DOI: 10.3842/umzh.v76i3.7289

UDC 512.64

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ABELIAN MODEL STRUCTURES ON COMMA CATEGORIES

АБЕЛЕВІ МОДЕЛЬНІ СТРУКТУРИ НА КАТЕГОРІЯХ КОМИ

Let A and B be bicomplete Abelian categories, which both have enough projectives and injectives and let $T\colon \mathsf{A} \to \mathsf{B}$ be a right exact functor. Under some mild conditions, we show that hereditary Abelian model structures on A and B can be amalgamated into a global hereditary Abelian model structure on the comma category $(T \downarrow \mathsf{B})$. As an application of this result, we give an explicit description of a subcategory that consists of all trivial objects of the Gorenstein flat model structure on the category of modules over a triangular matrix ring.

Нехай A і B — біповні абелеві категорії, які мають достатню кількість проєктивних та ін'єктивних об'єктів. Крім того, нехай $T\colon \mathsf{A} \to \mathsf{B}$ — точний правий функтор. За деяких м'яких умов показано, що спадкові абелеві модельні структури на A і B можна об'єднати в глобальну спадкову абелеву модельну структуру на категорії коми $(T \downarrow \mathsf{B})$. Як застосування цього результату, наведено чіткий опис підкатегорії, що складається з усіх тривіальних об'єктів структури плоскої моделі Горенштейна на категорії модулів над трикутним матричним кільцем.

Introduction. Hovey studied extensively in [11] the model structures on Abelian categories. The most celebrated result in [11], which is now known as Hovey's correspondence, says that an Abelian model structures on a bicomplete Abelian category G is equivalent to a triple of subcategories $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ of G such that W is thick and $(\mathsf{C}, \mathsf{W} \cap \mathsf{F})$ and $(\mathsf{C} \cap \mathsf{W}, \mathsf{F})$ form two complete cotorsion pairs (here, W (resp., C and F) is the subcategory of G consisting of all trivial (resp., cofibrant and fibrant) objects associated to the corresponding Abelian model structure). Hovey's correspondence makes it clear that an Abelian model structures on G can be succinctly represented by the triple $(\mathsf{C}, \mathsf{W}, \mathsf{F})$. Therefore, one often refers to such a triple as an Abelian model structure in literatures, and called it a *Hovey triple*.

Let $T\colon \mathsf{A} \to \mathsf{B}$ be a right exact functor between Abelian categories A and B. Then there exists an Abelian category, denoted by $(T \downarrow \mathsf{B})$, whose objects are the morphisms $\sigma\colon T(A) \to B$ with $A \in \mathsf{A}$ and $B \in \mathsf{B}$, and morphisms from the object $\varphi\colon T(A) \to B$ to the object $\varphi'\colon T(A') \to B'$ are the pair $(a\colon A \to A', b\colon B \to B')$ of morphisms such that $b \circ \varphi = \varphi' \circ T(a)$. Such an Abelian category is called a *comma category* in literature (see [6]). Examples of comma categories include but are not limited to the category of modules or complexes over a triangular matrix ring, the morphism category of an Abelian category and so on.

Recently, Hu and Zhu [12] showed under some mild conditions that (complete and hereditary) cotorsion pairs shared by local Abelian categories A and B can be amalgamated to a global (complete and hereditary) cotorsion theory in $(T \downarrow \mathsf{B})$. More explicitly, let (X, Y) and $(\mathsf{X}', \mathsf{Y}')$ be (complete and hereditary) cotorsion pairs in A and B, respectively. They proved that if T is X-exact then $\left(\Phi^{\mathsf{X}}_{\mathsf{X}'}, \Psi^{\mathsf{Y}}_{\mathsf{Y}'}\right)$ forms a (complete and hereditary) cotorsion pair in $(T \downarrow \mathsf{B})$ (see [12, Proposition 2.5, Lemma 3.3 and Proposition 3.4]). Here, T is called X-*exact* if T preserves the exactness of the exact sequence $0 \to M \to N \to X \to 0$ in A with $X \in \mathsf{X}$, and one can refer to 1.4 for the definitions of the subcategories $\Phi^{\mathsf{X}}_{\mathsf{X}'}$ and $\Psi^{\mathsf{Y}}_{\mathsf{Y}'}$.

