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FIXED-POINT THEOREM FOR AN INFINITE TOEPLITZ MATRIX AND ITS EXTENSION TO GENERAL INFINITE MATRICES

ТЕОРЕМА ПРО НЕРУХОМУ ТОЧКУ ДЛЯ НЕСКІНЧЕННОЇ МАТРИЦІ ТЕПЛІЦА ТА ЇЇ ПОШИРЕННЯ НА ЗАГАЛЬНІ НЕСКІНЧЕННІ МАТРИЦІ

In [V. M. Abramov, *Bull. Austral. Math. Soc.*, **104**, 108 – 117 (2021)], the fixed-point equation was studied for an infinite nonnegative particular Toeplitz matrix. In the present paper, we provide an alternative proof for the existence of a positive solution in the general case. The presented proof is based on the application of a version of the M. A. Krasnosel'ski fixed-point theorem. The results are then extended to the equations with infinite matrices of a general type.

У [V. M. Abramov, *Bull. Austral. Math. Soc.*, **104**, 108 – 117 (2021)] досліджено рівняння з нерухомою точкою для нескінченної невід'ємної особливої матриці Тепліца. У цій статті ми наводимо альтернативне доведення існування додатного розв'язку в загальному випадку. Наведене доведення ґрунтується на застосуванні варіанту теореми М. О. Красносельського про нерухому точку. Результати поширено на рівняння з нескінченними матрицями загального вигляду.

1. Introduction. Let $\boldsymbol{x} = A\boldsymbol{x}$, where A is an infinite matrix with nonnegative entries, and $\boldsymbol{x}$ is an unknown vector-column, the entries of which are denoted $x_0, x_1, \ldots$. By positive solution of the aforementioned matrix equation we mean such a vector $\boldsymbol{x}$, the entries of which satisfy the condition $x_i \geq 0,\ i = 0, 1, \ldots,$ and $\sum_{i=0}^{\infty} x_i > 0$. Fixed-point matrix equations of the form $\boldsymbol{x} = A\boldsymbol{x}$ or $\boldsymbol{x} = A\boldsymbol{x} + \boldsymbol{f}$ have wide application in economics. They describe a quantitative economic model for the interdependencies between different sectors of a national economy or different regional economies [10]. All these models are typically studied under the assumption that $\|A\| < 1$, and their analysis uses principle of contraction mapping and iterative numerical procedures [9]. The detailed study of linear systems can also be found in [8]. For the infinite matrices, normed sequence spaces and related topics, the reader can refer to the textbook [3].

However, in a large number of problems in the areas of stochastic processes and applied probability (e.g., [2, 16, 17]) it is required to study the solutions of fixed-point matrix equations or convolution type recurrence relations, in which the assumption $\|A\| < 1$ becomes insufficient. The aim of the present paper is to find the conditions for a quite general class of infinite matrices (not necessarily obeying $\|A\| < 1$), under which the equation $\boldsymbol{x} = A\boldsymbol{x}$ has a positive solution. Our result is first demonstrated on the equation $\boldsymbol{x} = T\boldsymbol{x}$ with the infinite Toeplitz matrix with nonnegative entries. Then it is reformulated and proved for the equations with infinite matrices of general type, for which the basic details of the proof remain unchanged.

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The main novelty of the present paper is an analysis of the case $\|A\| > 1$ for linear operator equations that previously has not been considered in the literature. The fixed-point equations for nonlinear expansive operators has been earlier considered in [19, 21] and other papers.

Consider the equation

$$\boldsymbol{x} = T\boldsymbol{x}, \tag{1}$$

where

$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-n} & t_{-n-1} & \cdots \\ t_1 & t_0 & t_{-1} & \cdots & t_{-n+1} & t_{-n} & \cdots \\ t_2 & t_1 & t_0 & \cdots & t_{-n+2} & t_{-n+1} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ t_n & t_{n-1} & t_{n-2} & \cdots & t_0 & t_{-1} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2}$$

is an infinite Toeplitz matrix with nonnegative entries. Equation (1) with the matrix (2) has been studied in [1]. To initiate our study in the present paper, we need to recall the main theorem proved in [1].

Let $\tau_{-n}(z) = \sum_{i=0}^{\infty} t_{i-n} z^i$, $0 \leq z \leq 1$.

Theorem 1.1. *Assume that* $n = \max\{j : t_{-j} > 0\} < \infty$ *and*

$$\frac{d}{dz} \sqrt[n]{\tau_{-n}(z)} \quad \textit{increases}. \tag{3}$$

(i) *If* $\sum_{i=0}^{\infty} t_{i-n} > 1$, *then all positive solutions* (*if any*) *are bounded and* $\lim_{i\to\infty} x_i = 0$.

(ii) *If* $\sum_{i=0}^{\infty} t_{i-n} = 1$, *then all positive solutions are bounded if and only if* $\sum_{i=0}^{\infty} i t_{i-n} < n$. *In the case* $n = 1$, *if* $\sum_{i=0}^{\infty} i t_{i-1} < 1$, *then* $\lim_{i\to\infty} x_i$ *exists*, *and* $\lim_{i\to\infty} x_i = \dfrac{x_0 t_{-1}}{1 - \sum_{i=1}^{\infty} i t_{i-1}}$.

