DOI: 10.3842/umzh.v76i3.7296

UDC 511

**Yasemin Alp**[1] (Department of Education of Mathematics and Science, Faculty of Education, Selcuk University, Konya, Turkey),
**E. Gokcen Kocer** (Department of Mathematics and Computer Sciences, Faculty of Science, Necmettin Erbakan University, Konya, Turkey)

# LEONARDO AND HYPER-LEONARDO NUMBERS VIA RIORDAN ARRAYS

# ЧИСЛА ТА ГІПЕРЧИСЛА ЛЕОНАРДО В ТЕРМІНАХ МАСИВІВ РІОРДАНА

A generalization of the Leonardo numbers is defined and called the hyper-Leonardo numbers. Infinite lower triangular matrices, whose elements are Leonardo and hyper-Leonardo numbers are considered. Then the $A$- and $Z$-sequences of these matrices are obtained. Finally, the combinatorial identities between the hyper-Leonardo and Fibonacci numbers are obtained using the fundamental theorem of the Riordan arrays.

Визначено узагальнення чисел Леонардо, яке називається гіперчислами Леонардо. Розглянуто нескінченні нижчі трикутні матриці, елементами яких є числа Леонардо та гіперчисла Леонардо. Крім того, отримано $A$- та $Z$-послідовності цих матриць. Насамкінець за допомогою фундаментальної теореми про масиви Ріордана отримано комбінаторні тотожності між гіперчислами Леонардо та числами Фібоначчі.

**1. Introduction.** Special number sequences are one of the subjects that researchers study. One of the most studied special number sequences is the Fibonacci sequence. The Fibonacci sequence is defined by the following recurrent relation for $n \geq 2$:

$$F_n = F_{n-1} + F_{n-2}$$

with the initial conditions $F_0 = 0,\ F_1 = 1$. The generating function of the Fibonacci sequence is

$$g_F(t) = \sum_{n=1}^{\infty} F_n t^n = \frac{1}{1-t-t^2}.$$

More information on the Fibonacci numbers can be found in [9].

There are so many generalizations of the Fibonacci sequence. One of these generalizations is the hyper-Fibonacci sequence. The hyper-Fibonacci sequence is defined by Dil and Mezo as follows:

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}$$

such that $F_n^{(0)} = F_n,\ F_0^{(r)} = 0$ and $F_1^{(r)} = 1$ where $F_n^{(r)}$ is the $n$th hyper-Fibonacci number [6]. Also, the generating function of the hyper-Fibonacci sequence is given by

$$g_{F^{(r)}}(t) = \sum_{n=0}^{\infty} F_n^{(r)} t^n = \frac{t}{(1-t-t^2)(1-t)^r}.$$

The other properties and identities of the hyper-Fibonacci numbers can be found in [2, 6].

---

[1] Corresponding author, e-mail: yaseminalp66@gmail.com.

Now, we give information about the Leonardo numbers. In [4], the Leonardo sequence is introduced by Catarino and Borges. The Leonardo sequence is defined by the following recurrent relation for $n \geq 2$:

$$Le_n = Le_{n-1} + Le_{n-2} + 1$$

with the initial conditions $Le_0 = Le_1 = 1$. As well, the other recurrent relation for the Leonardo sequence is given as

$$Le_{n+1} = 2Le_n - Le_{n-2}$$

for $n \geq 2$. In addition, there is a relation between the Fibonacci and Leonardo numbers which is stated as follows:

$$Le_n = 2F_{n+1} - 1.$$

The generating function of the Leonardo sequence is given as

$$g_{Le}(t) = \sum_{n=0}^{\infty} Le_n t^n = \frac{1 - t + t^2}{1 - 2t + t^3}. \tag{1.1}$$

Also, some other properties and generalizations of the Leonardo numbers are given in [1, 16].

The Riordan arrays play an important role in combinatorics for obtaining combinatorial identities. In addition, the Riordan arrays have applications in many fields of mathematics. The Riordan arrays are infinite lower triangular matrices defined by the formal power series.

Let us consider the following formal power series:

$$g(t) = g_0 + g_1 t + g_2 t^2 + \ldots \qquad \text{and} \qquad f(t) = f_0 + f_1 t + f_2 t^2 + \ldots$$

with $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$. A Riordan matrix is represented as a pair of the formal power series with $D = (g(t), f(t))$. The generating function of $k$th column of the Riordan matrix is defined as $g(t)(f(t))^k$ for $k = 0, 1, 2, \ldots$. The elements of the Riordan matrix $D$ are given by

$$d_{n,k} = \left[t^n\right] g(t)(f(t))^k, \tag{1.2}$$

where $\left[t^n\right]$ is the coefficient operator. The multiplication of two Riordan matrices is defined by

$$(g(t), f(t)) * (h(t), l(t)) = (g(t)h(f(t)), l(f(t))). \tag{1.3}$$

The set of Riordan matrices is a group under the multiplication given by (1.3). This group is known as the Riordan group and it is denoted by $\mathcal{R}$. The identity element of the Riordan group is given by $I = (1, t)$ and the inverse of $(g(t), f(t))$ is defined by