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Inspired by the above results of Hu and Zhu, it is natural to ask that whether Hovey triples (or, equivalently, Abelian model structures) shared by local Abelian categories A and B can be amalgamated to a global Hovey triple in $(T \downarrow \mathsf{B})$. Indeed, by virtue of the work [7] of Gillespie, we obtain the following main result of the paper, which asserts under some mild conditions that hereditary Hovey triples $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ and $(\mathsf{C}', \mathsf{W}', \mathsf{F}')$ on A and B, respectively, induce a hereditary Hovey triple on $(T \downarrow \mathsf{B})$. Recall that a Hovey triple on a bicomplete Abelian category is said to be *hereditary* if both the associated cotorsion pairs in the Hovey's correspondence are hereditary.

Theorem A. *Let* A *and* B *be bicomplete Abelian categories which both have enough projectives and injectives and* $T\colon \mathsf{A} \to \mathsf{B}$ *be a right exact functor. Assume that* $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ *and* $(\mathsf{C}', \mathsf{W}', \mathsf{F}')$ *form hereditary Hovey triples on* A *and* B, *respectively. If* T *is* C*-exact and* $T(\mathsf{C} \cap \mathsf{W}) \subseteq \mathsf{W}'$, *then* $\big(\Phi^{\mathsf{C}}_{\mathsf{C}'}, \Psi^{\mathsf{W}}_{\mathsf{W}'}, \Psi^{\mathsf{F}}_{\mathsf{F}'}\big)$ *forms a hereditary Hovey triple on* $(T \downarrow \mathsf{B})$.

Auslander [1] introduced Gorenstein dimension (shortly G-dimension) for finitely generated modules over a commutative noetherian local ring. In particular, modules having G-dimension 0 can be regarded as a common generalization of finitely generated projective modules over commutative Noetherian rings and maximal Cohen – Macaulay modules over commutative Gorenstein local rings. Enochs and Jenda [3] called the modules having G-dimension 0 the Gorenstein projective and defined Gorenstein projective (whether finitely generated or not) modules over an arbitrary ring. Another important extension of the G-dimension is based on Gorenstein flat modules. These modules were introduced by Enochs et al. [5]. Further studies on Gorenstein flat modules can be found in [4] and [10] et al.

Let Λ be a triangular matrix ring. Šaroch and Šťovíček showed that there exists a Gorenstein flat Abelian model structure $\big(\mathsf{GF}(\Lambda), \mathsf{PGF}(\Lambda)^{\perp}, \mathsf{Cot}(\Lambda)\big)$ on the category of Λ-modules, where $\mathsf{GF}(\Lambda)$ (resp., $\mathsf{PGF}(\Lambda)$ and $\mathsf{Cot}(\Lambda)$) denote the category of Gorenstein flat (resp., projectively coresolved Gorenstein flat and cotorsion) Λ-modules (see [15, p. 27]). The category $\mathsf{PGF}(\Lambda)^{\perp}$ consists of all trivial objects, and hence, plays the most important role in the above Abelian model structure. As an application of Theorem A, we obtain the following result, which gives under some mild conditions that an explicit description of $\mathsf{PGF}(\Lambda)^{\perp}$ (see Section 3 for unexplained notation).

***Corollary* B.** *Let* $\Lambda = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ *be a triangular matrix ring. If* U_A *has finite flat or injective dimension and* ${}_BU$ *has finite flat dimension, then the equality* $\mathsf{PGF}(\Lambda)^{\perp} = \Psi^{\mathsf{PGF}(\mathsf{A})^{\perp}}_{\mathsf{PGF}(\mathsf{B})^{\perp}}$ *holds true.*

This paper is organized as follows. Section 1 contains some necessary notations and definitions for use throughout this paper. In Section 2, we give the proof of Theorem A. Section 3 is devoted to giving the proof of Corollary B.

1. Preliminaries. Throughout the paper, all rings are assumed to be associative. Let R be a ring. We adopt the convention that an R-module is a left R-module, refer to a right R-module as a module over the opposite ring R^{op}, and let $\operatorname{Mod} R$ denote the category of R-modules.

1.1. Comma categories. Let $T\colon \mathsf{A} \to \mathsf{B}$ be a right exact functor between Abelian categories A and B. Recall from [6] that the comma category $(T \downarrow \mathsf{B})$ is defined as follows:

(1) The objects are the triples $\begin{pmatrix} A \\ B \end{pmatrix}_{\varphi}$ or simply $\begin{pmatrix} A \\ B \end{pmatrix}$ if there is no possible confusion with $A \in \mathsf{A}$, $B \in \mathsf{B}$ and $\varphi\colon T(A) \to B$ is a morphism in B.

