N}$  with  $\max\{0, k - \alpha\} \leq i$ . It is immediate that  $x \notin f$  because  $x \notin \ell_\infty$  but

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{m + 1} \sum_{k=0}^m (C_\alpha x)_{n+k} &= \lim_{m \rightarrow \infty} \frac{1}{m + 1} \sum_{k=0}^m \sum_{j=0}^{n+k} \frac{\binom{n + k - j + \alpha - 1}{n + k - j}}{\binom{n + k + \alpha}{n + k}} x_j \\ &= \lim_{m \rightarrow \infty} \frac{1}{m + 1} \sum_{k=0}^m \sum_{j=0}^{n+k} \frac{\binom{n + k - j + \alpha - 1}{n + k - j}}{\binom{n + k + \alpha}{n + k}} \end{aligned}$$

$$\begin{aligned} & \times \sum_{i=0}^j (-1)^i \binom{j-i-\alpha-1}{j-i} \binom{i+\alpha}{i} \\ & = \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m (-1)^{n+k} = \lim_{m \rightarrow \infty} \frac{(-1)^n}{m+1} \left[ \frac{1 + (-1)^m}{2} \right] = 0, \end{aligned}$$

i.e.,  $C_\alpha x \in f$ , so  $x \in f(C_\alpha)$ . This means that the inclusion  $f \subset f(C_\alpha)$  is strict.

By similar arguments as above, we obtain similar results for the space  $f_0(C_\alpha)$ .

The theorem is proved.

**Theorem 2.6.** *The inclusion  $f_0(C_\alpha) \subset f(C_\alpha)$  strictly holds.*

**Proof.** Take a sequence  $x = (x_k)$  in the space  $f_0(C_\alpha)$ . Then we have  $C_\alpha x \in f_0$ . Since  $f_0 \subset f$ , we obtain  $C_\alpha x \in f$ , which means that  $x \in f(C_\alpha)$ . Therefore, the inclusion  $f_0(C_\alpha) \subset f(C_\alpha)$  holds.

Now, consider the sequence  $x = (x_k)$  defined by

$$x_k := \sum_{i=0}^k \binom{k-i-\alpha-1}{k-i} \binom{i+\alpha}{i}$$

for all  $k \in \mathbb{N}$ , where  $\max\{0, k - \alpha\} \leq i$ . Then we observe that

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m (C_\alpha x)_{n+k} &= \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^{n+k} \frac{\binom{n+k-j+\alpha-1}{n+k-j}}{\binom{n+k+\alpha}{n+k}} x_j \\ &= \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m \sum_{j=0}^{n+k} \frac{\binom{n+k-j+\alpha-1}{n+k-j}}{\binom{n+k+\alpha}{n+k}} \\ &\quad \times \sum_{i=0}^j \binom{j-i-\alpha-1}{j-i} \binom{i+\alpha}{i} \\ &= \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{k=0}^m 1 = \lim_{m \rightarrow \infty} \frac{m+1}{m+1} = 1, \end{aligned}$$

i.e.,  $C_\alpha x \in f$ , which means that  $x \in f(C_\alpha)$ . Obviously,  $x \in f(C_\alpha) \setminus f_0(C_\alpha)$ , i.e., the inclusion  $f_0(C_\alpha) \subset f(C_\alpha)$  is strict.

The theorem is proved.

**Theorem 2.7.** *The inclusions  $c(C_\alpha) \subset f(C_\alpha) \subset \ell_\infty(C_\alpha)$  strictly hold, where  $c(C_\alpha)$  was defined by Roopaei and Başar [22] as  $c(C_\alpha) := \{x = (x_k) \in \omega : C_\alpha x \in c\}$ .*