$$(g(t), f(t))^{-1} = \left(\frac{1}{g\left(\overline{f}(t)\right)}, \overline{f}(t)\right), \tag{1.4}$$

where $\overline{f}\ (t)$ is the compositional inverse of $f(t)$ [17]. Let $h(t)$ be the generating function of the sequence $\{h_k\}$ and $D = (g(t), f(t))$ be the Riordan matrix. From the fundamental theorem of the Riordan arrays, the following identity is held:

$$\sum_{k=0}^{n} d_{n,k} h_k = \left[t^n\right] g(t) h(f(t)). \tag{1.5}$$

The diagonal sum of the Riordan matrix is given as follows [18]:

$$\sum_{k=0}^{n} d_{n-k,k} = \left[t^n\right] \frac{g(t)}{1 - t f(t)}. \tag{1.6}$$

The coefficients method is an important technique for proving combinatorial identities. The coefficient shifting is defined by

$$[t^n] t g(t) = [t^{n-1}] g(t). \tag{1.7}$$

Also, the coefficient differentiation has been performed by following rule:

$$\left[t^n\right] g'(t) = (n+1) \left[t^{n+1}\right] g(t), \tag{1.8}$$

where $g'(t)$ is the first derivative of $g(t)$ [14].

There are some properties of the Riordan matrix, two of which are given in [13, 15]. Rogers has found that every element of a Riordan matrix $D = (d_{n,k})$ (except $d_{0,0}$) has been stated as a linear combination of the elements in the preceding row, starting from the preceding column in [15]. Merlini et al. have given that every element in column 0 can be expressed as a linear combination of all the elements in the preceding row in [13]. By these studies, it is seen that a Riordan array can be characterized by two sequences, known as sequence characterizations of the Riordan arrays. These sequence characterizations are given in the following theorem.

**Theorem 1.1.** *Let* $D = (d_{n,k})$ *be an infinite triangular matrix.* $D$ *is a Riordan matrix if and only if there are two sequences* $A = \{a_0, a_1, a_2, a_{3,\ldots}\}$ *and* $Z = \{z_0, z_1, z_2, z_{3,\ldots}\}$ *with* $a_0 \neq 0$ *such that*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \ldots$$

*and*

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \ldots .$$

$A = \{a_0, a_1, a_2, a_{3,\ldots}\}$ and $Z = \{z_0, z_1, z_2, z_{3,\ldots}\}$ are called the $A$- and $Z$-sequences of the Riordan matrix $D$, respectively. If $A(t)$ and $Z(t)$ are the generating functions of the $A$- and $Z$-sequences of the Riordan matrix $D = (g(t), f(t))$, respectively, then $f(t)$ and $g(t)$ are two solutions of the following equations [13]:

$$f(t) = t A(f(t)) \tag{1.9}$$

and

$$g(t) = \frac{d_{0,0}}{1 - t Z(f(t))}. \tag{1.10}$$

Also, some other properties of the $A$- and $Z$-sequences have been found in [7, 8, 18]. In [8], the characterization matrix of $D = (g(t), f(t))$ is given as

$$P=\begin{pmatrix} z_0 & a_0 & 0 & 0 & 0 & \dots \\ z_1 & a_1 & a_0 & 0 & 0 & \dots \\ z_2 & a_2 & a_1 & a_0 & 0 & \dots \\ z_3 & a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.11}$$

where $z_0,\ z_1,\ z_2,\ \dots$ and $a_0,\ a_1,\ a_2,\ \dots$ are elements of the $Z$- and $A$-sequences, respectively. The matrix $D^{-1}=(g(t),f(t))^{-1}$ is the inverse of $D$ with characterization matrix $\overline{P}$. Then we have

$$\overline{P}=D^2P\big(D^{-1}\big)^2. \tag{1.12}$$

Many researchers have studied the subgroups of the Riordan group. In [11], the algebraic structure of the Riordan group is considered and some properties of the Riordan subgroups are given. Moreover, new proofs for some known properties are obtained in [11]. Marshall and Nkwanta have considered the stochastic subgroup, which is the set of Riordan matrices, whose row sums are equal to 1 and they have defined stochastic Lucas arrays and Fibonacci arrays. Also, they have constructed pseudo-involutions of these arrays in [12]. Moreover, the applications of the Riordan arrays have been studied by some researchers. In [10], some combinatorial identities are obtained using the generalized Pascal, Fibonacci and Pell matrices. In [5], the combinatorial identities of the harmonic and hyperharmonic Fibonacci numbers are given. Also, many researchers have considered some analogues of the Riordan representation. Tuglu et al. have studied analogues of the Riordan representation of Pascal matrices via Fibonomial coefficients in [19]. In [20], the $q$-analogue of the fundamental theorem of the Riordan arrays are given and two new binary operations are defined. Baran and Tuglu have obtained $q$-matrices by using the $q$-Riordan representation in [3].