(iii) *If* $\sum_{i=0}^{\infty} t_{i-n} < 1$, *then any positive solution is unbounded.*

We have the following important comments on this theorem.

(C1) The condition $n = \max\{j : t_{-j} > 0\} < \infty$ means that we deal with

$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-n} & 0 & 0 & \cdots \\ t_1 & t_0 & t_{-1} & \cdots & t_{-n+1} & t_{-n} & 0 & \cdots \\ t_2 & t_1 & t_0 & \cdots & t_{-n+2} & t_{-n+1} & t_{-n} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \\ t_n & t_{n-1} & t_{n-2} & \cdots & t_0 & t_{-1} & t_{-2} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad t_{-n} > 0. \tag{4}$$

(C2) Claim (i) of Theorem 1.1 is true without technical condition (3), since from the infinite system of equations

$$0 = \underbrace{(t_0 - 1)x_j}_{\text{negative}} + \sum_{i=1}^{j} t_i x_{j-i} + \sum_{i=1}^{n} t_{-i} x_{j+i}, \quad j = 0, 1, \ldots, \tag{5}$$

under the assumption $\sum_{i=0}^{\infty} t_{i-n} > 1$, we arrive at $\sum_{i=0}^{\infty} x_i < \infty$. (In other words, the proof of Theorem 1.1(i) does not require technical assumption (3). The aforementioned technical assumption is used in the proof of Theorem 1.1(ii) only.)

(C3) The statements of Theorem 1.1 admit the extension for n increasing to infinity in a straightforward way.

(C4) If (1) with the matrix T defined by (4) or (2) has a positive solution, then there are infinitely many different positive solutions in general. For instance, if $\boldsymbol{x}^*$ is a positive solution, then $c\boldsymbol{x}^*$ is also a positive solution for any $c > 0$. However, if $x_0,\ x_1, \ldots, x_{n-1}$ are fixed, then the solution found by the recurrence relation is, of course, unique.

(C5) If $\boldsymbol{x}$ is a positive solution of (1), then every $x_i,\ i = 0, 1, \ldots,$ must be strictly positive. This follows directly from representation (5), where every term $x_i t_{-n}$ with $t_{-n} > 0$ can be expressed explicitly via the other terms, the sum of which is positive. If one of x_i's is set to zero, then it turns out that all $x_i,\ i = 0, 1, \ldots,$ are equal to zero, and we arrive at the contradiction.

(C6) Comment (C5) remains true in the situation when the assumption $n = \max\{j : t_{-j} > 0\} < \infty$ is not satisfied, since in this case we have the similar to (5) system of the equations

$$0 = \underbrace{(t_0 - 1)x_j}_{\text{negative}} + \sum_{i=1}^{j} t_i x_{j-i} + \sum_{i=1}^{\infty} t_{-i} x_{j+i}, \quad j = 0, 1, \ldots.$$

On the basis of comments (C2) and (C3) we have the following extension of Theorem 1.1(i).

***Corollary* 1.1.** *For equation* (1) *with the matrix T defined by* (2) *the following statement is true. Suppose that $\sum_{i=-\infty}^{\infty} t_i > 1$. If a positive solution of* (1) *exists, then $\sum_{i=0}^{\infty} x_i < \infty$.*

Conditions under which a solution of (1) with the matrix defined by (4) exists has been discussed in [1, Section 3.3]. It was shown that a positive solution exists if $\sum_{i=0}^{\infty} t_i < 1$. The proof provided there was entirely straightforward and based on a number of case studies. It cannot admit further extensions for more general types of matrix. In the alternative proof given in this paper for a more general situation, we show that the aforementioned condition can be easily obtained from the fixed-point theorem of Krasnosel'skii. The statement is then extended for a general class of infinite matrices.

Below we recall the fixed-point theorem of Krasnosel'skii (see [4, 7, 15]).

Theorem 1.2 (Krasnosel'skii [7]). *Let $\mathcal{M}$ be a closed convex nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map $\mathcal{M}$ into S such that*

(i) $A\boldsymbol{x} + B\boldsymbol{y} \in \mathcal{M}$ (*for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}$),

(ii) *A is continuous and $A\mathcal{M}$ is contained in a compact set*,

(iii) *B is a contraction mapping with constant $\alpha < 1$.*

Then there is a vector $\boldsymbol{y} \in \mathcal{M}$ with $A\boldsymbol{y} + B\boldsymbol{y} = \boldsymbol{y}$.

The fixed-point theorem of Krasnosel'skii was originated for the problems of nonlinear analysis, and all its applications are related to nonlinear problems (e.g., [5, 11, 13, 14]). There is a number of interesting development of this theorem, where some conditions of the theorem have been relaxed (see [11, 22, 23]). An application of Krasnosel'skii fixed-point theorem in the theory of fractional calculus for nonlocal fractional delay differential systems of order $1 < r < 2$ in Banach spaces has been provided in [20].

