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A NOTE ON THE MAPPING THEOREM OF ESSENTIAL PSEUDOSPECTRA ON A BANACH SPACE

ЗАУВАЖЕННЯ ДО ТЕОРЕМИ ПРО ВІДОБРАЖЕННЯ ІСТОТНИХ ПСЕВДОСПЕКТРІВ НА БАНАХОВОМУ ПРОСТОРІ

The main goal of the paper is to determine some basic properties of the essential pseudospectrum of a bounded linear operator A defined on a Banach space X. We also prove two different versions of the essential pseudospectral mapping theorem.

Основна мета цієї статті полягає в тому, щоб з'ясувати деякі основні властивості істотного псевдоспектра обмеженого лінійного оператора A, визначеного на банаховому просторі X. Крім того, доведено дві різні версії теореми про істотне псевдоспектральне відображення.

1. Introduction. Though several authors studied different aspects of the essential pseudospectrum of a unbounded linear operator (see [5, 6]), it is still important to examine certain basic properties of essential pseudospectrum of a bounded linear operator. Let $B(X)$ be the space of all bounded linear operators acting on a complex Banach space X. The spectrum of an element $A \in B(X)$ is defined as

$$\sigma(A) = \big\{\lambda \in \mathbb{C} \colon A - \lambda \text{ is not invertible}\big\}.$$

We know that spectrum of an operator reveals its nature. The behavior of normal matrix can be completely understandable by its spectrum. The spectrum of a non normal matrix, however, may not be very informative. Hence the concept of pseudo spectra has been introduced. Let $A \in B(X)$ and $\varepsilon > 0$. The ε-pseudospectrum of A is denoted by $\Lambda_\varepsilon(A)$ and defined as

$$\Lambda_\varepsilon(A) = \left\{\lambda \in \mathbb{C} \colon \|(A-\lambda)^{-1}\| > \frac{1}{\varepsilon}\right\}$$

by convention $\|(\lambda - A)^{-1}\| = +\infty$ if and only if $\lambda \in \sigma(A)$. In [4], E. B. Davies found the following characterization of the pseudospectrum of a closed linear operator A for every $\varepsilon > 0$:

$$\Lambda_\varepsilon(A) = \bigcup_{\|S\|<\varepsilon} \sigma(A+S). \tag{1}$$

One may refer to book [7] for the theory and application of pseudospectra in the fields of science and engineering. In the numerical computation point of view of pseudospectrum, the pseudospectral mapping theorem is a fundamental result and it plays a vital role.

In [2], A. Ammar and A. Jeribi introduced the concept of essential pseudospectra of a bounded linear operator.

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Definition 1.1. *Let X be a complex Banach space, $A \in B(X)$ and $\varepsilon > 0$. The essential pseudospectrum is defined by*

$$\sigma_{e,\varepsilon}(A) = \bigcap_{K \in \mathcal{K}(X)} \Lambda_\varepsilon(A+K),$$

where $\mathcal{K}(X)$ be the space of all compact linear operators defined on X.

In [1], the following characterization was established:

$$\sigma_{e,\varepsilon}(A) = \bigcup_{\|S\|<\varepsilon} \left(\bigcap_{K \in \mathcal{K}(X)} \sigma(A+S+K) \right).$$

This implies that

$$\sigma_{e,\varepsilon}(A) = \bigcup_{\|S\|<\varepsilon} \sigma_{e5}(A+S),$$

where $\sigma_{e5}(A)$ is the Schechter essential spectrum of the operator A defined by

$$\sigma_{e5}(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A+K).$$

In this paper, we extend the results established in [3, 8] to essential pseudospectra. The main objective of this note is to study the spectral mapping theorem for essential pseudospectrum.

Let $A \in B(X,Y)$. We design by $R(A)$ the range of A. The subspace $N(A)$ is called the null space of A. For $A \in B(X,Y)$, we write $\alpha(A) := \dim N(A)$, $\beta(A) := \dim X/R(A)$, the index of A is the quantity $i(A) := \alpha(A) - \beta(A)$ provided that $\alpha(A)$ and $\beta(A)$ are not both infinite. The set of upper semi-Fredholm operators from X into Y is defined by

$$\Phi_+(X,Y) = \Big\{ A \in B(X,Y) \colon \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } Y \Big\}$$

and the set of lower semi-Fredholm operators from X into Y is defined by

$$\Phi_-(X,Y) = \Big\{ A \in B(X,Y) \colon \beta(A) < \infty \text{ and } R(A) \text{ is closed in } Y \Big\}.$$

$\Phi_\pm(X,Y) = \Phi_+(X,Y) \cup \Phi_-(X,Y)$, denotes the set of semi-Fredholm operators from X into Y and $\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$ denotes the set of Fredholm operators on X into Y. If $X = Y$, then $\Phi_+(X,Y), \Phi_-(X,Y)$, $\Phi_\pm(X,Y)$ and $\Phi(X,Y)$ are replaced, respectively, by $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$ and $\Phi(X)$.

Following theorem gives an important characterization of $\sigma_{e,\varepsilon}(A)$.