Based on the preceding studies, we consider the Riordan array by the Leonardo numbers and find the $A$- and $Z$-sequences of the Riordan array. Then the hyper-Leonardo sequence is defined and the generating function of this sequence is given. Finally, the relationship between the Fibonacci matrix and a special matrix is given. Using this relation, some combinatorial identities are obtained.

## 2. Leonardo numbers with Riordan arrays.

In this section, we define an infinite lower triangular matrix with the Leonardo numbers and give the inverse of this matrix. Then the $A$- and $Z$-sequences of these matrices are obtained.

Let us consider an infinite lower triangular matrix defined by Leonardo numbers as follows:

$$Le=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 3 & 1 & 1 & 0 & 0 & 0 & \dots \\ 5 & 3 & 1 & 1 & 0 & 0 & \dots \\ 9 & 5 & 3 & 1 & 1 & 0 & \dots \\ 15 & 9 & 5 & 3 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.1}$$

The Riordan representation of the Leonardo matrix in (2.1) is given by

$$Le=\left(\frac{1-t+t^2}{1-2t+t^3},t\right). \tag{2.2}$$

**Proposition 2.1.** *The elements of the Leonardo matrix $Le$ are defined as follows*:

$$l_{n,k}=\begin{cases} Le_{n-k}, & n-k\geq 0,\\ 0, & \textit{otherwise}, \end{cases}$$

*for* $n,k\geq 0$.

***Proof.*** By (1.2), we have

$$l_{n,k}=\left[t^n\right]\frac{1-t+t^2}{1-2t+t^3}t^k.$$

By using (1.7), we get

$$l_{n,k}=\left[t^{n-k}\right]\frac{1-t+t^2}{1-2t+t^3}.$$

By (1.1), we obtain

$$l_{n,k}=\left[t^{n-k}\right]\sum_{k=0}^{\infty}Le_k t^k=Le_{n-k}.$$

Using (1.9) and (1.10), the generating functions of the $A$- and $Z$-sequences for the Leonardo matrix in (2.2) are obtained as follows:

$$A(t)=1 \qquad \text{and} \qquad Z(t)=\frac{1+t-t^2}{1-t+t^2},$$

respectively. It is clear that the elements of the $A$-sequence are stated as $1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ldots$ . Also, the first few elements of the $Z$-sequence can be given as $1,\ 2,\ 0,\ -2,\ -2,\ 0,\ 2,\ 2,\ 0,\ -2,\ -2,\ 0,\ 2,\ 2,\ldots$ .

If we consider (1.11), then the characterization matrix of the Leonardo matrix $Le$ is obtained as follows:

$$P_{Le}=\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots\\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots\\ -2 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots\\ -2 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots\\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, we give the inverse of the Leonardo matrix $Le$. Using (1.4) and (2.2), the Riordan representation of the inverse matrix $Le^{-1}$ is obtained as

$$Le^{-1}=\left(\frac{1-2t+t^3}{1-t+t^2},t\right).$$

Also, the inverse matrix $Le^{-1}$ is given as

$$Le^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -2 & -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & -2 & -1 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 0 & -2 & -1 & 1 & 0 & 0 & \ldots \\ 2 & 2 & 0 & -2 & -1 & 1 & 0 & \ldots \\ 0 & 2 & 2 & 0 & -2 & -1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Using (1.12), the characterization matrix of the matrix $Le^{-1}$ is obtained as follows:

$$\overline{P_{Le}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 3 & 1 & 1 & 0 & \ldots \\ 5 & 3 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^2 \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots \\ 2 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ -2 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & \ldots \\ -2 & -1 & 1 & 0 & \ldots \\ 0 & -2 & -1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^2$$

$$= \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -3 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ -5 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ -9 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\ -15 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\ -25 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\ -41 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By the definition of the characterization matrix, the elements of the $A$-sequence of the matrix $Le^{-1}$ are stated as $1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0, \ldots$ . Also, the first few elements of the $Z$-sequence of the matrix $Le^{-1}$ are given by $-Le_1, -Le_2, -Le_3, -Le_4, -Le_5, \ldots$ .

**Proposition 2.2.** *Let us consider the Riordan matrix* $D$ *as follows*:

$$D = (d_{n,k}) = \left(\frac{1}{1-t-t^2}, \frac{t}{1-t+t^2}\right) \tag{2.3}$$

*for* $n, k \geq 0$. *The diagonal sum of the matrix* $D$ *is the Leonardo number.*

***Proof.*** Using (1.6), we have

$$\sum_{k=0}^{n} d_{n-k,k} = \left[t^n\right]\left(\frac{1}{1-t-t^2}\right)\left(\frac{1}{1-t\dfrac{t}{1-t+t^2}}\right) = \left[t^n\right]\frac{1-t+t^2}{1-2t+t^3}.$$

By (1.1), we obtain

$$\sum_{k=0}^{n} d_{n-k,k} = \left[t^n\right]\sum_{k=0}^{\infty} Le_k t^k = Le_n.$$