**Proof.** Take a sequence  $x = (x_k)$  in  $c(C_\alpha)$ . Then  $C_\alpha x \in c$  and the inclusion  $c \subset f$  holds. Thus, we have  $C_\alpha x \in f$ . This means that  $x \in f(C_\alpha)$ . Hence, the inclusion  $c(C_\alpha) \subset f(C_\alpha)$  holds. Now, reconsider the sequence  $x = (x_k)$  defined by (2.5). Therefore, we get

$$(C_\alpha x)_n = \sum_{k=0}^n \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}} x_k$$

$$\begin{aligned}
 &= \sum_{k=0}^n \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}} \sum_{i=0}^k (-1)^i \binom{k-i-\alpha-1}{k-i} \binom{i+\alpha}{i} \\
 &= (-1)^n \text{ for all } n \in \mathbb{N},
 \end{aligned}$$

which shows that  $C_\alpha x \notin c$ , that is,  $x \notin c(C_\alpha)$ . But, it is known from the proof of Theorem 2.5 that  $x \in f(C_\alpha)$ . So, the inclusion  $c(C_\alpha) \subset f(C_\alpha)$  strictly holds.

Now we establish the second inclusion. Let  $y = (y_k) \in f(C_\alpha)$ . Then, since  $C_\alpha y \in f$  and  $f \subset \ell_\infty$ , we have  $C_\alpha y \in \ell_\infty$ . Therefore, the inclusion  $f(C_\alpha) \subset \ell_\infty(C_\alpha)$  holds.

Now consider the sequence  $z$  defined via the sequence  $y$  by  $y := C_\alpha z$ , where  $y := (0, 0, \dots, 1, \dots, 1, 0, \dots, 0, \dots)$  defined by Miller and Orhan [17] belonging to the set  $\ell_\infty \setminus f$ , where the blocks of 0's are increasing by factors of 100 and the blocks of 1's are increasing by factors of 10. Then  $z$  belongs to the space  $\ell_\infty(C_\alpha)$  but not in the space  $f(C_\alpha)$ , i.e.,  $z \in \ell_\infty(C_\alpha) \setminus f(C_\alpha)$ . This means that the inclusion  $f(C_\alpha) \subset \ell_\infty(C_\alpha)$  is strict.

The theorem is proved.

**3.  $\alpha$ -,  $\beta$ - and  $\gamma$ -duals of the spaces  $\ell_\infty(C_\alpha)$  and  $f(C_\alpha)$ .** In this section, we determine the  $\alpha$ -,  $\beta$ - and  $\gamma$ -duals of the spaces  $\ell_\infty(C_\alpha)$  and  $f(C_\alpha)$ .

We begin with the following lemmas.

**Lemma 3.1** (Kamphthan and Gupta [13]). *Let  $X$  and  $Y$  be two sequence spaces, and  $\xi \in \{\alpha, \beta, \gamma\}$ . If  $X \subset Y$ , then  $Y^\xi \subset X^\xi$ .*

**Lemma 3.2** (Stieglitz and Tietz [24]). *Let  $A = (a_{nk})$  be an infinite matrix. Then the following statements hold:*

(i)  $A = (a_{nk}) \in (\ell_\infty : \ell_1) = (c : \ell_1)$  if and only if

$$\sup_{K \in \mathcal{N}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty,$$

where  $\mathcal{N}$  denotes the collection of all finite subsets of  $\mathbb{N}$ ;

(ii)  $A = (a_{nk}) \in (\ell_\infty : \ell_\infty)$  if and only if

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty; \tag{3.1}$$

(iii)  $A = (a_{nk}) \in (\ell_\infty : c)$  if and only if

$$\exists(\eta_k) \in \omega \ni \lim_{n \rightarrow \infty} a_{nk} = \eta_k \text{ for all } k \in \mathbb{N}, \tag{3.2}$$

$$\sum_k |a_{nk}| \text{ converges uniformly in } n.$$

**Lemma 3.3** (Siddiqi [23]).  $A = (a_{nk}) \in (f : \ell_\infty)$  if and only if the relation in (3.1) holds.

**Lemma 3.4** (Siddiqi [23]).  $A = (a_{nk}) \in (f : c)$  if and only if (3.1) and (3.2) hold, and there are  $\eta_k, \eta \in \mathbb{C}$  such that

$$\lim_{n \rightarrow \infty} \sum_k a_{nk} = \eta, \tag{3.3}$$

$$\lim_{n \rightarrow \infty} \sum_k |\Delta(a_{nk} - \eta_k)| = 0. \tag{3.4}$$

**Theorem 3.1.** *The  $\alpha$ -dual of the spaces  $\ell_\infty(C_\alpha)$  and  $c(C_\alpha)$  is the set  $d$  defined by*

$$d := \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{N}} \sum_n \left| \sum_{k \in K} \binom{n-k-\alpha-1}{n-k} \binom{k+\alpha}{k} a_n \right| < \infty \right\}.$$