(2) A morphism from object $\begin{pmatrix} A \\ B \end{pmatrix}_{\varphi}$ to the object $\begin{pmatrix} A' \\ B' \end{pmatrix}_{\varphi'}$ is the pair $\begin{pmatrix} a \\ b \end{pmatrix}$ of morphisms, with $a\colon A \to A'$ a morphism in A, $b\colon B \to B'$ a morphism in B such that $b \circ \varphi = \varphi' \circ T(a)$.

1.2. *Example.* (1) Let A and B be two rings and U a B-A-bimodule. Denote by $\Lambda = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ the associated triangular matrix ring. Note that $\operatorname{Mod}\Lambda$ is equivalent to the comma category $((U \otimes_A -) \downarrow \operatorname{Mod} B)$, where $U \otimes_A - \colon \operatorname{Mod} A \to \operatorname{Mod} B$ is the usual tensor functor.

(2) Let A be an Abelian category and $T\colon \mathsf{A} \to \mathsf{A}$ the identity functor. Then the comma category $(T \downarrow \mathsf{A})$ coincides with $\operatorname{mor}(\mathsf{A})$, where $\operatorname{mor}(\mathsf{A})$ denotes the the morphism category of A.

For more examples of comma categories, one refers to [12].

1.3. A sequence

$$0 \longrightarrow \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}_{\varphi_1} \xrightarrow{\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}_{\varphi_2} \xrightarrow{\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}} \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}_{\varphi_3} \longrightarrow 0$$

in $(T \downarrow \mathsf{B})$ is exact if and only if both the sequence $0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0$ in A and the sequence $0 \to Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \to 0$ in B are exact.

1.4. Let X and X' be subcategories of Abelian categories A and B, respectively. Set

$$\Phi_{\mathsf{X}'}^{\mathsf{X}} = \left\{ \begin{pmatrix} A \\ B \end{pmatrix}_{\varphi} \in (T \downarrow \mathsf{B}) \;\middle|\; \varphi \text{ is a monomorphism, } A \in \mathsf{X} \text{ and } \operatorname{Coker}\varphi \in \mathsf{X}' \right\}$$

and

$$\Psi_{\mathsf{X}'}^{\mathsf{X}} = \left\{ \begin{pmatrix} A \\ B \end{pmatrix}_{\varphi} \in (T \downarrow \mathsf{B}) \;\middle|\; A \in \mathsf{X} \text{ and } B \in \mathsf{X}' \right\}.$$

1.5. Cotorsion pairs. Let G be an Abelian category. Recall from [4] that a pair (X, Y) of subcategories of G is called a *cotorsion pair* if $\mathsf{X}^{\perp} = \mathsf{Y}$ and ${}^{\perp}\mathsf{Y} = \mathsf{X}$, where

$$\mathsf{X}^{\perp} = \left\{ N \mid \operatorname{Ext}^1_{\mathsf{G}}(X, N) = 0 \text{ for all } X \in \mathsf{X} \right\}$$

and

$${}^{\perp}\mathsf{Y} = \left\{ M \mid \operatorname{Ext}^1_{\mathsf{G}}(M, Y) = 0 \text{ for all } Y \in \mathsf{Y} \right\}.$$

Following from [8] that a cotorsion pair (X, Y) is said to be *complete* if for any object M in G, there exist short exact sequences

$$0 \to M \to Y \to X \to 0 \qquad \text{and} \qquad 0 \to Y' \to X' \to M \to 0$$

in G with $X, X' \in \mathsf{X}$ and $Y, Y' \in \mathsf{Y}$. A cotorsion pair (X, Y) is called *hereditary* if X is closed under taking kernels of epimorphisms and Y is closed under taking cokernels of monomorphisms. The book [4] is a standard reference for cotorsion pairs.

1.6. Abelian model structures. A model category is said to be *Abelian* if its underlying category G is Abelian and the model structure is Abelian, that is, it is compatible with the Abelian structure of G (see [11]). The central result on Abelian model categories is now known as Hovey's correspondence: a correspondence between complete and compatible cotorsion pairs and Abelian model structures.