The first few entries of the matrix $D$ in (2.3) are given as follows:

$$D=\begin{pmatrix}1&0&0&0&0&0&0&0\\1&1&0&0&0&0&0&0\\2&2&1&0&0&0&0&0\\3&3&3&1&0&0&0&0\\5&4&5&4&1&0&0&0\\8&6&6&8&5&1&0&0\\13&10&7&10&12&6&1&0\\21&17&11&9&17&17&7&1\end{pmatrix}.$$

**Proposition 2.3.** *Let us consider the Riordan matrix* $H=(g'_{Le}(t),t)$, *where* $g_{Le}(t)$ *is the generating function of the Leonardo sequence in* (1.1). *The elements of the matrix* $H$ *are*

$$h_{n,k}=\begin{cases}(n-k+1)Le_{n-k+1}, & n-k\geq 0,\\ 0, & \textit{otherwise},\end{cases}\tag{2.4}$$

*for* $n,k\geq 0$.

***Proof.*** Using (1.2), we obtain $h_{n,k}=\left[t^n\right]g'_{Le}(t)t^k$. By (1.7), we get $h_{n,k}=\left[t^{n-k}\right]g'_{Le}(t)$. If we use (1.8), then

$$\begin{aligned}h_{n,k}&=\left[t^{n-k+1}\right](n-k+1)g_{Le}(t)\\&=\left[t^{n-k+1}\right](n-k+1)\frac{1-t+t^2}{1-2t+t^3}.\end{aligned}$$

By (1.1), we obtain

$$h_{n,k}=\left[t^{n-k+1}\right](n-k+1)\sum_{s=0}^{\infty}Le_st^s=(n-k+1)Le_{n-k+1}.$$

The few entries of the matrix $H$ in (2.4) are given as follows:

$$H=\begin{pmatrix}1&0&0&0&0&0&0\\6&1&0&0&0&0&0\\15&6&1&0&0&0&0\\36&15&6&1&0&0&0\\75&36&15&6&1&0&0\\150&75&36&15&6&1&0\\287&150&75&36&15&6&1\end{pmatrix}.$$

Let us consider the generating function of the Leonardo sequence in (1.1). If we take $h(t)=\dfrac{1}{1-t}$ in (1.5) to obtain the stochastic Leonardo number arrays, we have

$$h(f(t))=\frac{1-2t+t^3}{(1-t+t^2)(1-t)},$$

$$f(t)=\frac{2t\left(-t+t^2\right)}{1-2t+t^3}.$$

Then the stochastic Leonardo number array is

$$\left(\frac{1-t+t^2}{1-2t+t^3}, \frac{2t\big(-t+t^2\big)}{1-2t+t^3}\right). \tag{2.5}$$

The first few entries of (2.5) are obtained as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & -12 & 4 & 0 & 0 & 0 & 0 & 0 \\ 15 & -26 & 12 & 0 & 0 & 0 & 0 & 0 \\ 25 & -56 & 40 & -8 & 0 & 0 & 0 & 0 \\ 41 & -112 & 104 & -32 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

It is clear that the row sums of (2.6) are equal to 1. It is noted that the matrix in (2.6) does not have an inverse. Nevertheless, we can regulate this matrix so that it belongs to the Riordan group. If the generating function of the Leonardo sequence starts from $Le_1$, we obtain a stochastic Leonardo matrix as follows:

$$\left(\frac{-t^2+t+1}{t^3-2t+1}, \frac{2t}{t^2+t-1}\right). \tag{2.7}$$

Then, the first terms of (2.7) are given as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 & 0 & 0 & 0 \\ 5 & -8 & 4 & 0 & 0 & 0 & 0 \\ 9 & -20 & 20 & -8 & 0 & 0 & 0 \\ 15 & -46 & 64 & -48 & 16 & 0 & 0 \\ 25 & -96 & 176 & -184 & 112 & -32 & 0 \\ 41 & -192 & 432 & -584 & 496 & -256 & 64 \end{pmatrix}. \tag{2.8}$$

Using (1.9) and (1.10), the generating functions of the $A$- and $Z$-sequences of the matrix in (2.8) are given as

$$A(t) = \frac{2t^2}{-t+2-\sqrt{5t^2-4t+4}}$$

and

$$Z(t) = \frac{\Big(t-2+\sqrt{5t^2-4t+4}\Big)^2 - 2t\Big(t-2+\sqrt{5t^2-4t+4}\Big) - 12t^2}{\Big(t-2+\sqrt{5t^2-4t+4}\Big)^2 + 2t\Big(t-2+\sqrt{5t^2-4t+4}\Big) - 4t^2},$$

respectively. A few elements of the $A$-sequence can be given by

$$-2,\ 1,\ \frac{-1}{2},\ \frac{-1}{4},\ 0,\ \frac{1}{8},\ \frac{3}{32},\ \frac{-1}{64},\ \frac{-11}{128},\ \frac{-15}{256},\ldots .$$