The applications provided in the present paper are not standard and typical for this theorem. In addition, as it has been mentioned in [4], Krasnosel'skii's theorem in this form is hardly applicable, since the condition (i) is too stringent. We shall use another form of the theorem given in [4, Theorem 2]. Implicitly, the construction given in [4] was earlier used by O'Regan [12], from whom the required theorem was not formulated. Then Burton's theorem has been further developed in the various studies.

The formulation of Burton's theorem is as follows.

Theorem 1.3 (Burton [4]). *Let $\mathcal{M}$ be a closed, convex, and nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that $A\colon \mathcal{M} \to S$ and $B\colon S \to S$ such that*

(i) *B is a contraction with constant $\alpha < 1$,*

(ii) *A is continuous and $A\mathcal{M}$ is contained in a compact subset of S,*

(iii) *$[\boldsymbol{x} = B\boldsymbol{x} + A\boldsymbol{y}, \boldsymbol{y} \in \mathcal{M}]$ implies $\boldsymbol{x} \in \mathcal{M}$.*

Then there is a $\boldsymbol{y} \in \mathcal{M}$ with $A\boldsymbol{y} + B\boldsymbol{y} = \boldsymbol{y}$.

This paper is organized as follows. In Section 2, we prove the main result for infinite Toeplitz matrices. In Section 3, the result is extended to general matrices with strictly positive entries. In Section 4, we conclude the paper.

2. Fixed-point theorem for infinite Toeplitz matrix. We prove the following theorem.

Theorem 2.1. *For the fixed-point equation defined by* (1) *and* (2), *assume that $\sum_{i=0}^{\infty} t_i < 1$ and $\sum_{i=-\infty}^{\infty} t_i < \infty$. Then the fixed-point equation* (1) *has a positive solution.*

Proof. We define the norm of the infinite matrix T with nonnegative entries based on the known norms for finite-dimensional matrices given in [6, p. 294, 295]. Specifically,

$$\|T\| = \sup_{k \geq 0} \sum_{i=-\infty}^{k} t_i = \lim_{k\to\infty} \sum_{i=-\infty}^{k} t_i = \sum_{i=-\infty}^{\infty} t_i,$$

where the index value $k \geq 0$ is associated with the $(k+1)$st row of the matrix.

The Banach space $(S, \|\cdot\|)$ we deal with is the set of all absolutely convergent series of real numbers with ℓ^1 norm, i.e., it is the set of sequences $\boldsymbol{s} = \{s_0, s_1, \ldots\}$ obeying $\sum_{i=0}^{\infty} |s_i| < \infty$. The norm of a vector $\boldsymbol{x}$ is $\|\boldsymbol{x}\| = \sum_{i=0}^{\infty} |x_i|$.

Consider first the case $\|T\| \leq 1$. Assume first that $\max\{n : t_{-n} > 0\} < \infty$ (i.e., we first consider equation (1) with the matrix T given by (4)). The existence of the solution of the fixed-point equation considered in [1], and the conclusion about it was based on a wrong derivation. The corrected version of the proof is given below.

Consider first the case $n = 1$, in which x_0 is an arbitrary positive value. We have

$$x_1 = \frac{x_0}{t_{-1}} \geq x_0, \qquad x_2 = \frac{(1-t_0)x_1 - t_1 x_0}{t_{-1}} \geq \frac{(1-t_0-t_1)x_1}{t_{-1}} \geq x_0.$$

By induction, we arrive at the inequality

$$x_{N+1} \geq \frac{(1 - t_0 - t_1 - \ldots - t_N)x_N}{t_{-1}}.$$

So, $x_N \leq x_{N+1}$ for all $N \geq 0$.

In the case $n \geq 2$, the sequence x_N is no longer monotone increasing. Nevertheless, a positive solution of (1) exists. Indeed, consider the first equality of (5) when $j = 0$:

$$(1 - t_0)x_0 = t_{-1}x_1 + t_{-2}x_2 + \ldots + t_{-n}x_n.$$

Setting $x_0 = x_1 = \ldots = x_{n-1} > 0$ and using the notation $x^*_{n-1} = x_{n-1}$, we obtain

$$x_n = \frac{(1 - t_0 - t_{-1} - \ldots - t_{-n+1})x^*_{n-1}}{t_{-n}}$$

giving us $x_n \geq x^*_{n-1}$.