1.7. Theorem (Hovey's correspondence). *Let* G *be a bicomplete Abelian category. An Abelian model structure on* G *corresponds bijectively to a triple* $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ *of subcategories of* G *such that*

(a) $(\mathsf{C}, \mathsf{W} \cap \mathsf{F})$ *and* $(\mathsf{C} \cap \mathsf{W}, \mathsf{F})$ *are complete cotorsion pairs,*

(b) W *is thick, that is, it is closed under direct summands, extensions, and taking kernels of epimorphisms and cokernels of monomorphisms.*

Hovey's correspondence makes it clear that an Abelian model structure on G can be succinctly represented by the triple $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ of subcategories of G that satisfies the conditions (a) and (b) in Theorem 1.7. Therefore, one often refers to such a triple as an Abelian model structure in literatures, and called it a *Hovey triple*.

The next result gives characterization of the trivial objects (see [8, Proposition 3.2]).

1.8. Lemma. *Let* G *be a bicomplete Abelian category. Assume that* $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ *is a Hovey triple in* G. *Then the thick subcategory* W *can be characterized in each of the two following ways:*

$$\mathsf{W} = \{M \mid \exists \ \textit{s.e.s} \ 0 \to M \to A \to B \to 0 \ \textit{with} \ A \in \mathsf{W} \cap \mathsf{F} \ \textit{and} \ B \in \mathsf{C} \cap \mathsf{W}\}$$
$$= \big\{M \mid \exists \ \textit{s.e.s} \ 0 \to A' \to B' \to M \to 0 \ \textit{with} \ A' \in \mathsf{W} \cap \mathsf{F} \ \textit{and} \ B' \in \mathsf{C} \cap \mathsf{W}\big\}.$$

Consequently, W *is unique in the sense that whenever* $(\mathsf{C}, \mathsf{V}, \mathsf{F})$ *is a Hovey triple, then necessarily* $\mathsf{V} = \mathsf{W}$.

A Hovey triple $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ on G is said to be *hereditary* if both the cotorsion pairs $(\mathsf{C}, \mathsf{W} \cap \mathsf{F})$ and $(\mathsf{C} \cap \mathsf{W}, \mathsf{F})$ are hereditary. The next result supplements Hovey's correspondence, making it easier than ever to construct Abelian model structures from complete and hereditary cotorsion pairs (see [7, Theorem 1.1]).

1.9. Lemma. *Let* G *be a bicomplete Abelian category. Assume that* $(\mathsf{C}, \widetilde{\mathsf{F}})$ *and* $(\widetilde{\mathsf{C}}, \mathsf{F})$ *are complete and hereditary cotorsion pairs in* G *such that* $\widetilde{\mathsf{C}} \subseteq \mathsf{C}$ *(or, equivalently,* $\widetilde{\mathsf{F}} \subseteq \mathsf{F}$*) and* $\mathsf{C} \cap \widetilde{\mathsf{F}} = \widetilde{\mathsf{C}} \cap \mathsf{F}$. *Then there exists a unique thick subcategory* W *for which* $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ *forms a Hovey triple. Moreover, this thick subcategory* W *can be described as follows:*

$$\mathsf{W} = \Big\{M \mid \exists \ \text{s.e.s} \ 0 \to M \to A \to B \to 0 \ \text{with} \ A \in \widetilde{\mathsf{F}} \ \text{and} \ B \in \widetilde{\mathsf{C}}\Big\}.$$

One refers to [9] for more background materials on Abelian model structures.

2. Proof of Theorem A. In this section, we aim at proving Theorem A in the introduction. Throughout the section, let A and B be bicomplete Abelian categories which both have enough projectives and injectives, $T \colon \mathsf{A} \to \mathsf{B}$ be a right exact additive functor, and let $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ and $(\mathsf{C}', \mathsf{W}', \mathsf{F}')$ be hereditary Hovey triples on A and B, respectively.

We begin with the following result, which is from [12, Proposition 2.5, Lemma 3.3 and Proposition 3.4].

2.1. Theorem. *Let* X *and* Y *be subcategories in* A, *and let* X' *and* Y' *be subcategories in* B. *Assume that* T *is* X*-exact. If* (X, Y) *and* $(\mathsf{X}', \mathsf{Y}')$ *form complete and hereditary cotorsion pairs in* A *and* B, *respectively, then* $\big(\Phi^{\mathsf{X}}_{\mathsf{X}'}, \Psi^{\mathsf{Y}}_{\mathsf{Y}'}\big)$ *forms a complete and hereditary cotorsion pair in* $(T \downarrow \mathsf{B})$.