Also, the first few elements of the $Z$-sequence can be given by

$$3,\ 2,\ \frac{5}{2},\ \frac{11}{4},\ \frac{11}{4},\ \frac{21}{8},\ \frac{81}{32},\ \frac{163}{64},\ \frac{337}{128},\ \frac{689}{256},\ldots .$$

**Proposition 2.4.** *Let us consider the Riordan matrix as follows*:

$$B = (b_{n,k}) = (l(t), tl(t)),$$

*where* $l(t) = \dfrac{1-t+t^2}{1-t-t^2+2t^3}$. *The row sum of the Riordan matrix* $B$ *is Leonardo number.*

***Proof.*** Taking $h(t) = \dfrac{1}{1-t}$ in (1.5), we obtain

$$\begin{aligned}\sum_{k=0}^{n} b_{n,k} &= \left[t^n\right] l(t) \frac{1}{1-tl(t)} \\ &= \left[t^n\right]\left(\frac{1-t+t^2}{1-t-t^2+2t^3}\right)\left(\frac{1-t-t^2+2t^3}{1-2t+t^3}\right) = \left[t^n\right]\frac{1-t+t^2}{1-2t+t^3}.\end{aligned}$$

By (1.1), we have

$$\sum_{k=0}^{n} b_{n,k} = \left[t^n\right]\sum_{k=0}^{\infty} Le_k t^k = Le_n.$$

The first few entries of the matrix $B$ is obtained as follows:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 6 & 0 & 1 & 0 & 0 \\ -2 & 8 & 0 & 8 & 0 & 1 & 0 \\ 0 & -4 & 18 & 0 & 10 & 0 & 1 \end{pmatrix}.$$

**3. Hyper-Leonardo numbers and Riordan arrays.** In this section, we define the hyper-Leonardo numbers and give the generating function of the hyper-Leonardo sequence. Then some identities are obtained by using Riordan arrays.

**Definition 3.1.** *Let* $r$ *be a positive integer. Then the* $n$*th hyper-Leonardo number* $Le_n^{(r)}$ *is defined as follows*:

$$Le_n^{(r)} = \sum_{k=0}^{n} Le_k^{(r-1)} \tag{3.1}$$

*with* $Le_n^{(0)} = Le_n$, $Le_0^{(r)} = 1$ *and* $Le_1^{(r)} = r+1$.

**Theorem 3.1.** *The generating function of the hyper-Leonardo sequence is*

$$g_{Le_n^{(r)}}(t) = \sum_{n=0}^{\infty} Le_n^{(r)} t^n = \frac{1-t+t^2}{(1-2t+t^3)(1-t)^r}. \tag{3.2}$$

***Proof.*** Using mathematical induction, the formula is true for $r = 0$. Assume that the formula in (3.2) is true for $r = k$. We will prove that the formula above is true for $r = k + 1$. By using (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} Le_n^{(k+1)} t^n &= Le_0^{(k+1)} + Le_1^{(k+1)} t + Le_2^{(k+1)} t^2 + \ldots \\ &= Le_0^{(k)} + \Big(Le_0^{(k)} + Le_1^{(k)}\Big)t + \Big(Le_0^{(k)} + Le_1^{(k)} + Le_2^{(k)}\Big)t^2 + \ldots \\ &= \left(\sum_{n=0}^{\infty} t^n\right)\left(\sum_{n=0}^{\infty} Le_n^{(k)} t^n\right) \\ &= \left(\frac{1}{1-t}\right)\left(\frac{1-t+t^2}{(1-2t+t^3)(1-t)^k}\right) = \frac{1-t+t^2}{(1-2t+t^3)(1-t)^{k+1}}.\end{aligned}$$

Let us consider an infinite lower triangular matrix defined by the hyper-Leonardo numbers as follows:

$$Le^{(r)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ r+1 & 1 & 0 & 0 & \ldots \\ \frac{1}{2}r^2 + \frac{3}{2}r + 3 & r+1 & 1 & 0 & \ldots \\ \frac{1}{6}r^3 + r^2 + \frac{23}{6}r + 5 & \frac{1}{2}r^2 + \frac{3}{2}r + 3 & r+1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.3}$$

The Riordan representation of the hyper-Leonardo matrix $Le^{(r)}$ in (3.3) is given by

$$Le^{(r)} = \left(\frac{1-t+t^2}{(1-2t+t^3)(1-t)^r}, t\right). \tag{3.4}$$

**Proposition 3.1.** *The elements of the hyper-Leonardo matrix* $Le^{(r)}$ *are given as follows*:

$$l_{n,k}^{(r)} = \begin{cases} Le_{n-k}^{(r)}, & n-k \geq 0, \\ 0, & \textit{otherwise}, \end{cases}$$

*for* $n, k \geq 0$.