From the second equality of (5) when $j = 1$, we have

$$(1 - t_0)x_1 = t_1x_0 + t_{-1}x_2 + \ldots + t_{-n}x_{n+1}.$$

It yields

$$x_{n+1} = \frac{(1 - t_1 - t_0 - t_{-1} - \ldots - t_{-n+2})x^*_{n-1} + t_{-n+1}x_n}{t_{-n}}.$$

Denoting $(1 - t_1 - t_0 - t_{-1} - \ldots - t_{-n+2})x_{n-1} + t_{-n+1}x_n = x^*_n(1 - t_1 - t_0 - t_{-1} - \ldots - t_{-n+1})$, we obtain

$$x_{n+1} = \frac{(1 - t_1 - t_0 - t_{-1} - \ldots - t_{-n+1})x^*_n}{t_{-n}}$$

giving us $x_{n+1} \geq x^*_n$. The procedure continues, and, for x_{n+k}, we obtain

$$x_{n+k} = \frac{(1 - t_k - t_{k-1} - \ldots - t_{-n+1})x^*_{n+k-1}}{t_{-n}}$$

giving us $x_{n+k} \geq x^*_{n+k-1}$. It follows from this procedure that all x_n, $n = 0, 1, \ldots,$ are positive, which means that a positive solution of (1) exists.

Assume now that $\max\{n : t_{-n} > 0\} < \infty$ is not satisfied. Now we prove that under the assumption $\sum_{i=-\infty}^{\infty} t_i \leq 1$ a positive solution exists. Assume first that $\max\{n : t_{-n} > 0\} = N$, where N is a sufficiently large integer number. Following the arguments provided above, under this assumption a positive solution of (1) exists. Due to Comment (C4), it can be reckoned that the first entry of a solution x_0 is fixed and, say, equal to 1. Then in a series of operator equations under different values of parameter N, the first entry x_0 of the vector of a positive solution can be set to 1. From this we arrive at the conclusion that the first entry x_0 of the limiting solution, as $N \to \infty$, is equal to 1 as well, and, therefore, this limiting solution is positive. Hence, in the case when $\max\{n : t_{-n} > 0\} < \infty$ is not satisfied the existence of a positive solution is proved. Note that according to Comment (C6) all entries of a positive solution vector must be positive. Therefore, all other entries $x_1, x_2, \ldots$ of the vector $\boldsymbol{x}$ are positive too.

So, the rest of the proof is provided under the assumption that $1 < \|T\| < \infty$. From Corollary 1.1 we know that any positive solution $\boldsymbol{x}$ (if exists) satisfies the property $\lim_{i\to\infty} x_i = 0$. From the proof of that theorem we have $\sum_{i=0}^{\infty} x_i < \infty$. Based on this, in order to apply Theorem 1.3 we use the following construction. We represent the matrix T as a sum $T_1 + T_2$ of two matrices T_1 and T_2 defined by

$$T_1 = \begin{pmatrix} t_0 & 0 & 0 & 0 & \cdots \\ t_1 & t_0 & 0 & 0 & \cdots \\ t_2 & t_1 & t_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0 & t_{-1} & t_{-2} & \cdots & t_{-n} & t_{-n-1} & \cdots \\ 0 & 0 & t_{-1} & \cdots & t_{-n+1} & t_{-n} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Next, let $K \subset S$ be a set of all nonnegative vectors $\boldsymbol{s}$, $T_1 : K \to K$, where K is a cone of the aforementioned Banach space $(S, \|\cdot\|)$. We aimed to apply Theorem 1.3. In our case, $\mathcal{M}$ is the set of vectors $\boldsymbol{m}$ with positive entries having norm $\|\boldsymbol{m}\| = 1$. Apparently, $\mathcal{M}$ is closed, convex and nonempty set. (The convexity follows, since for any $\boldsymbol{m}_1,\ \boldsymbol{m}_2 \in \mathcal{M}$ and positive $\lambda_1,\ \lambda_2$ satisfying $\lambda_1 + \lambda_2 = 1$, we have $\|\lambda_1\boldsymbol{m}_1 + \lambda_2\boldsymbol{m}_2\| = \lambda_1\|\boldsymbol{m}_1\| + \lambda_2\|\boldsymbol{m}_2\| = 1$.) In addition, the set $T_2\mathcal{M}$ is compact. Indeed, following [18, p. 451 – 458] a set $\mathcal{S} \subset S$ of elements having ℓ^1-norm to be compact should satisfy the following conditions. It should be bounded, closed and equismall at infinity. Recall (see [18, p. 451 – 458]) that a set $\mathcal{S}$ is equismall at infinity, if for any $\epsilon > 0$ there exists a natural number n_ϵ such that $\sum_{i=n_\epsilon}^{\infty} |s_i| < \epsilon$ for all $\boldsymbol{s} = (s_i)_{i=0}^{\infty} \in \mathcal{S}$. In our case, for any $\boldsymbol{m} \in \mathcal{M}$,

$$T_2\boldsymbol{m} = \begin{pmatrix} \sum_{i=1}^{\infty} t_{-i}m_i \\ \sum_{i=1}^{\infty} t_{-i}m_{i+1} \\ \vdots \\ \sum_{i=1}^{\infty} t_{-i}m_{i+N} \\ \vdots \end{pmatrix}.$$

Since the set $\mathcal{M}$ consists of vectors $\boldsymbol{m}$, the entries of which are convergent positive sequences with $\|\boldsymbol{m}\| = 1$, and $\sum_{i=1}^{\infty} t_{-i} < \infty$, then the problem reduces to show that, for any $\epsilon > 0$ and all $\boldsymbol{m} \in \mathcal{M}$, there exists N such that

$$\sum_{k=N}^{\infty}\sum_{i=1}^{\infty} t_{-i}m_{i+k} < \epsilon.$$

The last inequality reduces to

$$\sum_{k=N+1}^{\infty} m_k < \delta$$

for some positive small δ. That is, the compactness of $T_2\mathcal{M}$ reduces to the compactness of $\mathcal{M}$. Since $\mathcal{M}$ is known to be compact, then $T_2\mathcal{M}$ is compact too.