Theorem 1.1 [2, Theorem 2.1]. *Let $A \in B(X)$ and $\varepsilon > 0$. Then $\lambda \in \sigma_{e,\varepsilon}(A)$ if and only if there exists $S \in B(X)$ with $\|S\| < \varepsilon$ such that $A + S - \lambda \notin \Phi(X)$ or $i(A+S-\lambda) \neq 0$.*

This paper is organized as follows. We divide the subsequent sections into three parts. First part carries some basic definitions, notions and results. The second part is devoted to prove some basic properties of ε-essential pseudospectrum. The third part has two essential pseudospectrum mapping theorems. First it is given that a general essential pseudospectrum mapping theorem in the form of two set inclusions is stated and proved. It is shown that the set inclusions reduce to an equality for all bounded linear operators, if the mapping is an linear function. Further, a weaker version of the pseudospectrum mapping theorem is also proved.

2. Preliminary and auxiliary results.

This section contains some basic definitions, notions and related results that will be used in the next sections.

We use the concept of Frechet differentiable and implicit mapping theorem for normed vector spaces to prove the weaker version of essential pseudospectrum mapping theorem, they are as follows.

Definition 2.1 (Frechet differentiable)**.** *Let V and W be two normed vector spaces and $U \subseteq V$ be an open subset of V. A function $f\colon U \to W$ is called Frechet differentiable at $x \in U$ if there exists a bounded linear operator $A\colon V \to W$ such that*

$$\lim_{\|h\|\to 0} \frac{\|f(x+h) - f(x) - Ah\|_W}{\|h\|_V} = 0.$$

***Remark* 2.1** [9, Section 3.2]**.** Let $E_1,\ E_2,\ E_3, \ldots, E_n$ and F be normed vector spaces. We set $E = E_1 \times E_2 \times E_3 \times \ldots \times E_n$ and defines a norm on E as usual by

$$\big\|(x_1, x_2, x_3, \ldots, x_n)\big\|_E = \max\big\{\|x_1\|_{E_1}, \|x_2\|_{E_2}, \ldots, \|x_n\|_{E_n}\big\}.$$

Let O be a open subset of E and f be a mapping from O into F. If we take a point $a \in O$, the kth coordinate vary and fix the others, then we obtain a mapping $f_{a,k}$ from E_k into F defined on an open set of E_k containing a_k. If $f_{a,k}$ is Frechet differentiable at a_k, then we call the differential as the kth partial differential of f at a and write it as $\partial_k f(a)$. Moreover, $\partial_k f(a)$ is a bounded linear operator from E_k to F.

Theorem 2.1 [9, Theorem 8.2]**.** *Let X, Y and Z be Banach spaces, $\mathcal{O}$ be an open subset of $X \times Y$. Let the mapping $f\colon \mathcal{O} \to Z$ be continuously Frechet differentiable. If $(x_0, y_0) \in \mathcal{O}$, $f(x_0, y_0) = 0$ and the map $\partial_2 f(x_0, y_0)\colon Y \to Z$ is invertible, then there exist a neighbourhood $\mathcal{O}'$ of (x_0, y_0) included in $\mathcal{O}$, a neighbourhood U of x_0 in X and a continuously Frechet differentiable function $g\colon U \to Y$ such that $f(x, g(x)) = 0$ and $(x, y) \in \mathcal{O}'$ if and only if $x \in U$ and $y = g(x)$.*

***Remark* 2.2** [10]**.** Let X be a complex Banach space and $A \in B(X)$. Define

$$f(A) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\psi)}{\psi - A} d\psi,$$

where f is a holomorphic function defined on an open set $D \subseteq \mathbb{C}$ which contains $\sigma(A)$, and $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ is a collection of disjoint Jordan curves in D bounding an inside set U such that $\sigma(A) \subset U$ and each γ_i is oriented in the boundary sense.

The following two versions of the pseudospectral mapping theorem were proved by S. H. Lui in [8].

Theorem 2.2. *Let A be a matrix and f be an analytic function defined on $\mathcal{D}$ an open set containing $\sigma(A)$. For each $\varepsilon, s \geq 0$ and sufficiently small, define*

$$\phi(\varepsilon) = \sup_{\psi \in \Lambda_\varepsilon(A)} \inf\big\{r \geq 0\colon f(\psi) \in \Lambda_r(f(A))\big\}$$

and

$$\psi(s) = \sup_{z \in \Lambda_s(f(A))} \inf\big\{r \geq 0\colon z \in f(\Lambda_r(A))\big\}.$$

Then

$$f(\Lambda_\varepsilon(A)) \subseteq \Lambda_{\phi(\varepsilon)}(f(A)) \subseteq f\big(\Lambda_{\psi(\phi(\varepsilon))}(A)\big).$$

Theorem 2.3. *Let* A *be a matrix and* f *be an analytic function defined on an open set containing* $\sigma(A)$. *For each* $\varepsilon \geq 0$ *and sufficiently small, define*

$$\gamma(\varepsilon) = \sup_{\|E\| \leq \varepsilon} \big\| f(A+E) - f(A) \big\|.$$

Then

$$f(\Lambda_\varepsilon(A)) \subseteq \Lambda_{\gamma(\varepsilon)}\big(f(A)\big) \subseteq f\big(\Lambda_{\delta(\gamma(\varepsilon)))}(A)\big),$$

where $\delta = \sup_{\|C\| \leq s} \|\mathcal{E}(C)\|$, $s \leq \alpha$, $\mathcal{E}$ *is an analytic matrix function and* α *is a constant exists by the implicit mapping theorem.*

Remark **2.3.** Unless otherwise stated, we assume that the underlying Banach space is infinite dimensional one.