In what follows, denote by $\widetilde{\mathsf{F}}$ (resp., $\widetilde{\mathsf{F}'}$, $\widetilde{\mathsf{C}}$ and $\widetilde{\mathsf{C}'}$) the subcategory $\mathsf{W}\cap\mathsf{F}$ (resp., $\mathsf{W}'\cap\mathsf{F}'$, $\mathsf{C}\cap\mathsf{W}$ and $\mathsf{C}'\cap\mathsf{W}'$). As an application of Lemma 1.9, we obtain the following result.

2.2. Lemma. *Assume that T is C-exact and $T(\widetilde{\mathsf{C}}\cap\mathsf{F})\subseteq\mathsf{W}'$. Then there exists a hereditary Hovey triple*

$$\left(\Phi^{\mathsf{C}}_{\mathsf{C}'},\Omega,\Psi^{\mathsf{F}}_{\mathsf{F}'}\right)$$

on $(T\downarrow\mathsf{B})$, where Ω consists of all objects $\begin{pmatrix}A\\B\end{pmatrix}$ such that there exists a short exact sequence

$$0\to\begin{pmatrix}A\\B\end{pmatrix}\to\begin{pmatrix}U\\V\end{pmatrix}\to\begin{pmatrix}M\\N\end{pmatrix}\to 0$$

in $(T\downarrow\mathsf{B})$ with $\begin{pmatrix}U\\V\end{pmatrix}\in\Psi^{\widetilde{\mathsf{F}}}_{\widetilde{\mathsf{F}'}}$ and $\begin{pmatrix}M\\N\end{pmatrix}\in\Phi^{\widetilde{\mathsf{C}}}_{\widetilde{\mathsf{C}'}}$.

Proof. By Theorems 1.7 and 2.1, we obtain complete and hereditary cotorsion pairs $\left(\Phi^{\mathsf{C}}_{\mathsf{C}'},\Psi^{\widetilde{\mathsf{F}}}_{\widetilde{\mathsf{F}'}}\right)$ and $\left(\Phi^{\widetilde{\mathsf{C}}}_{\widetilde{\mathsf{C}'}},\Psi^{\mathsf{F}}_{\mathsf{F}'}\right)$ in $(T\downarrow\mathsf{B})$. It is evident that $\Psi^{\widetilde{\mathsf{F}}}_{\widetilde{\mathsf{F}'}}\subseteq\Psi^{\mathsf{F}}_{\mathsf{F}'}$ as $\widetilde{\mathsf{F}}\subseteq\mathsf{F}$ and $\widetilde{\mathsf{F}'}\subseteq\mathsf{F}'$. According to Lemma 1.9, if we prove that

$$\Phi^{\mathsf{C}}_{\mathsf{C}'}\cap\Psi^{\widetilde{\mathsf{F}}}_{\widetilde{\mathsf{F}'}}=\Phi^{\widetilde{\mathsf{C}}}_{\widetilde{\mathsf{C}'}}\cap\Psi^{\mathsf{F}}_{\mathsf{F}'},$$

then there exists a unique thick subcategory Ω with the desired form such that $\left(\Phi^{\mathsf{C}}_{\mathsf{C}'},\Omega,\Psi^{\mathsf{F}}_{\mathsf{F}'}\right)$ forms a hereditary Hovey triple on $(T\downarrow\mathsf{B})$.

For the inclusion $\Phi^{\mathsf{C}}_{\mathsf{C}'}\cap\Psi^{\widetilde{\mathsf{F}}}_{\widetilde{\mathsf{F}'}}\subseteq\Phi^{\widetilde{\mathsf{C}}}_{\widetilde{\mathsf{C}'}}\cap\Psi^{\mathsf{F}}_{\mathsf{F}'}$, take an object $\begin{pmatrix}A\\B\end{pmatrix}_{\varphi}\in\Phi^{\mathsf{C}}_{\mathsf{C}'}\cap\Psi^{\widetilde{\mathsf{F}}}_{\widetilde{\mathsf{F}'}}$. Then we have an exact sequence

$$0\to T(A)\xrightarrow{\varphi}B\to\operatorname{Coker}\varphi\to 0$$

in B with $A\in\mathsf{C}\cap\widetilde{\mathsf{F}}$, $B\in\widetilde{\mathsf{F}'}$ and $\operatorname{Coker}\varphi\in\mathsf{C}'$. Note that $T(A)\in W'$ by assumption. This implies that $\operatorname{Coker}\varphi\in\mathsf{W}'$ as W' is a thick subcategory of A. Therefore, we see that $\begin{pmatrix}A\\B\end{pmatrix}_{\varphi}\in\Phi^{\widetilde{\mathsf{C}}}_{\widetilde{\mathsf{C}'}}\cap\Psi^{\mathsf{F}}_{\mathsf{F}'}$. It follows that $\Phi^{\mathsf{C}}_{\mathsf{C}'}\cap\Psi^{\widetilde{\mathsf{F}}}_{\widetilde{\mathsf{F}'}}\subseteq\Phi^{\widetilde{\mathsf{C}}}_{\widetilde{\mathsf{C}'}}\cap\Psi^{\mathsf{F}}_{\mathsf{F}'}$. The other direction is similar.