Let $\boldsymbol{x} \in K$ be a vector. We have $\|T_1\boldsymbol{x}\| = \sum_{i=0}^{\infty} t_i \sum_{j=0}^{\infty} x_j = \alpha\|\boldsymbol{x}\|,\ \alpha < 1$. Hence, T_1 is a contraction mapping.

So, (i) and (ii) of Theorem 1.3 are justified. Let us justify (iii).

It follows from Corollary 1.1 that if a positive solution of equation (1) exists, then it must belong to K. According to Comment (C4) one can reckon that among the solutions of (1) there is a solution $\boldsymbol{x}^*$ obeying $\|\boldsymbol{x}^*\| = 1$. Hence, an existing solution can be assumed to belong to $\mathcal{M}$. Consider the equation $\boldsymbol{x} = T_1\boldsymbol{x} + T_2\boldsymbol{y},\ \boldsymbol{y} \in \mathcal{M}$. Show that if $\boldsymbol{x}^{**} = \boldsymbol{x}^{**}(\boldsymbol{y})$ is a solution of this equation, then $\boldsymbol{x}^*$ is a solution of (1) belonging to $\mathcal{M}$ and vice versa. Indeed, suppose that $\boldsymbol{x}^* - (T_1 + T_2)\boldsymbol{x}^* = \mathbf{0}$, where $\mathbf{0}$ is the vector of zeros. Then, taking into account Comments (C5) and (C6), we arrive at the conclusion that there is a vector $\boldsymbol{\lambda}$ such that $\boldsymbol{y} = (\boldsymbol{\lambda}I)\boldsymbol{x}^*$, I is the infinite unit matrix, and

$$\mathbf{0} = (\boldsymbol{\lambda}I)\boldsymbol{x}^* - (\boldsymbol{\lambda}I)(T_1 + T_2)\boldsymbol{x}^* = \boldsymbol{x}^{**} - T_1\boldsymbol{x}^{**} - T_2\boldsymbol{y}.$$

Therefore, if $\boldsymbol{x}^* \in \mathcal{M}$, then $\boldsymbol{x}^{**} \in \mathcal{M}$. The inverse statement is also true, because of the one-to-one correspondence between $\boldsymbol{x}^*$ and $\boldsymbol{x}^{**}$ for any fixed $\boldsymbol{y} \in \mathcal{M}$.

Theorem 2.1 is proved.

3. Fixed-point theorem for general infinity matrices. Theorem 2.1 can be extended for any arbitrary matrix T with nonnegative entries, the entries of which for convenience of the further formulations of the main result are denoted:

$$T = \begin{pmatrix} t_0^{(1)} & t_{-1}^{(1)} & t_{-2}^{(1)} & \cdots \\ t_1^{(2)} & t_0^{(2)} & t_{-1}^{(2)} & \cdots \\ t_2^{(3)} & t_1^{(3)} & t_0^{(3)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For this matrix, the theorem below covers the case $\|T\| > 1$ only.

Theorem 3.1. *Assume that* $t_i^{(j)} > 0$ *for all* $i = \ldots, -1, 0, 1, \ldots$ *and* $j = 1, 2, \ldots$. *Assume also that*

$$\sum_{i=0}^{\infty} \sup_k t_i^{(k+1)} < 1, \tag{6}$$

$$\sup_k \sum_{i=1}^{\infty} t_{-i}^{(k)} < \infty, \tag{7}$$

$$\liminf_{k\to\infty} \left[\sum_{i=0}^{k} t_i^{(k+1)} + \sum_{i=1}^{\infty} t_{-i}^{(k+1)}\right] > 1. \tag{8}$$

Then the equation $\boldsymbol{x} = T\boldsymbol{x}$ *has a positive solution satisfying the property* $\|\boldsymbol{x}\| < \infty$.

Proof. To start the proof let us first discuss our assumptions. The assumption that the terms $t_i^{(j)}$ all are strictly positive, $i = \ldots, -1, 0, 1, \ldots,\ j = 1, 2, \ldots$, excludes the situation when any positive solution is impossible because of inconsistency of the left- and right-hand sides of the equations for the coordinates of the vector. For instance, if (6) is satisfied but $t_{-i}^{(j)} = 0$ for all $i = 1, 2, \ldots$, then, from the first equation, we have $t_0^{(1)}x_0 = x_0$ or $x_0 = 0$, and, from the following equations of the recursion, we obtain $x_n = 0$ for all $n \geq 0$. That is, no positive solution exists.