Theorem 2.4. *Let* X *be a complex Banach space,* $A \in B(X)$ *and* $\varepsilon > 0$. *If* $B(a,r) = \{\lambda \in \mathbb{C} : |\lambda - a| < r\}$, *then*:

(1) $\sigma_{e5}(A) + B(0,\varepsilon) \subseteq \sigma_{e,\varepsilon}(A)$,

(2) $\sigma_{e,\varepsilon}(A)$ *is an open set*,

(3) *if* $A = \mu I$, *then* $\sigma_{e,\varepsilon}(A) = B(\mu,\varepsilon)$,

(4) *if* $A \in \mathcal{K}(X)$, *then* $\sigma_{e,\varepsilon}(A) = B(0,\varepsilon)$,

(5) $\sigma_{e,\varepsilon}(A)$ *is a bounded set*,

(6) $\sigma_{e,\varepsilon}(A+B) \subseteq \sigma_{e,\varepsilon+\|B\|}(A)$ *for any* $B \in B(X)$,

(7) $\sigma_{e,\varepsilon}(kA) = k\sigma_{e,\frac{\varepsilon}{|k|}}(A)$ *for any* $k \in \mathbb{C}$,

(8) $\sigma_{e,\varepsilon}(A+k) = k + \sigma_{e,\varepsilon}(A)$ *for any* $k \in \mathbb{C}$,

(9) *if* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *such that* $\varepsilon_1 < \varepsilon_2$, *then* $\sigma_{e,\varepsilon_1}(A) \subseteq \sigma_{e,\varepsilon_2}(A)$.

Proof. (1) Let $\mu \in B(\lambda,\varepsilon)$ where $\lambda \in \sigma_{e5}(A)$. Consider the operator $S = (\mu - \lambda)I$. Clearly that $\|S\| < \varepsilon$ and $A + S - \mu = A - \lambda$. Hence, either $A + S - \mu$ is non Fredholm or $A + S - \mu$ is Fredhlom of index zero. By using Theorem 1.1, we have $\mu \in \sigma_{e,\varepsilon}(A)$.

(2) Let $\lambda \in \sigma_{e,\varepsilon}(A)$. There exists $D \in B(X)$ with $\|D\| < \varepsilon$ such that $A+D-\lambda$ is non Fredholm or $i(A + D - \lambda) \neq 0$. We first assume that $A + D - \lambda$ is non Fredholm. Take $r = \varepsilon - \|D\|$. Let $\mu \in B(\lambda, r)$. Consider the operator $S = D - \lambda + \mu$. Then we have

$$\|S\| \leq \|D\| + |\mu - \lambda| < \varepsilon - r + r = \varepsilon$$

and

$$A + S - \mu = A + D - \lambda + \mu - \mu = A + D - \lambda. \tag{2}$$

Hence $A + S - \mu$ is non Fredholm and so $\mu \in \sigma_{e,\varepsilon}(A)$. Suppose that $i(A + D - \lambda) \neq 0$. Then, from equation (2), it is easy to see that $B(\lambda, r) \subseteq \sigma_{e,\varepsilon}(A)$ for $r = \varepsilon - \|D\|$.

(3) Note that if $A = \mu$, then

$$\Lambda_\varepsilon(A) = \left\{\lambda \in \mathbb{C} : \frac{1}{|\lambda - \mu|} > \frac{1}{\varepsilon}\right\} = B(\mu,\varepsilon).$$

Since $\sigma_{e,\varepsilon}(A) \subseteq \Lambda_\varepsilon(A)$, we have $\sigma_{e,\varepsilon}(A) \subseteq B(\mu,\varepsilon)$. Clearly that $\mu \in \sigma_{e5}(A)$. By (1), $\sigma_{e,\varepsilon}(A) = B(\mu,\varepsilon)$.

(4) By the definition of $\sigma_{e,\varepsilon}(A)$ and A is compact, we get $\sigma_{e,\varepsilon}(A) \subseteq \Lambda_\varepsilon(0)$. Thus, $\sigma_{e,\varepsilon}(A) \subseteq B(0,\varepsilon)$. Since $0 \in \sigma_{e5}(A)$, by (1) we obtain $\sigma_{e,\varepsilon}(A) = B(0,\varepsilon)$.

(5) We know that $\Lambda_\varepsilon(A) \subset B(0, \|A\| + \varepsilon)$ for any $\varepsilon > 0$. Let

$$m = \inf_{K \in \mathcal{K}(X)} \|A + K\|.$$

From Definition 1.1, it is easy to see that $|\lambda| < m + \varepsilon$ for all $\lambda \in \sigma_{e,\varepsilon}(A)$.

(6) If $\lambda \in \sigma_{e,\varepsilon}(A+B)$, then there exists $D \in B(X)$ with $\|D\| < \varepsilon$ such that either $A+B+D-\lambda \notin \Phi(X)$ or $i(A+B+D-\lambda) \neq 0$. Take $S = B+D$, then either $A+S-\lambda \notin \Phi(X)$ or $i(A+S-\lambda) \neq 0$ with $S \in B(X)$ and $\|S\| < \varepsilon + \|B\|$. Hence, $\lambda \in \sigma_{e,\varepsilon+\|B\|}(A)$.