***Proof.*** By (1.2), we have

$$l_{n,k}^{(r)} = \big[t^n\big] \frac{1-t+t^2}{(1-2t+t^3)(1-t)^r} t^k.$$

By using (1.7), we obtain

$$l_{n,k}^{(r)} = \big[t^{n-k}\big] \frac{1-t+t^2}{(1-2t+t^3)(1-t)^r}.$$

By (3.2), we get

$$l_{n,k}^{(r)} = \left[t^{n-k}\right] \sum_{k=0}^{\infty} Le_k^{(r)} t^k = Le_{n-k}^{(r)}.$$

The first few terms of the $A$-sequence of the hyper-Leonardo matrix $Le^{(r)}$ are stated as $1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ldots$. Also, the first few terms of the $Z$-sequence of the hyper-Leonardo matrix $Le^{(r)}$ are given by

$$r+1,\ -\frac{1}{2}r^2-\frac{1}{2}r+2,\ \frac{1}{6}r\left(r^2-13\right),\ -\frac{1}{24}r^4+\frac{1}{12}r^3+\frac{25}{24}r^2-\frac{13}{12}r-2,\ldots.$$

**Proposition 3.2.** *Let $Le_n^{(r)}$ be the $n$th hyper-Leonardo number. Then we have*

$$Le_n^{(r+2)} = \sum_{k=0}^{n} (k+1) Le_{n-k}^{(r)}. \tag{3.5}$$

***Proof.*** Taking $h(t) = \dfrac{1}{(1-t)^2}$ in (1.5) and using (3.4), we get

$$\begin{aligned}\sum_{k=0}^{n} (k+1) Le_{n-k}^{(r)} &= [t^n] \left( \frac{1-t+t^2}{(1-2t+t^3)(1-t)^r} \right) \left( \frac{1}{1-t} \right)^2 \\ &= \left[t^n\right] \frac{1-t+t^2}{(1-2t+t^3)(1-t)^{r+2}}.\end{aligned}$$

By (3.2), we obtain

$$\sum_{k=0}^{n} (k+1) Le_{n-k}^{(r)} = \left[t^n\right] \sum_{n=0}^{\infty} Le_n^{(r+2)} t^n = Le_n^{(r+2)}.$$

If $r=0$ in (3.5), we have

$$Le_n^{(2)} = \sum_{k=0}^{n} (k+1) Le_{n-k},$$

where $Le_n$ is the $n$th Leonardo number.

**Lemma 3.1.** *Let us consider the generating function $\dfrac{1}{(1-t)^r}$. Then we have*

$$\left( \frac{1}{(1-t)^r}, t \right) = \begin{cases} \dbinom{r+n-k-1}{r-1}, & n-k \geq 0, \\ 0, & \textit{otherwise}. \end{cases} \tag{3.6}$$

**Proposition 3.3.** *The following equation is satisfied*:

$$Le_n^{(r+m)} = \sum_{k=0}^{n} \binom{r+n-k-1}{r-1} Le_k^{(m)} \tag{3.7}$$

*for $n \geq 1$, $m \geq 0$ and $r > 0$.*

***Proof.*** Let $h(t) = \dfrac{1-t+t^2}{(1-2t+t^3)(1-t)^m}$ be in (1.5). By using (3.6), we have

$$\sum_{k=0}^{n}\binom{r+n-k-1}{r-1}Le_k^{(m)} = [t^n]\left(\frac{1}{1-t}\right)^r\left(\frac{1-t+t^2}{(1-2t+t^3)(1-t)^m}\right)$$
$$= \left[t^n\right]\frac{1-t+t^2}{(1-2t+t^3)(1-t)^{r+m}}.$$

By (3.2), we obtain

$$\sum_{k=0}^{n}\binom{r+n-k-1}{r-1}Le_k^{(m)} = \left[t^n\right]\sum_{n=0}^{\infty}Le_n^{(r+m)}t^n = Le_n^{(r+m)}.$$

Taking $m = 0$ in (3.7), we get

$$Le_n^{(r)} = \sum_{k=0}^{n}\binom{r+n-k-1}{r-1}Le_k,$$

where $Le_n^{(r)}$ and $Le_k$ are the $n$th hyper-Leonardo number and the $k$th Leonardo number, respectively.

**Proposition 3.4.** *Let us consider the Riordan representation with*

$$S = (s_{n,k}) = \left(\frac{1-t+t^2}{1-2t+t^3}, \frac{t}{1-t}\right) \tag{3.8}$$

*for* $n, k \geq 0$. *The elements of the matrix* $S$ *are* $Le_{n-k}^{(k)}$.