Assumption (6) is a basic assumption. It guarantees that the matrix

$$T_1 = \begin{pmatrix} t_0^{(1)} & 0 & 0 & 0 & \cdots \\ t_1^{(2)} & t_0^{(2)} & 0 & 0 & \cdots \\ t_2^{(3)} & t_1^{(3)} & t_0^{(3)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a contraction mapping. The last is true since $\|T_1\boldsymbol{x}\| \leq \|\boldsymbol{x}\| \sum_{i=0}^{\infty} \sup_j t_i^{(j)} = \alpha\|\boldsymbol{x}\|$, $\alpha < 1$. The matrix T_1 is similar to that was used in the proof of Theorem 2.1, when the matrix T was presented as $T = T_1 + T_2$.

Inequality (7) keeps the norm of matrix T finite. Otherwise, a positive solution does not need to exist.

Condition (8) is an important assumption that enables us to prove the following claim $(*)$: if a solution of the equation $\boldsymbol{x} = T\boldsymbol{x}$ exists, then it satisfies the property $\sum_{i=0}^{\infty} x_i < \infty$, where x_0, $x_1, \ldots$ are the coordinates of that solution. The proof of $(*)$, as the most important one, is given below.

Prove $(*)$. Suppose that $\boldsymbol{x} = \boldsymbol{x}^*$ is a positive fixed-point solution. Since $\|T\| > 1$, then $\boldsymbol{x}^*$ must belong to some 'good' subset $\mathcal{M} \subset \mathbb{R}_+^{\infty}$. Our aim is to characterize that subset and prove that $\|\boldsymbol{x}^*\| < \infty$.

For a fixed-point solution $\boldsymbol{x}^*$, the fixed-point equation in the matrix-operator form is as follows:

$$\boldsymbol{x}^* = \begin{pmatrix} x_0^* \\ x_1^* \\ \vdots \end{pmatrix} = \begin{pmatrix} t_0^{(1)} & t_{-1}^{(1)} & t_{-2}^{(1)} & \cdots \\ t_1^{(2)} & t_0^{(2)} & t_{-1}^{(2)} & \cdots \\ t_2^{(3)} & t_1^{(3)} & t_0^{(3)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_0^* \\ x_1^* \\ \vdots \end{pmatrix}. \tag{9}$$

Then, for the jth coordinate of $\boldsymbol{x}^*$, its expansion is

$$0 = \underbrace{(t_0^{(j+1)} - 1)x_j^*}_{\text{negative value}} + \sum_{i=1}^{j} t_i^{(j+1)} x_{j-i}^* + \sum_{i=1}^{\infty} t_{-i}^{(j+1)} x_{j+i}. \tag{10}$$

Assuming that j in the equation is large enough and (8) is satisfied, i.e., $\sum_{i=-\infty}^{\infty} t_i^{(j+1)} > 1$ for each j and not approaching 1 as $j \to \infty$, we are aimed to prove that the tail of the series

$$x_{j+1}^* + \ldots + x_{j+n}^* + \ldots$$

is finite, and, hence, $\|\boldsymbol{x}^*\| < \infty$. That is, the 'good' subset $\mathcal{M}$ is the set of all positive sequences $x_0, x_1, \ldots$ such that $\sum_{i=0}^{\infty} x_i < \infty$.

Indeed, under condition (8), consider the series of Toeplitz matrices $T^{(j)}$ given by

$$T^{(j)} = \begin{pmatrix} t_0^{(j+1)} & t_{-1}^{(j+1)} & t_{-2}^{(j+1)} & \cdots & t_{-j}^{(j+1)} \\ t_1^{(j+1)} & t_0^{(j+1)} & t_{-1}^{(j+1)} & \cdots & t_{1-j}^{(j+1)} \\ t_2^{(j+1)} & t_1^{(j+1)} & t_0^{(j+1)} & \cdots & t_{2-j}^{(j+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_j^{(j+1)} & t_{j-1}^{(j+1)} & t_{j-2}^{(j+1)} & \cdots & t_0^{(j+1)} \end{pmatrix}$$

with different $j \geq j_0$, where j_0 is large.