(7) If $D \in B(X)$ with $\|D\| < \varepsilon$, then

$$kA + D - \lambda \ \text{ is Fredholm } \iff A + \frac{D}{k} - \frac{\lambda}{k} \ \text{ is Fredholm}$$

and

$$i(kA + D - \lambda) = 0 \iff i\left(A + \frac{D}{k} - \frac{\lambda}{k}\right) = 0.$$

Hence, $\sigma_{e,\varepsilon}(kA) = k\sigma_{e,\frac{\varepsilon}{|k|}}(A)$ for any $k \in \mathbb{C}$.

The proof of (8) and (9) follows immediately from the definition of essential pseudospectrum.

3. Essential pseudospectral mapping theorem. This section is devoted to the study of essential pseudospectral mapping theorem. The central idea of all the results in this section follows [8].

The following is an essential pseudospectral mapping theorem for complex analytic functions. Since the complex functions (ϕ, ψ) defining the sizes of the pseudospectra are optimal, it is believed that this theorem is sharp. This theorem is an easy consequence of the definitions of these functions.

The theory of essential spectrum, its characterization and its numerical computation are studied by various authors. One can see [3] for various types of essential spectra and the corresponding essential spectral mapping theorem. The following theorem will be used in the subsequent results.

Theorem 3.1 [3, Theorem 7]. *Let $A \in B(X)$ with a nonempty resolvent set and f be a complex-valued function that is locally holomorphic on an open set containing the extended spectrum of A. Then*

$$\sigma_{e5}(f(A)) \subseteq f(\sigma_{e5}(A)). \tag{3}$$

The following illustration implies that Theorem 3.1 does holds for the essential pseudospectrum.

***Example* 3.1.** Consider the Banach space $\ell^2(\mathbb{N})$. Let

$$S\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \quad \text{by} \quad S(e_i) = e_{i+1} \quad \forall i \in \mathbb{N}.$$

The following holds for S:

$$SS^* \le S^*S, \quad \sigma(S) = \big\{\lambda \in \mathbb{C}\colon |\lambda| \le 1\big\}.$$

It is clear that $\sigma(S) + B(0,\varepsilon) \subseteq \Lambda_\varepsilon(S)$. Since $\|S\| = 1$, we have $\Lambda_\varepsilon(S) \subseteq B(0, 1+\varepsilon)$. Moreover, $\sigma(S) + B(0,\varepsilon) = B(0, 1+\varepsilon)$. Hence, $\Lambda_\varepsilon(S) = \sigma(S) + B(0,\varepsilon)$ and

$$\Lambda_\varepsilon(S) = \big\{\lambda \in \mathbb{C}\colon |\lambda| < 1 + \varepsilon\big\}$$

for any $\varepsilon > 0$. By using the fact that $\sigma_{e5}(S) = \sigma(S)$ (see [10]) and Theorem 2.4 (1), we have

$$\sigma_{e5}(A)+B(0,\varepsilon)\subseteq\sigma_{e,\varepsilon}(A)\subseteq\Lambda_{\epsilon}(S).$$

Consequently, we obtain

$$\sigma_{e,\varepsilon}(S)=\Lambda_{\epsilon}(S)=B(0,1+\varepsilon).$$

Let $\mathcal{D}$ be an open subset of $\mathbb{C}$ such that $\mathcal{D}$ contains $\sigma(A)$. Now, consider the function $f\colon\mathcal{D}\to\mathbb{C}$ such that $f(z)=z^2$. Clearly that f is analytic, $f(S)=S^2$ and $f(S)f(S)^*\leq f(S)^*f(S)$. From [11, Remark 3.8] and the spectral mapping theorem yield us the following:

$$\sigma_{e5}(f(S))=\sigma(f(S))=\big\{\lambda\in\mathbb{C}\colon|\lambda|\leq 1\big\}$$

and

$$\Lambda_{\varepsilon}(f(S))=\sigma(f(S))+B(0,\varepsilon)$$

for any $\varepsilon>0$. Moreover,

$$\sigma_{e,\varepsilon}(f(S))=B(0,1+\varepsilon).$$

We observe that

$$f\big(\sigma_{e,\varepsilon}(S)\big)=\big\{f(\lambda)\in\mathbb{C}\colon\lambda\in\sigma_{e,\varepsilon}(S)\big\}=\big\{\lambda^2\in\mathbb{C}\colon\lambda\in B(0,1+\varepsilon)\big\}\neq\sigma_{e,\varepsilon}(f(S)).$$

Following is a version of essential pseudospectral mapping theorem.

Theorem 3.2. *Let* $A\in B(X)$ *such that* $\sigma_{e5}(A)\neq\varnothing$. *Consider an analytic function* $f\colon\mathcal{D}\to\mathbb{C}$ *where* $\mathcal{D}$ *is an open subset of* $\mathbb{C}$ *such that* $\mathcal{D}$ *contains* $\sigma(A)$. *For* $\varepsilon,s>0$ *are sufficiently small define*

$$\phi(\varepsilon)=\sup_{\zeta\in\sigma_{e,\varepsilon}(A)}\inf\big\{r\geq 0\colon f(\zeta)\in\sigma_{e,r}(f(A))\big\}$$

and

$$\psi(s)=\sup_{z\in\sigma_{e,s}(f(A))}\inf\big\{r\geq 0\colon z\in f(\sigma_{e,r}(A))\big\}.$$