***Proof.*** Considering (3.2) and (1.2), we have

$$s_{n,k} = \left[t^n\right]\left(\frac{1-t+t^2}{1-2t+t^3}\right)\left(\frac{t}{1-t}\right)^k = \left[t^n\right]\left(\frac{1-t+t^2}{1-2t+t^3}\right)\frac{t^k}{(1-t)^k}.$$

By (1.7), we obtain

$$s_{n,k} = \left[t^{n-k}\right]\frac{1-t+t^2}{(1-2t+t^3)(1-t)^k}.$$

By using (3.2), we get

$$s_{n,k} = \left[t^{n-k}\right]\sum_{n=0}^{\infty}Le_n^{(k)}t^n = Le_{n-k}^{(k)}.$$

The first few elements of the Riordan matrix $S$ in (3.8) are obtained as follows:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 5 & 5 & 3 & 1 & 0 & 0 & 0 \\ 9 & 10 & 8 & 4 & 1 & 0 & 0 \\ 15 & 19 & 18 & 12 & 5 & 1 & 0 \\ 25 & 34 & 37 & 30 & 17 & 6 & 1 \end{pmatrix}.$$

Also, the first few terms of the $A$-sequence of the matrix $S$ can be given as $1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,$ $0,\ldots$ . Furthermore, the first few terms of the $Z$-sequence of the matrix $S$ can be given as $1,\ 2,\ -2,$ $0,\ 2,\ -2,\ 0,\ 2,-2,\ldots$ .

**Proposition 3.5.** *The following identity is satisfied*:

$$Le_n^{(s)}=\sum_{k=0}^{n}\binom{s}{k}Le_{n-k}^{(k)},$$

*where $s$ is a nonnegative integer.*

***Proof.*** If we use (3.8) and take $h(t)=(1+t)^s$ in (1.5), then we get

$$\begin{aligned}\sum_{k=0}^{n}\binom{s}{k}Le_{n-k}^{(k)}&=\left[t^n\right]\left(\frac{1-t+t^2}{1-2t+t^3}\right)h\left(\frac{t}{1-t}\right)\\&=\left[t^n\right]\left(\frac{1-t+t^2}{1-2t+t^3}\right)\left(1+\frac{t}{1-t}\right)^s\\&=\left[t^n\right]\frac{1-t+t^2}{(1-2t+t^3)(1-t)^s}.\end{aligned}$$

By (3.2), we have

$$\sum_{k=0}^{n}\binom{s}{k}Le_{n-k}^{(k)}=\left[t^n\right]\sum_{n=0}^{\infty}Le_n^{(s)}t^n=Le_n^{(s)}.$$

Now, let us give the factorization of the matrix $S$ in (3.8). We know that the Riordan representation of the Fibonacci matrix is given by

$$F=\left(\frac{1}{1-t-t^2},t\right). \tag{3.9}$$

We may define a new matrix $M=(m_{i,j})$ as an infinite triangular matrix as follows:

$$m_{i,j}=\binom{i}{j}-\binom{i-1}{j}+\binom{i-2}{j}. \tag{3.10}$$

Considering the definition of the matrix $M$, we obtain

$$m_{i,j}=m_{i-1,j-1}+m_{i-1,j} \tag{3.11}$$

with $i,j\geq 1$.

**Theorem 3.2.** *Let $M$ be the matrix which is defined in* (3.10). *Then we have*

$$S=F*M, \tag{3.12}$$

*where the matrices $S$ and $F$ are defined in* (3.8) *and* (3.9).

***Proof.*** Firstly, let us find the Riordan representation of the matrix $M$. By (3.10), we get

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 3 & 4 & 3 & 1 & 0 & 0 & \dots \\ 1 & 4 & 7 & 7 & 4 & 1 & 0 & \dots \\ 1 & 5 & 11 & 14 & 11 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The first column vector of the matrix $M$ is $(1, 0, 1, 1, 1, 1, 1, \dots)^T$. Then the generating function of the first column is obtained as

$$\begin{aligned} g_M(t) &= 1 + 0.t + 1.t^2 + 1.t^3 + \dots \\ &= 1 + t^2\big(1 + 1.t + 1.t^2 + 1.t^3 + \dots\big) = 1 + t^2\frac{1}{1-t} = \frac{t^2 - t + 1}{1-t}. \end{aligned}$$

By (3.11), we obtain

$$g_M(t)(f_M(t))^j = tg_M(t)(f_M(t))^{j-1} + tg_M(t)(f_M(t))^j.$$

Then we have

$$f_M(t) = \frac{t}{1-t}.$$

Therefore, we find the Riordan representation of the matrix $M$ as follows:

$$M = \left(\frac{t^2 - t + 1}{1-t}, \frac{t}{1-t}\right).$$

Considering the Riordan product given in (1.3), we get

$$\begin{aligned} F * M &= \left(\frac{1}{1-t-t^2}, t\right) * \left(\frac{t^2 - t + 1}{1-t}, \frac{t}{1-t}\right) \\ &= \left(\frac{1}{1-t-t^2}\frac{t^2-t+1}{1-t}, \frac{t}{1-t}\right) = \left(\frac{1-t+t^2}{1-2t+t^3}, \frac{t}{1-t}\right). \end{aligned}$$

This is the Riordan representation of $S$ in (3.8).