For each of the systems $\boldsymbol{x} = T^{(j)}\boldsymbol{x}$, $j \geq j_0$, a solution $\boldsymbol{x}_j$ exists ($\boldsymbol{x}_j$ is $(j+1)$-dimensional vector). Its coordinates are asymptotically close to the corresponding coordinates of the solution of the fixed-point equation

$$\boldsymbol{x} = \begin{pmatrix} t_0^{(j+1)} & t_{-1}^{(j+1)} & t_{-2}^{(j+1)} & \cdots & t_{-j}^{(j+1)} & 0 & 0 & \cdots \\ t_1^{(j+1)} & t_0^{(j+1)} & t_{-1}^{(j+1)} & \cdots & t_{1-j}^{(j+1)} & t_{-j}^{(j+1)} & 0 & \cdots \\ t_2^{(j+1)} & t_1^{(j+1)} & t_0^{(j+1)} & \cdots & t_{2-j}^{(j+1)} & t_{1-j}^{(j+1)} & t_{-j}^{(j+1)} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \\ t_j^{(j+1)} & t_{j-1}^{(j+1)} & t_{j-2}^{(j+1)} & \cdots & t_0^{(j+1)} & t_{-1}^{(j+1)} & t_{-2}^{(j+1)} & \cdots \\ 0 & t_j^{(j+1)} & t_{j-1}^{(j+1)} & \cdots & t_1^{(j+1)} & t_0^{(j+1)} & t_{-1}^{(j+1)} & \cdots \\ 0 & 0 & t_j^{(j+1)} & \cdots & t_2^{(j+1)} & t_1^{(j+1)} & t_0^{(j+1)} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \boldsymbol{x} \qquad (11)$$

with infinite Toeplitz matrix. The last is true, since according to Theorem 1.1 and Comment (C2) the solution of the last equation satisfies the property $\sum_{i=0}^{\infty} x_i < \infty$, and if we denote the vector of corresponding solution of the equation $\boldsymbol{x} = T^{(j)}\boldsymbol{x}$ by $\boldsymbol{x}^{(j)}$, then the system of equations for its entries $x_0^{(j)}, x_1^{(j)}, \ldots, x_j^{(j)}$ is

$$0 = (t_0^{(j+1)} - 1)x_k^{(j)} + \sum_{i=1}^{k} t_i^{(j+1)} x_{k-i}^{(j)} + \sum_{i=1}^{j-k} t_{-i}^{(j+1)} x_{k+i}^{(j)}, \quad k = 0, 1, \ldots, j.$$

The similar system of equations for (11) is

$$0 = (t_0^{(j+1)} - 1)x_k + \sum_{i=1}^{k} t_i^{(j+1)} x_{k-i} + \sum_{i=1}^{j} t_{-i}^{(j+1)} x_{k+i}, \quad k = 0, 1, \ldots, j.$$

It is readily seen from these systems of equations that given that $\sum_{i=0}^{\infty} x_i < \infty$, under the appropriate normalization condition we obtain $x_i^{(j)} \to x_i$ as $j \to \infty$. Hence, under the assumptions given in the theorem, we have $\limsup_{j\to\infty} \sum_{i=0}^{j} x_i^{(j)} < \infty$, where $x_0^{(j)}, x_1^{(j)}, \ldots, x_j^{(j)}$ are the entries of the vector-solution of the equation $\boldsymbol{x} = T^{(j)}\boldsymbol{x}$.

We shall now show that there exists a solution $\boldsymbol{x}^*$ of the original equation $\boldsymbol{x} = T\boldsymbol{x}$ obeying $\|\boldsymbol{x}^*\| < \infty$ as well. Indeed, the required solution can be approached by the series of solutions of the equations $\boldsymbol{x} = \tilde{T}^{(j)}\boldsymbol{x}$, where

$$\tilde{T}^{(j)} = \begin{pmatrix} t_0^{(1)} & t_{-1}^{(1)} & t_{-2}^{(1)} & \cdots & t_{-j}^{(1)} \\ t_1^{(2)} & t_0^{(2)} & t_{-1}^{(2)} & \cdots & t_{1-j}^{(2)} \\ t_2^{(3)} & t_1^{(3)} & t_0^{(3)} & \cdots & t_{2-j}^{(3)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_j^{(j+1)} & t_{j-1}^{(j+1)} & t_{j-2}^{(j+1)} & \cdots & t_0^{(j+1)} \end{pmatrix},$$

as j increases to infinity. According to the convention, a solution of the original equation $\boldsymbol{x} = T\boldsymbol{x}$ exists. Therefore, for sufficiently large j, a solution of the equation $\boldsymbol{x} = \tilde{T}^{(j)}\boldsymbol{x}$ exists too, and the sequence of solutions $\tilde{\boldsymbol{x}}_j$ as $j \to \infty$ must approach a solution of the original equation. As well, any series of solutions $\tilde{\boldsymbol{x}}_j$ of the equation $\boldsymbol{x} = \tilde{T}^{(j)}\boldsymbol{x}$ obeys $\limsup_{j\to\infty} \|\tilde{\boldsymbol{x}}_j\| < \infty$, since the asymptotic behavior of the solution $\tilde{\boldsymbol{x}}_j$ is specified by the similar asymptotic behavior of the solutions $\boldsymbol{x}_j$ for large j. The required statement follows.