Then

$$f\big(\sigma_{e,\varepsilon}(A)\big)\subseteq\sigma_{e,\phi(\varepsilon)}(f(A))\subseteq f\big(\sigma_{e,\psi(\phi(\varepsilon))}(A)\big).$$

Proof. We first show that ϕ is well defined. If $\zeta\in\sigma_{e,\varepsilon}(A)$, then there exists $E\in B(X)$ with $\|E\|<\varepsilon$ such that

$$\zeta\in\sigma_{e5}(A+E).$$

By using Theorem 3.1, we have

$$f(\zeta)\in\sigma_{e5}(f(A+E))=\sigma_{e5}(f(A)+F),$$

where $F=f(A+E)-f(A)$ and satisfies $\|F\|<r$ for some positive r. We may take r to be independent of ζ since $\zeta\in\sigma_{e,\varepsilon}(A)$. Thus,

$$f(\zeta)\in\sigma_{e,r}(f(A))$$

which implies that the infimum in the definition of ϕ is taken over a nonempty set. The first set inclusion now follows directly from the definition of ϕ and by (9) of Theorem 2.4. Next we show that ψ is well defined. Let $z\in\sigma_{e,s}\big(f(A)\big)$. Then there exists an $F\in B(X)$ with $\|F\|<s$ such that

$$z \in \sigma_{e5}(f(A) + F).$$

Let $z = f(\zeta)$ for some $\zeta \in \mathcal{D}$ (such ζ always exists because the value of s is very small, the map $A \mapsto \sigma(A)$ is upper semicontinuous at $f(A)$ and $f(\mathcal{D})$ is open). We shall show that there exists $r > 0$ such that

$$\zeta \in \sigma_{e,r}(A).$$

Suppose that $\zeta - A + D$ is Fredholm for all $D \in B(X)$. Then A is Fredholm. This is a contradiction to the assumption. Hence, $z \in f(\sigma_{e,r}(A))$ for some $r > 0$. Thus, the infimum in the definition of ψ is taken over a nonempty set and so ψ is well defined and $\psi(s) \leq r$. The second set inclusion now follows directly from the definition of ψ with $s = \phi(\varepsilon)$. This completes the proof.

Next, we see an example that illustrates the above theorem.

Example **3.2.** Consider the operator $S \in B\big(\ell^2(\mathbb{N})\big)$, open subset $\mathcal{D}$ and analytic function $f(z) = z^2$ defined on Example 3.1. From Example 3.1, for any $r > 0$ we know that

$$\sigma_{e,r}(S) = \sigma_{e,r}(f(S)) = B(0, 1 + r).$$

We find $\phi(\varepsilon), \psi(\phi(\varepsilon))$ for sufficiently small $\varepsilon > 0$. We have

$$\begin{aligned} \phi(\varepsilon) &= \sup_{\zeta \in \sigma_{e,\varepsilon}(S)} \inf \big\{ r \geq 0 \colon f(\zeta) \in \sigma_{e,r}(f(S)) \big\} \\ &= \sup_{\zeta \in B(0,1+\varepsilon)} \inf \big\{ r \geq 0 \colon f(\zeta) \in B(0, 1 + r) \big\} \\ &= \sup_{\zeta \in B(0,1+\varepsilon)} \inf \big\{ r \geq 0 \colon \zeta^2 \in B(0, 1 + r) \big\}. \end{aligned}$$

Consider a sequence $\{\zeta_n\}$ such that

$$\zeta_n \in \sigma_{e,\varepsilon}(S), \quad 1 < |\zeta_n| < |\zeta_{n+1}| \qquad \text{and} \qquad \lim_{n \to \infty} |\zeta_n| = 1 + \varepsilon.$$

Then we have

$$|\zeta_n^2| < |\zeta_{n+1}^2| \qquad \text{and} \qquad \lim_{n \to \infty} |\zeta_n^2| = (1 + \varepsilon)^2.$$

For each fixed n, we observe that

$$\zeta_n^2 \in \sigma_{e,1+(|\zeta_{n+1}^2|-1)}(f(S)).$$

Hence,

$$|\zeta_n^2| - 1 \leq \inf \big\{ r \geq 0 \colon \zeta_n^2 \in B(0, 1 + r) \big\} \leq |\zeta_{n+1}^2| - 1$$

and

$$\lim_{n \to \infty} \big\{ r \geq 0 \colon \zeta_n^2 \in \sigma_{e,r}(f(S)) \big\} = \varepsilon^2 + 2\varepsilon.$$

Thus, $\phi(\varepsilon) = \varepsilon^2 + 2\varepsilon$. Then

$$\begin{aligned} \psi(\phi(\varepsilon)) = \psi(\varepsilon^2 + 2\varepsilon) &= \sup_{z \in \sigma_{e,\varepsilon^2+2\varepsilon}(f(S))} \inf \big\{ r \geq 0 \colon z \in f(\sigma_{e,r}(A)) \big\} \\ &= \sup_{z \in B(0,1+\varepsilon^2+2\varepsilon)(f(S))} \inf \big\{ r \geq 0 \colon z \in \{\lambda^2 \colon \lambda \in B(0, 1 + r)\} \big\} \end{aligned}$$

$$= \sup_{z\in B(0,(1+\varepsilon)^2)(f(S))} \inf\big\{r \geq 0\colon z \in \{\lambda^2\colon \lambda\in B(0,1+r)\}\big\}.$$