**Proof.** Let us take  $a = (a_k) \in \omega$ . We easily derive with (2.3) that

$$a_n x_n = \sum_{k=0}^n \binom{n-k-\alpha-1}{n-k} \binom{k+\alpha}{k} a_n y_k = (B_\alpha y)_n \tag{3.5}$$

for all  $n \in \mathbb{N}$ , where  $B_\alpha = (b_{nk}^{(\alpha)})$  is defined by

$$b_{nk}^{(\alpha)} = \begin{cases} \binom{n-k-\alpha-1}{n-k} \binom{k+\alpha}{k} a_n, & \max\{0, n-\alpha\} \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all  $k, n \in \mathbb{N}$ . Thus, we observe by combining (3.5) with Part (i) of Lemma 3.2 that  $ax = (a_n x_n) \in \ell_1$ , whenever  $x = (x_n) \in \ell_\infty(C_\alpha)$ , if and only if  $B_\alpha y \in \ell_1$ , whenever  $y = (y_n) \in \ell_\infty$ . This gives that  $\{\ell_\infty(C_\alpha)\}^\alpha = d$ .

In a similar way, we can show that the  $\alpha$ -dual of the space  $c(C_\alpha)$  is the set  $d$ . So, we omit details.

**Theorem 3.2.** *The  $\alpha$ -dual of the space  $f(C_\alpha)$  is the set  $d$ .*

**Proof.** The inclusions  $c(C_\alpha) \subset f(C_\alpha) \subset \ell_\infty(C_\alpha)$  hold by Theorem 2.7. Using this fact with Lemma 3.1, we obtain the desired result.

**Theorem 3.3.** *Let  $\max\{0, j-\alpha\} \leq k$ . Then define the sets  $d_1, d_2, d_3, d_4$  and  $d_5$  as follows:*

$$d_1 := \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_k \left| \sum_{j=k}^n \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_j \right| < \infty \right\},$$

$$d_2 := \left\{ a = (a_k) \in \omega : \exists (\eta_k) \in \omega \ni \lim_{n \rightarrow \infty} \sum_{j=k}^n \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_j = \eta_k \text{ for all } k \in \mathbb{N} \right\},$$

$$d_3 := \left\{ a = (a_k) \in \omega : \exists \eta \in \mathbb{C} \ni \lim_{n \rightarrow \infty} \sum_k \sum_{j=k}^n \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_j = \eta \right\},$$

$$d_4 := \left\{ a = (a_k) \in \omega : \exists (\eta_k) \in \omega \ni \lim_{n \rightarrow \infty} \sum_k \left| \sum_{j=k}^n \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_j - \eta_k \right| = 0 \right\},$$

$$d_5 := \left\{ a = (a_k) \in \omega : \sum_k \left| \sum_{j=k}^n \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_j \right| \text{ converges uniformly in } n \right\}.$$

Then the following statements hold:

- (i)  $\{C_\alpha(f)\}^\gamma = d_1, \{C_\alpha(f)\}^\beta = \bigcap_{i=1}^4 d_i,$
- (ii)  $\{\ell_\infty(C_\alpha)\}^\gamma = d_1, \{\ell_\infty(C_\alpha)\}^\beta = d_2 \cap d_5.$