The next result gives a clear characterization of the thick subcategory Ω in the hereditary Hovey triple $\left(\Phi^{\mathsf{C}}_{\mathsf{C}'},\Omega,\Psi^{\mathsf{F}}_{\mathsf{F}'}\right)$ on $(T\downarrow\mathsf{B})$ established in Lemma 2.2.

2.3. Lemma. *Assume that T is C-exact and $T(\widetilde{\mathsf{C}})\subseteq\mathsf{W}'$. Then the equality $\Omega=\Psi^{\mathsf{W}}_{\mathsf{W}'}$ holds true.*

Proof. For the inclusion $\Omega\subseteq\Psi^{\mathsf{W}}_{\mathsf{W}'}$. Let $\begin{pmatrix}A\\B\end{pmatrix}$ be an object in Ω. We will prove that $A\in\mathsf{W}$ and $B\in\mathsf{W}'$. Note that there exist a short exacts sequence

$$0\to\begin{pmatrix}A\\B\end{pmatrix}\to\begin{pmatrix}U\\V\end{pmatrix}\to\begin{pmatrix}M\\N\end{pmatrix}_{\psi}\to 0$$

in $(T\downarrow\mathsf{B})$ with $\begin{pmatrix}U\\V\end{pmatrix}\in\Psi^{\widetilde{\mathsf{F}}}_{\widetilde{\mathsf{F}'}}$ and $\begin{pmatrix}M\\N\end{pmatrix}_{\psi}\in\Phi^{\widetilde{\mathsf{C}}}_{\widetilde{\mathsf{C}'}}$. Hence, there exists a short exact sequence $0\to A\to U\to M\to 0$ in A with $U\in\widetilde{\mathsf{F}}$ and $M\in\widetilde{\mathsf{C}}$. This implies that $A\in\mathsf{W}$ as W is a thick subcategory of A. Furthermore, note that there exists also a short exact sequence

$$0 \to T(M) \xrightarrow{\psi} N \to \operatorname{Coker}\psi \to 0$$

in B with $M \in \widetilde{\mathsf{C}}$ and $\operatorname{Coker}\psi \in \widetilde{\mathsf{C}'}$. Since $T(M) \in \mathsf{W}'$ by assumption and W' is a thick subcategory of B, we conclude that $N \in \mathsf{W}'$. Consider the short exact sequence $0 \to B \to V \to N \to 0$ in B with $V \in \widetilde{\mathsf{F}'}$. Since W' is a thick subcategory of B, we conclude that $B \in \mathsf{W}'$.

Conversely, let $\begin{pmatrix} A \\ B \end{pmatrix}$ be an object in $\Psi^{\mathsf{W}}_{\mathsf{W}'}$. Then there exists an exact sequence $0 \to A \to U \to M \to 0$ in A with $U \in \widetilde{\mathsf{F}}$ and $M \in \widetilde{\mathsf{C}}$, as $\left(\widetilde{\mathsf{C}}, \mathsf{F}\right)$ is a complete cotorsion pairs in A and W is a thick subcategory of A. Note that T is C-exact, there exists the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(A) & \longrightarrow & T(U) & \longrightarrow & T(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(M) & \longrightarrow & 0 \end{array}$$

in which $C \in \mathsf{W}'$. There exists also an exact sequence $0 \to C \to V \to X \to 0$ in B with $V \in \widetilde{\mathsf{F}'}$ and $X \in \widetilde{\mathsf{C}'}$, as $\left(\widetilde{\mathsf{C}'}, \mathsf{F}'\right)$ is a complete cotorsion pairs in B and W' is a thick subcategory of B. Consider the pushout diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(M) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & B & \longrightarrow & V & \longrightarrow & N & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & X & = & X & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Thus, we get an exact sequence

$$0 \to \begin{pmatrix} A \\ B \end{pmatrix} \to \begin{pmatrix} U \\ V \end{pmatrix} \to \begin{pmatrix} M \\ N \end{pmatrix}_{\psi} \to 0$$

in $(T \downarrow \mathsf{B})$ with $\begin{pmatrix} U \\ V \end{pmatrix} \in \Psi^{\widetilde{\mathsf{F}}}_{\widetilde{\mathsf{F}'}}$ and $\begin{pmatrix} M \\ N \end{pmatrix}_{\psi} \in \Phi^{\widetilde{\mathsf{C}}}_{\widetilde{\mathsf{C}'}}$. This implies that $\begin{pmatrix} A \\ B \end{pmatrix} \in \Omega$.