Consider a sequence $\{z_n\}$ such that

$$z_n \in \sigma_{e,(1+\varepsilon)^2}(f(S)), \qquad 1<|z_n|<|z_{n+1}| \qquad \text{and} \qquad \lim_{n\to\infty}|z_n| = (1+\varepsilon)^2.$$

Then

$$\sqrt{z_n}\in B(0,1+\varepsilon), \qquad |\sqrt{z_n}|<|\sqrt{z_{n+1}}| \qquad \text{and} \qquad \lim_{n\to\infty}|\sqrt{z_n}| = 1+\varepsilon.$$

For each fixed n, we observe that

$$z_n \in \big\{\lambda^2\colon \lambda\in B(0,1+(|\sqrt{z_{n+1}}|-1))\big\}.$$

Hence,

$$|\sqrt{z_n}|-1 \leq \inf\big\{r\geq 0\colon z_n\in\big\{\lambda^2\colon \lambda\in B(0,r)\big\}\big\} \leq |\sqrt{z_{n+1}}|-1$$

and

$$\lim_{n\to\infty}\inf\big\{r\geq 0\colon z_n\in\{\lambda^2\colon\lambda\in B(0,r)\}\big\} = \varepsilon.$$

Thus, $\psi(\phi(\varepsilon))=\varepsilon$. From Theorem 3.2,

$$f\big(\sigma_{e,\varepsilon}(A)\big)\subseteq\sigma_{e,\varepsilon^2}(f(A))\subseteq f\big(\sigma_{e,\varepsilon}(A)\big),$$

consequently,

$$f\big(\sigma_{e,\varepsilon}(A)\big)=\sigma_{e,\varepsilon^2+2\varepsilon}(f(A)).$$

Here are some remarks about Theorem 3.2.

Remark **3.1.** (i) We observe that $\lim_{\varepsilon\to 0}\phi(\varepsilon)=0$ and, consequently, $\lim_{\phi(\varepsilon)\to 0}\psi(\varepsilon)=0$. Moreover, $\sigma_{e,\varepsilon}(A)\to\sigma_{e5}(A)$ as $\varepsilon\to 0$. Hence, Theorem 3.2 reduces to spectral mapping theorem of essential spectrum when $\varepsilon\to 0$ (see Theorem 3.1).

(ii) Consider the affine function $f(z)=\alpha+\beta z$, where α,β are complex numbers. Then

$$\begin{aligned}\phi(\varepsilon) &= \sup_{\zeta\in\sigma_{e,\varepsilon}(A)}\inf\big\{r\geq 0\colon f(\zeta)\in\sigma_{e,r}(f(A))\big\}\\ &= \sup_{\zeta\in\sigma_{e,\varepsilon}(A)}\inf\big\{r\geq 0\colon \alpha+\beta\zeta\in\sigma_{e,r}(\alpha+\beta A)\big\}\\ &= \sup_{\zeta\in\sigma_{e,\varepsilon}(A)}\inf\big\{r\geq 0\colon \alpha+\beta\zeta\in\sigma_{e}(\alpha+\beta A+F)\ \text{ for some }\ \|F\|<r\big\}\\ &= \sup_{\zeta\in\sigma_{e,\varepsilon}(A)}\inf\left\{r\geq 0\colon \alpha+\beta\zeta\in\sigma_{e5}\left(\alpha+\beta\left(A+\frac{F}{\beta}\right)\right)\ \text{ for some }\ \|F\|<r\right\}\\ &= \sup_{\zeta\in\sigma_{e,\varepsilon}(A)}\inf\left\{r\geq 0\colon \alpha+\beta\zeta\in\alpha+\beta\sigma_{e5}\left(A+\frac{F}{\beta}\right)\ \text{ for some }\ \|F\|<r\right\}\\ &= \sup_{\zeta\in\sigma_{e,\varepsilon}(A)}\inf\Big\{r\geq 0\colon \alpha+\beta\zeta\in\alpha+\beta\sigma_{e,\frac{r}{|\beta|}}(A)\Big\}.\end{aligned}$$

By the essential spectral mapping theorem, one can see that

$$\alpha+\beta\zeta\in\alpha+\beta\sigma_{e,\frac{r}{|\beta|}}(A) \quad \text{if and only if} \quad \zeta\in\sigma_{e,\frac{r}{|\beta|}}(A). \tag{4}$$

Hence, for each $\zeta\in\sigma_{e,\varepsilon}(A)$,

$$\inf\bigl\{r\geq 0\colon \alpha+\beta\zeta\in\alpha+\beta\sigma_{e,\frac{r}{|\beta|}}(A)\bigr\}\leq\varepsilon|\beta|.$$

Thus, $\phi(\varepsilon)\leq\varepsilon|\beta|$ and $\sigma_{e,\frac{\phi(\varepsilon)}{|\beta|}}(\alpha+\beta A)\subseteq\sigma_{e,\varepsilon}(\alpha+\beta A)$. We show that $\phi(\varepsilon)=\varepsilon|\beta|$. Let $\zeta\in\sigma_{e,\varepsilon}(A)$. By the definition of $\phi(\varepsilon)$, $\alpha+\beta\zeta\in\alpha+\beta\sigma_{e,\frac{\phi(\varepsilon)}{|\beta|}}(A)$. From (4), it follows that $\zeta\in\sigma_{e,\frac{\phi(\varepsilon)}{|\beta|}}(A)$. Hence, $\sigma_{e,\varepsilon}(A)\subseteq\sigma_{e,\frac{\phi(\varepsilon)}{|\beta|}}(A)$. Thus, $\phi(\varepsilon)=\varepsilon|\beta|$.