**Proof.** (i) Let us take  $a = (a_n) \in \omega$ . By using (2.3), we obtain

$$\begin{aligned} \sum_{k=0}^n a_k x_k &= \sum_{k=0}^n \sum_{j=0}^k \binom{k-j-\alpha-1}{k-j} \binom{j+\alpha}{j} a_k y_j \\ &= \sum_{k=0}^n \sum_{j=k}^n \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_j y_k = (E_\alpha y)_n \end{aligned} \tag{3.6}$$

for all  $n \in \mathbb{N}$ , where  $E_\alpha = (e_{nk}^{(\alpha)})$  is defined by

$$e_{nk}^{(\alpha)} = \begin{cases} \sum_{j=k}^n \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_j, & \max\{0, j-\alpha\} \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all  $k, n \in \mathbb{N}$ . Thus, we deduce from Lemma 3.3 with (3.6) that  $ax = (a_k x_k) \in bs$ , whenever  $x = (x_k) \in C_\alpha(f)$ , if and only if  $E_\alpha y \in \ell_\infty$ , whenever  $y = (y_k) \in f$ . Therefore, we obtain  $\{C_\alpha(f)\}^\gamma = d_1$ .

Similarly, we have  $ax = (a_k x_k) \in cs$ , whenever  $x = (x_k) \in C_\alpha(f)$ , if and only if  $E_\alpha y \in c$ , whenever  $y = (y_k) \in f$  by Lemma 3.4. We conclude  $\{C_\alpha(f)\}^\beta = d_1 \cap d_2 \cap d_3 \cap d_4$ .

(ii) The proof is obvious from Parts (i) and (ii) of Lemma 3.2 using the calculations in Part (i). Therefore, the proof is immediate.

**4. Characterization of some classes of matrix transformations.** In this section, we characterize the classes  $(\ell_\infty(C_\alpha) : \ell_\infty)$ ,  $(C_\alpha(f) : Y)$ , and  $(Y : C_\alpha(f))$  of matrix transformations. Since  $Y_A \cong Y$  for any triangle  $A$  and any sequence space  $Y$ , it is trivial that the equivalence " $x \in Y_A$  if and only if  $y \in Y$ " holds, where  $y = Ax$ .

Throughout this section, we assume that the entries of the infinite matrices  $A = (a_{nk})$  and  $F_\alpha = (f_{nk}^{(\alpha)})$  are connected with the relation

$$f_{nk}^{(\alpha)} := \sum_{j=k}^\infty \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_{nj} \tag{4.1}$$

for all  $k, n \in \mathbb{N}$ , where  $\max\{0, j-\alpha\} \leq k$ .

**Theorem 4.1.** *Suppose that the entries of the infinite matrices  $A = (a_{nk})$  and  $F_\alpha = (f_{nk}^{(\alpha)})$  are connected with the relation in (4.1) for all  $k, n \in \mathbb{N}$ . Then  $A \in (\ell_\infty(C_\alpha) : \ell_\infty)$  if and only if*

$$A_n \in \{\ell_\infty(C_\alpha)\}^\beta \text{ for each } n \in \mathbb{N}, \tag{4.2}$$

$$\sup_{n \in \mathbb{N}} \sum_k \left| \sum_{j=k}^\infty \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_{nj} \right| < \infty. \tag{4.3}$$

**Proof.** Suppose that  $A = (a_{nk}) \in (\ell_\infty(C_\alpha) : \ell_\infty)$  and take  $x = (x_k) \in \ell_\infty(C_\alpha)$ . Also, assume that (4.1) holds between the entries of the matrices  $A$  and  $F_\alpha$ , and take into account that the spaces  $\ell_\infty(C_\alpha)$  and  $\ell_\infty$  are linearly norm isomorphic.