As mentioned above, under assumption (6) $T_1 : K \to K$ is a contraction mapping. The map $T_2 : \mathcal{M} \to K$, being linear, is continuous. As in the proof of Theorem 2.1, $\mathcal{M}$ is the set of all positive convergent sequences $\boldsymbol{m}$ obeying $\|\boldsymbol{m}\| = 1$. According to an assumption of the theorem, $t_i^j > 0$ for all $i = \ldots, -1, 0, 1, \ldots,$ $j = 1, 2, \ldots.$ Therefore, it follows from presentations (9) and (10), any positive solution $\boldsymbol{x}^*$ satisfies the property $x_j^* > 0,$ $j = 0, 1, \ldots.$ This property is an analogue of the properties given in Comments (C5) and (C6) that have been used in the proof of Theorem 2.1. Hence, statement (iii) of Theorem 1.3 is justified similarly to that in the proof of Theorem 2.1. Thus, we arrive at the conclusion that a positive solution of the fixed-point equation $\boldsymbol{x} = T\boldsymbol{x}$ exists.

Theorem 3.1 is proved.

4. Concluding remarks. The circle of research problems of the present paper and the earlier one [1] of the author were originated from the applied problems of the theories of stochastic processes and applied probability mentioned in the introduction, where the certain recurrence relations of convolution type have been considered. The perspectives of the future research seem to be in further applications of the results obtained in these two papers to advanced telecommunication systems, that in turn may initiate novel studies of operator equations.

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References

1. V. M. Abramov, *Fixed point theorem for an infinite Toeplitz matrix*, Bull. Aust. Math. Soc., **104**, 108 – 117 (2021).
2. V. M. Abramov, *Optimal control of a large dam with compound Poisson input and costs depending on water levels*, Stochastics, **91**, № 3, 433 – 483 (2019).
3. F. Başar, *Summability theory and its applications*, 2nd ed., CRC Press/Taylor & Francis Group, Boca Raton etc. (2022).

4. T. A. Burton, *A fixed point theorem of Krasnosel'skii*, Appl. Math. Lett., **11**, 85 – 88 (1998).
5. T. A. Burton, T. Furumochi, *Krasnosel'skii fixed point theorem and stability*, Nonlinear Anal., **49**, 445 – 454 (2002).
6. R. A. Horn, C. R. Johnson, *Matrix analysis*, Cambridge Univ. Press, Cambridge (1988).
7. M. A. Krasnosel'skii, *Some problems of nonlinear analysis*, Amer. Math. Soc. Transl. Ser. 2, **10**, 345 – 409 (1958).
8. M. A. Krasnosel'skii, J. A. Lifshits, A. V. Sobolev, *Positive linear systems*, Heldermann Verlag, Berlin (1989).
9. M. A. Krasnosel'skii, G. M. Vainikko, P. P. Zabreiko, Y. B. Rutitskii, V. Y. Stetsenko, *Approximate solutions of operator equations*, Wolters-Noordhoff, Groningen (1972).
10. W. Leontief, *Input-output economics*, 2nd ed., Oxford Univ. Press, Oxford (1986).
11. L. Liu, Z. Li, *Krasnosel'skii type fixed point theorems and applications*, Proc. Amer. Math. Soc., **136**, 1213 – 1220 (2008).
12. D. O'Regan, *Fixed point theory for the sum of two operators*, Appl. Math. Lett., **9**, 1 – 8 (1996).
13. V. M. Sehgal, S. P. Singh, *On a fixed point theorem of Krasnosel'skii for locally convex spaces*, Pacific J. Math., **62**, 561 – 567 (1976).
14. E. K. Shah, M. Sarvar, D. Beleanu, *Study of Krasnoselkii's fixed point theorem for Caputo – Fabrizio fractional differentional equations*, Adv. Difference Equat., **2020**, Article 178 (2020).
15. D. R. Smart, *Fixed point theorems*, Cambridge Univ. Press, Cambridge (1980).
16. L. Takács, *Combinatorial methods in the theory of stochastic processes*, John Wiley, New York (1967).
17. L. Takács, *On the busy periods of single-server queues with Poisson input and general service times*, Oper. Res., **24**, № 3, 564 – 571 (1976).
18. F. Trèves, *Topological vector spaces, distributions and kernels*, Dover Publ., New York (2006).
19. F. Wang, F. Wang, *Krasnosel'skii type fixed point theorem for nonlinear expansion*, Fixed Point Theory, **13**, 285 – 291 (2012).
20. W. K. Williams, V. Vijayakumar, R. Udhayakumar, K. S. Nisar, *A new study on existence and uniqueness of nonlocal fractional delay differential systems of order* $1 < r < 2$ *in Banach spaces*, Numer. Methods Partial Different. Equat., **37**, 949 – 961 (2021).
21. T. Xiang, R. Yuan, *A class of expansive-type Krasnosel'skii fixed point theorems*, Nonlinear Anal., **71**, 3229 – 3239 (2009).
22. T. Xiang, R. Yuan, *Critical type of Krasnosel'skii fixed point theorem*, Proc. Amer. Math. Soc., **139**, 1033 – 1044 (2011).
23. T. Xiang, R. Yuan, *A note on Krasnosel'skii fixed point theorem*, Fixed Point Theory and Appl., **2015**, Article 99 (2015).

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