Now Theorem A in the introduction holds by Lemmas 2.2 and 2.3.

3. Proof of Corollary B. This section is devoted to giving the proof of Corollary B in the introduction. Throughout the section. Let A and B be two rings and U a B-A-bimodule. Denote by $\Lambda = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ the associated triangular matrix ring.

We begin with the following result, which is an immediate consequence of Theorem A (see Example 1.2 (1)).

3.1. ***Corollary.*** *Assume that* $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ *and* $(\mathsf{C}', \mathsf{W}', \mathsf{F}')$ *are hereditary Hovey triples on* $\operatorname{Mod} A$ *and* $\operatorname{Mod} B$, *respectively. If* $U \otimes_A -$ *is* C*-exact and* $U \otimes_A (\mathsf{C} \cap \mathsf{W}) \subseteq \mathsf{W}'$, *then* $\left(\Phi^{\mathsf{C}}_{\mathsf{C}'}, \Psi^{\mathsf{W}}_{\mathsf{W}'}, \Psi^{\mathsf{F}}_{\mathsf{F}'}\right)$ *is a hereditary Hovey triple on* $\operatorname{Mod} \Lambda$.

Let R be an associative ring. Recall from [15] that an R-module M is called *Gorenstein flat* (resp., *projectively coresolved Gorenstein flat*) if there exists an exact sequence

$$\cdots \to F^{-1} \to F^0 \to F^1 \to \cdots$$

of flat (resp., projective) R-modules with $M = \operatorname{Ker}\left(F^0 \to F^1\right)$ such that $E \otimes_R$-leaves the sequence exact whenever E is an injective R^{op}-module. In what follows, the subcategory of $\operatorname{Mod} R$ consisting of all Gorenstein flat (resp., projectively coresolved Gorenstein flat, flat) R-modules is denoted by $\mathsf{GF}(R)$ (resp., $\mathsf{PGF}(R)$, $\mathsf{Flat}(R)$).

According to [15, Theorem 4.11], we know that $\mathsf{Flat}(R) \subseteq \mathsf{PGF}(R)^{\perp}$. This fact enables us to obtain following result, which will be applied in the proof of Lemma 3.3.

3.2. Lemma. *The equality* $\operatorname{Ext}^{\geq 1}_R(G, F) = 0$ *holds.*

Proof. We proceed by induction on $n \geq 1$. If $n = 1$, then the equality holds by [15, Theorem 4.11]. Assume now that the equality holds for $n - 1$. We show next that the equality also holds for n.

Note that there exists a short exact sequence $0 \to K \to P \to G \to 0$ in $\operatorname{Mod} R$ with $K, G \in \mathsf{PGF}(R)$ and P a projective R-module. Applying the functor $\operatorname{Hom}_R(-, F)$ to the above short exact sequence, we deduce that $\operatorname{Ext}^n_R(G, F) \cong \operatorname{Ext}^{n-1}_R(K, F)$. Note that $\operatorname{Ext}^{n-1}_R(K, F) = 0$ by the induction assumption. Thus, $\operatorname{Ext}^n_R(G, F) = 0$, as desired.

The next result will play an important role in the proof of Corollary B.

3.3. Lemma. *Assume that* U_A *has finite flat or injective dimension and* ${}_BU$ *has finite flat dimension. Then the following hold:*

(a) *the functor* $U \otimes_A -$ *is* $\mathsf{GF}(A)$*-exact,*

(b) $U \otimes_A \left(\mathsf{GF}(A) \cap \mathsf{PGF}(A)^{\perp}\right) \subseteq \mathsf{PGF}(B)^{\perp}$.

Proof. (a) Let M be a Gorenstein flat A-module. According to [2, Lemma 2.3] or [14, Lemma 2.1], we see that $\operatorname{Tor}^A_1(U, M) = 0$. This implies that the functor $U \otimes_A -$ is $\mathsf{GF}(A)$-exact.