We also observe the following:

$$\begin{aligned}
\psi(s)&=\sup_{z\in\sigma_{e,s}(f(A))}\inf\bigl\{r\geq 0\colon z\in f(\sigma_{e,r}(A))\bigr\}\\
&=\sup_{z\in\sigma_{e,s}(\alpha+\beta A)}\inf\bigl\{r\geq 0\colon z\in f(\sigma_{e5}(A+F)) \ \text{ for some } \ \|F\|<r\bigr\}\\
&=\sup_{z\in\sigma_{e,s}(\alpha+\beta A)}\inf\bigl\{r\geq 0\colon z\in\sigma_{e5}(f(A+F)) \ \text{ for some } \|F\|<r\bigr\}\\
&=\sup_{z\in\sigma_{e,s}(\alpha+\beta A)}\inf\bigl\{r\geq 0\colon z\in\sigma_{e5}(\alpha+\beta(A+F)) \ \text{ for some } \ \|F\|<r\bigr\}\\
&=\sup_{z\in\sigma_{e,s}(\alpha+\beta A)}\inf\bigl\{r\geq 0\colon z\in\sigma_{e5}(\alpha+\beta A+\beta F) \ \text{ for some } \ \|F\|<r\bigr\}\\
&=\sup_{z\in\sigma_{e,s}(\alpha+\beta A)}\inf\bigl\{r\geq 0\colon z\in\sigma_{e,|\beta|r}(\alpha+\beta A)\bigr\}.
\end{aligned}$$

Since $z\in\sigma_{e,s}(\alpha+\beta A)$ we have that $\inf\bigl\{r\geq 0\colon z\in\sigma_{e,|\beta|r}(\alpha+\beta A)\bigr\}\leq\dfrac{s}{|\beta|}$. Thus, $\psi(s)\leq\dfrac{s}{|\beta|}$, consequently, $\sigma_{e,\psi(s)|\beta|}(\alpha+\beta A)\subseteq\sigma_{e,s}(\alpha+\beta A)$. Let $z\in\sigma_{e,s}(\alpha+\beta A)$. By the definition of $\psi(s)$, we have $z\in f\bigl(\sigma_{e5,\psi(s)}(A)\bigr)$. Since $f\bigl(\sigma_{e,\psi(s)}(A)\bigr)=\sigma_{e,|\beta|\psi(s)}(\alpha+\beta A)$, we get $\sigma_{e,s}(\alpha+\beta A)\subseteq\sigma_{e,|\beta|\psi(s)}(\alpha+\beta A)$. Thus, $\psi(s)|\beta|=s$. Further, $\psi(\phi(\varepsilon))=\varepsilon$. From the last theorem

$$f\bigl(\sigma_{e,\varepsilon}(A)\bigr)=\sigma_{e,\varepsilon|\beta|}(f(A)).$$

Next, we can derive simpler functions which give an upper bound to ϕ and ψ leading to a weak essential pseudospectral mapping theorem.

Theorem 3.3. *Let $A\in B(X)$ such that $\sigma_{e5}(A)\neq\varnothing$. Consider an analytic function $f:\mathcal{D}\to\mathbb{C}$, where $\mathcal{D}$ is an open subset of $\mathbb{C}$ such that $\mathcal{D}$ contains $\sigma_\varepsilon(A)$, $\sigma_{e5}(f(A))=f(\sigma_{e5}(A))$. In addition, we have the following assumptions*:

1. *For sufficiently small $r>0$ and for any $B\in\bigl\{E\in\mathcal{L}(X)\colon\|A-E\|<r\bigr\}$, the Frechet derivative of f is continuous.*

2. *The Frechet derivative Df evaluated at A is nonsingular.*

For each $\varepsilon > 0$ *and sufficiently small, define*

$$\gamma(\varepsilon) = \sup_{\|E\| \le \varepsilon} \|f(A+E) - f(A)\|.$$

Then, for sufficiently small $\varepsilon > 0$,

$$f\big(\sigma_{e,\varepsilon}(A)\big) \subseteq \sigma_{e,\gamma(\varepsilon)}\big(f(A)\big) \subseteq f\big(\sigma_{e,\delta(\gamma(\varepsilon))}(A)\big),$$

where δ *is defined in* (5).

Proof. We first show that

$$f\big(\sigma_{e,\varepsilon}(A)\big) \subseteq \sigma_{e,\gamma(\varepsilon)}\big(f(A)\big).$$