Now, let us give a identity between the hyper-Leonardo and Fibonacci numbers. By (3.8), we know that the elements of the matrix $S$ are $Le_{n-k}^{(k)}$. Considering (3.12) and the matrix product, we obtain the following identity between the hyper-Leonardo and Fibonacci numbers:

$$Le_{n-j}^{(j)} = \sum_{k=0}^{\infty} F_{n-k+1}\left(\binom{k}{j} - \binom{k-1}{j} + \binom{k-2}{j}\right). \tag{3.13}$$

For $0 \le k < j$, we have

$$\sum_{k=0}^{j-1} F_{n-k+1}\left(\binom{k}{j}-\binom{k-1}{j}+\binom{k-2}{j}\right)=0.$$

Since the matrix $S$ is an infinite lower triangular matrix, we get

$$\sum_{k=n+1}^{\infty} F_{n-k+1}\left(\binom{k}{j}-\binom{k-1}{j}+\binom{k-2}{j}\right)=0,$$

where $n<k<\infty$. So, the expression in (3.13) is reduced

$$Le_{n-j}^{(j)}=\sum_{k=j}^{n} F_{n-k+1}\left(\binom{k}{j}-\binom{k-1}{j}+\binom{k-2}{j}\right).$$

Finally, we have the following identity between the hyper-Leonardo and Fibonacci numbers:

$$Le_{n-j}^{(j)}=\sum_{k=j}^{n} F_{n-k+1}\frac{(k-2)!}{(k-j)!j!}(k(k-1)+j(j-k)).$$

On behalf of all authors, the corresponding author states that there is no conflict of interest.

## References

1. Y. Alp, E. G. Kocer, *Some properties of Leonardo numbers*, Konuralp J. Math., **9**, № 1, 183 – 189 (2021).
2. M. Bahsi, I. Mezo, S. Solak, *A symmetric algorithm for hyper-Fibonacci and hyper-Lucas numbers*, Ann. Math. et Inform., **43**, 19 – 27 (2014).
3. F. Y. Baran, N. Tuglu, *q-Riordan representation*, Linear Algebra and Appl., **525**, 105 – 117 (2017).
4. P. Catarino, A. Borges, *On Leonardo numbers*, Acta Math. Univ. Comenian, **89**, № 1, 75 – 86 (2019).
5. M. Cetin, C. Kizilates, F. Y. Baran, N. Tuglu, *Some identities of harmonic and hyperharmonic Fibonacci numbers*, Gazi Univ. J. Sci., **34**, № 2, 493 – 504 (2021).
6. A. Dil, I. Mezo, *A symmetric algorithm for hyperharmonic and Fibonacci numbers*, Appl. Math. and Comput., **206**, 942 – 951 (2008).
7. T. X. He, R. Sprugnoli, *Sequence characterization of Riordan arrays*, Discrete Math., **309**, 3962 – 3974 (2008).
8. T. X. He, *Matrix characterizations of Riordan arrays*, Linear Algebra and Appl., **465**, 15 – 42 (2015).
9. T. Koshy, *Fibonacci and Lucas numbers with applications*, John Wiley & Sons, Inc., Hoboken, NJ (2018).
10. G. Y. Lee, S. H. Cho, *The generalized Pascal matrix via the generalized Fibonacci matrix and the generalized Pell matrix*, J. Korean Math. Soc., **45**, № 2, 479 – 491 (2008).
11. C. J. Louis, A. Nkwanta, *Some algebraic structure of the Riordan group*, Linear Algebra and Appl., **438**, 2018 – 2035 (2012).
12. C. Marshall, A. Nkwanta, *Fibonacci and Lucas Riordan arrays and construction of pseudo-involutions*, Appl. Anal.; https://doi.org/10.1080/00036811.2021.1989418, (2021).
13. D. Merlini, D. G. Rogers, R. Sprugnoli, M. C. Verri, *On some alternative characterizations of Riordan arrays*, Canad. J. Math., **49**, № 2, 301 – 320 (1997).
14. D. Merlini, R. Sprugnoli, M. C. Verri, *Lagrange inversion: when and how*, Acta Appl. Math., **94**, 233 – 249 (2006).
15. D. G. Rogers, *Pascal triangles, Catalan numbers and renewal arrays*, Discrete Math., **22**, 301 – 310 (1978).
16. A. G. Shannon, *A note on generalized Leonardo numbers*, Notes Number Theory Discrete Math., **25**, № 3, 97 – 101 (2019).
17. L. W. Shapiro, S. Getu, W. J. Woan, L. C. Woodson, *The Riordan group*, Discrete Appl. Math., **34**, 229 – 239 (1991).
18. R. Sprugnoli, *Riordan arrays and combinatorial sums*, Discrete Math., **132**, 267 – 290 (1994).
19. N. Tuglu, F. Yesil, E. G. Kocer, M. Dziemianczuk, *The $F$-analogue of Riordan representation Pascal matrices via Fibonomial coefficents*, J. Appl. Math.; https://dx.doi.org/10.1155/2014/841826, (2014).
20. N. Tuglu, F. Yesil, M. Dziemianczuk, E. G. Kocer, *q-Riordan array for q-Pascal matrix and its inverse matrix*, Turkish J. Math., **40**, № 5, 1038 – 1048 (2016).

Received 23.08.22