Since  $Ax$  exists and belongs to the space  $\ell_\infty$  by the hypothesis, the necessity of the condition in (4.2) is obvious. With some calculations, we observe that

$$(F_\alpha C_\alpha)_{nk} = \sum_{j=k}^\infty f_{nj}^{(\alpha)} c_{jk}^{(\alpha)} = \sum_{j=k}^\infty \sum_{i=j}^\infty \binom{i-j-\alpha-1}{i-j} \binom{j-k+\alpha-1}{j-k} a_{nj} = a_{nk}, \tag{4.4}$$

i.e.,  $F_\alpha C_\alpha$  exists and  $A_n \in \{\ell_\infty(C_\alpha)\}^\beta$  which yields that  $\{F_\alpha\}_n \in \ell_1$  for each  $n \in \mathbb{N}$ . Hence,  $F_\alpha y$  exists and, thus, we see from the relation in (4.4) that

$$(Ax)_n = \sum_k a_{nk} x_k = (F_\alpha C_\alpha x)_n = \sum_k f_{nk}^{(\alpha)} y_k = (F_\alpha y)_n \tag{4.5}$$

for all  $n \in \mathbb{N}$ . Therefore, we conclude that  $F_\alpha y \in \ell_\infty$ . This means  $F_\alpha \in (\ell_\infty : \ell_\infty)$ . Hence,  $F_\alpha$  satisfies the condition in (3.1) which is equivalent to (4.3). This shows the necessity of the condition in (4.3).

Conversely, suppose that conditions in (4.2) and (4.3) are satisfied, and take  $x = (x_k) \in \ell_\infty(C_\alpha)$ . Then  $Ax$  exists and the condition in (3.1) is satisfied for the matrix  $F_\alpha$ . Therefore, we conclude from the relation in (4.5) that  $Ax = F_\alpha y \in \ell_\infty$ , which means  $A \in (\ell_\infty(C_\alpha) : \ell_\infty)$ .

The theorem is proved.

**Theorem 4.2.** *Suppose that the entries of the infinite matrices  $A = (a_{nk})$  and  $F_\alpha = (f_{nk}^{(\alpha)})$  are connected with the relation in (4.1) for all  $k, n \in \mathbb{N}$ , and  $Y$  be any given sequence space. Then  $A \in (f(C_\alpha) : Y)$  if and only if  $A_n \in \{f(C_\alpha)\}^\beta$  for all  $n \in \mathbb{N}$  and  $F_\alpha \in (f : Y)$ .*

**Proof.** Let  $Y$  be any given sequence space. Suppose that (4.1) holds between the entries of the matrices  $A = (a_{nk})$  and  $F_\alpha = (f_{nk}^{(\alpha)})$ , and take into account that the spaces  $f(C_\alpha)$  and  $f$  are norm isomorphic.

Let  $A \in (f(C_\alpha) : Y)$  and take any  $y \in f$ . Then, from the relation in (4.4),  $F_\alpha y$  exists and, thus,

$$\begin{aligned} \sum_k f_{nk}^{(\alpha)} y_k &= \sum_k \sum_{j=k}^\infty \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_{nj} \left[ \frac{1}{\binom{k+\alpha}{k}} \sum_{i=0}^k \binom{k-i+\alpha-1}{k-i} x_i \right] \\ &= \sum_k \sum_{j=k}^\infty \binom{j-k-\alpha-1}{j-k} a_{nj} \sum_{i=0}^k \binom{k-i+\alpha-1}{k-i} x_i = \sum_k a_{nk} x_k \end{aligned}$$

for all  $n \in \mathbb{N}$ . So, we derive that  $F_\alpha y = Ax$ , which leads us to the consequence  $F_\alpha \in (f : Y)$ .

Conversely, let  $A_n \in \{f(C_\alpha)\}^\beta$  for each  $n \in \mathbb{N}$  and  $F_\alpha \in (f : Y)$ , and take  $x = (x_k) \in f(C_\alpha)$ . Then  $Ax$  exists. Therefore, we again obtain from the equality

$$\begin{aligned} \sum_k a_{nk} x_k &= \sum_k a_{nk} \sum_{j=0}^k \binom{k-j-\alpha-1}{k-j} \binom{j+\alpha}{j} y_j \\ &= \sum_k \sum_{j=k}^\infty \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_{nj} y_k \end{aligned}$$

for all  $n \in \mathbb{N}$  that  $Ax = F_\alpha y$ . We conclude  $A \in (f(C_\alpha) : Y)$ .

The theorem is proved.