Let $z \in f\big(\sigma_{e,\varepsilon}(A)\big)$. Then there is some complex number $\psi \in \sigma_{e,\varepsilon}(A)$ such that $z = f(\psi)$. It is clear that $\psi \in \sigma_{e5}(A+E)$ for some $\|E\| < \varepsilon$. By spectral mapping theorem for essential pseudospectra,

$$f(\psi) \in \sigma_{e5}\big(f(A+E)\big) = \sigma_{e5}\big(f(A) + F\big),$$

where $F = f(A+E) - f(A)$ and satisfies $\|F\| \le \gamma(\varepsilon)$. Thus, $z \in \sigma_{e,\gamma(\varepsilon)}\big(f(A)\big)$. We now prove that the second set of inclusion. We assumed that the derivative map Df at A is nonsingular. For sufficiently small $r > 0$, we define

$$\mathcal{G} \colon B_r(0) \times B(X) \to B(X) \quad \text{by} \quad \mathcal{G}(B, C) = f(A+B) - f(A) - C,$$

where $B_r(0) = \big\{E \in \mathcal{L}(X) \colon \|E\| < r\big\}$. For any $(B, C) \in B_r(0) \times B(X)$, we have

$$\mathcal{G}(B+h, C+k) - \mathcal{G}(B, C) = f(A+B+h) - f(A+B).$$

If we assume that the Frechet derivative of f at $A+B$ is the linear map D', then from the last equation we have the Frechet derivative of $\mathcal{G}$ at (B, C) is also D'. Consequently, our assumption yield us the derivative is also continuous. Hence, $\mathcal{G}$ is continuously differentiable in its domain. Clearly,

$$\mathcal{G}(0, 0) = 0.$$

We now calculate $\partial_1 \mathcal{G}(0, 0)$. We observe the following:

$$\mathcal{G}(0+h, 0) - \mathcal{G}(0, 0) = f(A+h) - f(A).$$

From the above equation, $\partial_1 \mathcal{G}(0, 0) = Df(A)$. Since we assumed that $Df(A)$ is nonsingular, this gives us $\partial_1 \mathcal{G}(0, 0)$ is an isomorphism. By implicit mapping theorem, there exist a continuously Frechet differentiable function $\mathcal{E}$ and a positive constant α such that

$$\mathcal{G}(\mathcal{E}(C), C) = 0 \quad \text{for all} \quad \|C\| \le \alpha \qquad \text{and} \qquad \mathcal{E}(0) = 0.$$

Define

$$\delta(s) = \sup_{\|C\| \le s} \|\mathcal{E}(C)\|, \quad s \le \alpha. \tag{5}$$

Suppose that $z \in \sigma_{e,\gamma(\varepsilon)}(f(A))$. Then there exists $F \in B(X)$ such that $\|F\| \le \gamma(\varepsilon)$ and

$$z \in \sigma_{e5}(f(A) + F).$$

It is assumed that ε is small enough such that $\gamma(\varepsilon) \leq \alpha$. Let $E = \mathcal{E}(F)$ From the fact of $\mathcal{G}(\mathcal{E}(F), F) = 0$ implies

$$f(A+E) - f(A) = F$$

and $\|E\| \leq \delta(\gamma(\varepsilon))$. Since $z \in \sigma_{e5}(f(A)+F)$, we have $z \in \sigma_{e5}(f(A+E))$. By our assumption, for essential pseudospectra $z \in f(\sigma_{e5}(A+E))$, thus $z = f(\psi)$ for some $\psi \in \sigma_{e5}(A+E)$ or $\psi \in \sigma_{e5,\delta(\gamma(\varepsilon))}(A)$. This implies $z \in f\big(\sigma_{e5,\delta(\gamma(\varepsilon))}(A)\big)$.

***Remark* 3.2.** 1. We observe that $\lim_{\varepsilon\to 0}\gamma(\varepsilon) = 0$ and consequently $\lim_{\gamma(\varepsilon)\to 0}\delta(\gamma(\varepsilon)) = 0$. Moreover, $\sigma_{e,\varepsilon}(A) \to \sigma_{e5}(A)$ as $\varepsilon \to 0$. Hence, the above theorem reduces to the usual spectral mapping theorem for the essential spectrum.

2. If (a) $f(z) = \alpha + \beta z$ where α, β are complex numbers, (b) $\sigma_{e,\varepsilon}(A)$ is nonempty, (c) $Df(A)$ is invertible, then

$$\gamma(\varepsilon) = \sup_{\|E\|\leq\varepsilon} \big\|\alpha + \beta(A+E) - \alpha + \beta A\big\| = \sup_{\|E\|\leq\varepsilon} \|\beta E\| = |\beta|\varepsilon.$$

If $G(B,C) = 0$, then

$$\begin{aligned}
& f(A+B) - f(A) - C = 0, \\
& \alpha + \beta(A+B) - \alpha - \beta A - C = 0, \\
& \beta B - C = 0, \\
& B = \frac{C}{\beta}.
\end{aligned}$$

Consequently, by Theorem 3.3,

$$\delta(s) = \sup_{\|C\|\leq s} \|\mathcal{E}(C)\| = \sup_{\|C\|\leq s} \|B\| = \sup_{\|C\|\leq s} \left\|\frac{C}{\beta}\right\| = \frac{s}{|\beta|}.$$

Hence, $\delta(\gamma(\varepsilon)) = \varepsilon$. Thus, we have

$$f\big(\sigma_{e,\varepsilon}(A)\big) = \sigma_{e,\varepsilon|\beta|}\big(f(A)\big).$$

Problem 1. *Theorems* 3.2 *and* 3.3 *holds for the operators* $A \in B(X)$ *which has the property* $\sigma_{e5}(A) \neq \varnothing$. *Is there any other assumption on* $A \in B(X)$ *which is much simpler than the condition* $\sigma_{e5}(A) \neq \varnothing$, *so that Theorems* 3.2 *and* 3.3 *are still true*?

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