(b) By [15, Theorem 4.11], it suffices to show that $U \otimes_A (\mathsf{Flat}(A)) \subseteq \mathsf{PGF}(B)^{\perp}$. Let $Q \in \mathsf{Flat}(A)$ and $G \in \mathsf{PGF}(B)$. We claim that $\operatorname{Ext}^1_B(G, U \otimes_A Q) = 0$. Since $\mathsf{fd}({}_BU) = n < \infty$ by assumption, where $\mathsf{fd}({}_BU)$ denotes the flat dimension of B-module U, there exists an exact sequence

$$0 \to F^n \to F^{n-1} \to \ldots \to F^0 \to U \to 0$$

with each $F^i \in \mathsf{Flat}(B)$. Since $Q \in \mathsf{Flat}(A)$, we get a flat resolution

$$0 \to F^n \otimes_A Q \to F^{n-1} \otimes_A Q \to \ldots \to F^0 \otimes_A Q \to U \otimes_A Q \to 0$$

of $U \otimes_A Q$. Applying the functor $\mathrm{Hom}_B(G, -)$ to the above flat resolution, by dimension-shifting argument and Lemma 3.2, we conclude that

$$\mathrm{Ext}^1_B(G, U \otimes_A Q) \cong \mathrm{Ext}^{n+1}_B(G, F^n \otimes_A Q) = 0,$$

as desired.

Mao gave in [14, Theorem 2.3] the structure of Gorenstein flat Λ-modules. More precisely, he showed that if U_A has finite flat or injective dimension, ${}_BU$ has finite flat dimension and Λ is a right coherent ring, then the equality $\mathsf{GF}(\Lambda) = \Phi^{\mathsf{GF}(A)}_{\mathsf{GF}(B)}$ holds true. Note that the assumption Λ is a right coherent ring is to guarantee that $\mathsf{GF}(\Lambda)$ is closed under extensions, which plays a key role in the proof of the above result. However, by virtue of the work [15] of Šaroch and Šťovíček, we see that for any associative ring R, $\mathsf{GF}(R)$ is closed under extensions (see [15, p. 25]). Hence, we obtain the following result, which removes the assumption Λ is a right coherent ring.

3.4. Proposition. *Assume that U_A has finite flat or injective dimension and ${}_BU$ has finite flat dimension. Then there exists an equality* $\mathsf{GF}(\Lambda) = \Phi^{\mathsf{GF}(A)}_{\mathsf{GF}(B)}$.

3.5. *Proof of Corollary* B. Note that

$$\Big(\mathsf{GF}(A), \mathsf{PGF}(A)^{\perp}, \mathsf{Cot}(A)\Big) \qquad \text{and} \qquad \Big(\mathsf{GF}(B), \mathsf{PGF}(B)^{\perp}, \mathsf{Cot}(B)\Big)$$

form hereditary Hovey triples on $\mathrm{Mod}\,A$ and $\mathrm{Mod}\,B$, respectively (see [15, p. 27]). It follows from Corollary 3.1 and Lemma 3.3 that

$$\Big(\Phi^{\mathsf{GF(A)}}_{\mathsf{GF(B)}}, \Psi^{\mathsf{PGF(A)}^{\perp}}_{\mathsf{PGF(B)}^{\perp}}, \Psi^{\mathsf{Cot(A)}}_{\mathsf{Cot(B)}}\Big)$$

forms a hereditary Hovey triple on $\mathrm{Mod}\,\Lambda$. On the other hand,

$$\Big(\mathsf{GF}(\Lambda), \mathsf{PGF}(\Lambda)^{\perp}, \mathsf{Cot}(\Lambda)\Big)$$

forms a hereditary Hovey triple on $\mathrm{Mod}\,\Lambda$ (see also [15, p. 27]). However, by Proposition 3.4 and [13, Corollary 4.3], respectively, we see that $\mathsf{GF}(\Lambda) = \Phi^{\mathsf{GF(A)}}_{\mathsf{GF(B)}}$ and $\mathsf{Cot}(\Lambda) = \Psi^{\mathsf{Cot(A)}}_{\mathsf{Cot(B)}}$. Thus, we conclude that $\mathsf{PGF}(\Lambda)^{\perp} = \Psi^{\mathsf{PGF(A)}^{\perp}}_{\mathsf{PGF(B)}^{\perp}}$ by Lemma 1.8.

The author states that there is no conflict of interest.

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Received 16.08.22