**Theorem 4.3.** *Let  $Y$  be any given sequence space and the entries of the infinite matrices  $A = (a_{nk})$  and  $G_\alpha = (g_{nk}^{(\alpha)})$  are connected with the relation*

$$g_{nk}^{(\alpha)} = \frac{1}{\binom{n+\alpha}{n}} \sum_{j=0}^n \binom{n-j+\alpha-1}{n-j} a_{jk} \tag{4.6}$$

for all  $k, n \in \mathbb{N}$ . Then  $A \in (Y : f(C_\alpha))$  if and only if  $G_\alpha \in (Y : f)$ .

**Proof.** Let  $u = (u_k) \in Y$  and the entries of the infinite matrices  $A = (a_{nk})$  and  $G_\alpha = (g_{nk}^{(\alpha)})$  are connected with the relation (4.6) for all  $n, k \in \mathbb{N}$ . Consider the equality

$$\begin{aligned} \sum_{k=0}^m g_{nk}^{(\alpha)} u_k &= \sum_{k=0}^m \frac{1}{\binom{n+\alpha}{n}} \sum_{j=0}^n \binom{n-j+\alpha-1}{n-j} a_{jk} u_k \\ &= \frac{1}{\binom{n+\alpha}{n}} \sum_{j=0}^n \binom{n-j+\alpha-1}{n-j} \sum_{k=0}^m a_{jk} u_k, \end{aligned} \tag{4.7}$$

which holds for all  $m \in \mathbb{N}$ . Then, by letting  $m \rightarrow \infty$  in (4.7), we have  $(G_\alpha u)_n = \{C_\alpha(Au)\}_n$  for all  $n \in \mathbb{N}$ . Since  $Au \in f(C_\alpha)$ , we get  $C_\alpha(Au) = G_\alpha u \in f$ .

The theorem is proved.

Of course, Theorems 4.2 and 4.3 have several consequences depending on the choice of the sequence space  $Y$ . Define  $a(n, k)$ ,  $a(n, k, m)$ , and  $\Delta a_{nk}$  for all  $k, m, n \in \mathbb{N}$  as follows:

$$a(n, k) := \sum_{j=0}^n a_{jk}, \quad a(n, k, m) := \frac{1}{m+1} \sum_{j=0}^m a_{n+j,k}, \quad \text{and} \quad \Delta a_{nk} := a_{nk} - a_{n,k+1}.$$

Prior to giving some results as an application of this idea, we give the following basic lemma, which is the collection of the characterizations of some classes of matrix transformations related to almost convergence.

**Lemma 4.1.** *Let  $A = (a_{nk})$  be an infinite matrix. Then the following statements hold:*

(i) (Duran [11])  $A = (a_{nk}) \in (\ell_\infty : f)$  if and only if (3.1) holds and

$$\exists \eta_k \in \mathbb{C} \ni f - \lim a_{nk} = \eta_k \quad \text{for all } k \in \mathbb{N}, \tag{4.8}$$

$$\exists \eta_k \in \mathbb{C} \ni \lim_{m \rightarrow \infty} \sum_k |a(n, k, m) - \eta_k| = 0 \quad \text{uniformly in } n \tag{4.9}$$

also hold.

(ii) (Duran [11])  $A = (a_{nk}) \in (f : f)$  if and only if (3.1) and (4.8) hold, and

$$\exists \eta \in \mathbb{C} \ni f - \lim \sum_k a_{nk} = \eta, \tag{4.10}$$

$$\exists \eta_k \in \mathbb{C} \ni \lim_{m \rightarrow \infty} \sum_k |\Delta[a(n, k, m) - \eta_k]| = 0 \quad \text{uniformly in } n \tag{4.11}$$

also hold.

(iii) (King [14])  $A = (a_{nk}) \in (c : f)$  if and only if (3.1), (4.8), and (4.10) hold.

(iv) (Başar and Çolak [4])  $A = (a_{nk}) \in (cs : f)$  if and only if (4.8) holds, and

$$\sup_{n \in \mathbb{N}} \sum_k |\Delta a_{nk}| < \infty \quad (4.12)$$

also holds.

(v) (Başar and Solak [5])  $A = (a_{nk}) \in (bs : f)$  if and only if (4.8) and (4.12) hold, and

$$\lim_{k \rightarrow \infty} a_{nk} = 0 \quad \text{for all } n \in \mathbb{N}, \quad (4.13)$$

$$\exists \eta_k \in \mathbb{C} \ni \lim_{q \rightarrow \infty} \sum_k \frac{1}{q+1} \left| \sum_{i=0}^q \Delta[a(n+i, k) - \eta_k] \right| = 0 \quad \text{uniformly in } n \quad (4.14)$$

also hold.

(vi) (Başar [2])  $A = (a_{nk}) \in (f : cs)$  if and only if (3.1)–(3.4) hold with  $a(n, k)$  instead of  $a_{nk}$ .

**Corollary 4.1.** *The following statements hold:*

(i)  $A = (a_{nk}) \in (f(C_\alpha) : \ell_\infty)$  if and only if  $A_n \in \{f(C_\alpha)\}^\beta$  and (3.1) holds with  $f_{nk}^{(\alpha)}$  instead of  $a_{nk}$ ;

(ii)  $A = (a_{nk}) \in (f(C_\alpha) : c)$  if and only if  $A_n \in \{f(C_\alpha)\}^\beta$  and (3.1)–(3.4) hold with  $f_{nk}^{(\alpha)}$  instead of  $a_{nk}$ ;

(iii)  $A = (a_{nk}) \in (f(C_\alpha) : bs)$  if and only if  $A_n \in \{f(C_\alpha)\}^\beta$  and (3.1) holds with  $f^{(\alpha)}(n, k)$  instead of  $a_{nk}$ ;

(iv)  $A = (a_{nk}) \in (f(C_\alpha) : cs)$  if and only if  $A_n \in \{f(C_\alpha)\}^\beta$  and (3.1)–(3.4) hold with  $f^{(\alpha)}(n, k)$  instead of  $a_{nk}$ ;

(v)  $A = (a_{nk}) \in (f(C_\alpha) : f)$  if and only if  $A_n \in \{f(C_\alpha)\}^\beta$  and (3.1), (4.8), (4.10), and (4.11) hold with  $f_{nk}^{(\alpha)}$  instead of  $a_{nk}$ .

**Corollary 4.2.** *The following statements hold:*

(i)  $A = (a_{nk}) \in (\ell_\infty : f(C_\alpha))$  if and only if (3.1), (4.8), and (4.9) hold with  $g_{nk}^{(\alpha)}$  instead of  $a_{nk}$ ;

(ii)  $A = (a_{nk}) \in (f : f(C_\alpha))$  if and only if (3.1), (4.8), (4.10), and (4.11) hold with  $g_{nk}^{(\alpha)}$  instead of  $a_{nk}$ ;

(iii)  $A = (a_{nk}) \in (c : f(C_\alpha))$  if and only if (3.1), (4.8), and (4.10) hold with  $g_{nk}^{(\alpha)}$  instead of  $a_{nk}$ ;

(iv)  $A = (a_{nk}) \in (bs : f(C_\alpha))$  if and only if (4.8) and (4.12)–(4.14) hold with  $g_{nk}^{(\alpha)}$  instead of  $a_{nk}$ ;

(v)  $A = (a_{nk}) \in (cs : f(C_\alpha))$  if and only if (4.8) and (4.12) hold with  $g_{nk}^{(\alpha)}$  instead of  $a_{nk}$ .

**5. Conclusion.** The algebraic and topological properties of the domain  $\widehat{f} = (f)_{B(r,s)}$  of the double band matrix  $B(r, s)$  in the space  $f$  was investigated by Başar and Kirişçi [6]. In this work, we essentially deal with the domains  $f_0(C_\alpha)$  and  $f(C_\alpha)$  of the Cesàro matrix  $C_\alpha$  of order  $\alpha$  in the spaces  $f_0$  and  $f$ , respectively. The main results of this study have potential for researchers and will fill a gap in the existing literature.

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