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Sung Guen Kim[1] (Department of Mathematics, Kyungpook National University, Republic of Korea)

THE NORMING SETS OF $\mathcal{L}\big({}^m l_1^n\big)$

НОРМУЮЧІ МНОЖИНИ В $\mathcal{L}\big({}^m l_1^n\big)$

Let $n \in \mathbb{N},\ n \geq 2.$ An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}({}^n E)$ if $\|x_1\| = \ldots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|,$ where $\mathcal{L}({}^n E)$ denotes the space of all continuous n-linear forms on $E.$ For $T \in \mathcal{L}({}^n E),$ we define

$$\mathrm{Norm}(T) = \big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \ \text{is a norming point of}\ T\big\}.$$

The $\mathrm{Norm}(T)$ is called the *norming set* of $T.$ For $m \in \mathbb{N},\ m \geq 2,$ we characterize $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}\big({}^m l_1^n\big),$ where $l_1^n = \mathbb{R}^n$ with the l_1-norm. As applications, we classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}\big({}^m l_1^n\big)$ with $n = 2, 3$ and $m = 2.$

Нехай $n \in \mathbb{N},\ n \geq 2.$ Елемент $(x_1, \ldots, x_n) \in E^n$ називається *нормуючою точкою* $T \in \mathcal{L}({}^n E),$ якщо $\|x_1\| = \ldots = \|x_n\| = 1$ і $|T(x_1, x_1, \ldots, x_n)| = \|T\|,$ де $\mathcal{L}({}^n E)$ – простір усіх неперервних n-лінійних форм на $E.$ Для $T \in \mathcal{L}({}^n E)$ визначаємо

$$\mathrm{Norm}(T) = \big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \ \text{– точка нормування в}\ T\big\}.$$

Множина $\mathrm{Norm}(T)$ називається *нормуючою множиною в* $T.$ Для $m \in \mathbb{N},\ m \geq 2,$ ми характеризуємо $\mathrm{Norm}(T)$ для кожного $T \in \mathcal{L}\big({}^m l_1^n\big),$ де $l_1^n = \mathbb{R}^n$ з нормою $l_1.$ Як застосування, ми класифікуємо $\mathrm{Norm}(T)$ для кожного $T \in \mathcal{L}\big({}^m l_1^n\big)$ при $n = 2, 3$ і $m = 2.$

1. Introduction. In 1961 Bishop and Phelps [2] showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon – Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon – Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N},\ n \geq 2.$ We write S_E for the unit sphere of a Banach space $E.$ We denote by $\mathcal{L}({}^n E)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{(x_1, \ldots, x_n) \in S_E \times \ldots \times S_E} |T(x_1, \ldots, x_n)|.$ $\mathcal{L}_s({}^n E)$ denote the closed subspace of all continuous symmetric n-linear forms on $E.$ An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \ldots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|.$

For $T \in \mathcal{L}({}^n E),$ we define

$$\mathrm{Norm}(T) = \big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \ \text{is a norming point of}\ T\big\}.$$

$\mathrm{Norm}(T)$ is called the *norming set* of $T.$ Note that $(x_1, \ldots, x_n) \in \mathrm{Norm}(T)$ if and only if $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \mathrm{Norm}(T)$ for some $\epsilon_k = \pm 1,\ k = 1, \ldots, n.$ Indeed, if $(x_1, \ldots, x_n) \in \mathrm{Norm}(T),$ then

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[1] E-mail: sgk317@knu.ac.kr.

$$|T(\epsilon_1 x_1, \ldots, \epsilon_n x_n)| = |\epsilon_1 \ldots \epsilon_n T(x_1, \ldots, x_n)| = |T(x_1, \ldots, x_n)| = \|T\|,$$

which shows that $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \mathrm{Norm}(T)$. If $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \mathrm{Norm}(T)$ for some $\epsilon_k = \pm 1$, $k = 1, \ldots, n$, then

$$(x_1, \ldots, x_n) = \big(\epsilon_1(\epsilon_1 x_1), \ldots, \epsilon_n(\epsilon_n x_n)\big) \in \mathrm{Norm}(T).$$

The following examples show that $\mathrm{Norm}(T) = \varnothing$ or an infinite set.

Examples. **(a)** Let

$$T\big((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\big) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s(^2 c_0).$$

We claim that $\mathrm{Norm}(T) = \varnothing$. Obviously, $\|T\| = 1$. Assume that $\mathrm{Norm}(T) \neq \varnothing$. Let $\big((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\big) \in \mathrm{Norm}(T)$. Then

$$1 = \big|T\big((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\big)\big| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i|\,|y_i| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

which shows that $|x_i| = |y_i| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, $\mathrm{Norm}(T) = \varnothing$.

(b) Let

$$T\big((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\big) = x_1 y_1 \in \mathcal{L}_s(^2 c_0).$$

Then

$$\mathrm{Norm}(T) = \Big\{\big((\pm 1, x_2, x_3, \ldots), (\pm 1, y_2, y_3, \ldots)\big) \in c_0 \times c_0 \colon |x_j| \leq 1, |y_j| \leq 1 \text{ for } j \geq 2\Big\}.$$

A mapping $P \colon E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \ldots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^n E)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^n E)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^n E)$, we define

$$\mathrm{Norm}(P) = \big\{x \in E \colon x \text{ is a norming point of } P\big\}.$$

$\mathrm{Norm}(P)$ is called the *norming set* of P. Note that $\mathrm{Norm}(P) = \varnothing$ or a finite set or an infinite set.

Kim classified in [7] $\mathrm{Norm}(P)$ for every $P \in \mathcal{P}\big(^2 l_\infty^2\big)$, where $l_\infty^2 = \mathbb{R}^2$ with the supremum norm.

If $\mathrm{Norm}(T) \neq \varnothing$, $T \in \mathcal{L}(^n E)$ is called a *norm attaining* n-linear form and if $\mathrm{Norm}(P) \neq \varnothing$, $P \in \mathcal{P}(^n E)$ is called a *norm attaining* n-homogeneous polynomial (see [3]).

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to study about $\mathrm{Norm}(T)$ for $T \in \mathcal{L}(^n E)$. For $m \in \mathbb{N}$, let $l_1^m := \mathbb{R}^m$ with the the l_1-norm and $l_\infty^2 = \mathbb{R}^2$ with the supremum norm. Note that if $E = l_1^m$ or l_∞^2 and $T \in \mathcal{L}(^n E)$, $\mathrm{Norm}(T) \neq \varnothing$ since S_E is compact. Kim [6, 8–10] classified $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}_s\big(^2 l_\infty^2\big)$, $\mathcal{L}\big(^2 l_\infty^2\big)$, $\mathcal{L}\big(^2 l_1^2\big)$, $\mathcal{L}_s\big(^2 l_1^3\big)$ or $\mathcal{L}_s\big(^3 l_1^2\big)$. Recently, Kim [11] classified

$\mathrm{Norm}(T)$ for every $T \in \mathcal{L}\big({}^2\mathbb{R}^2_{h(w)}\big)$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm $\|(x,y)\|_{h(w)} = \max\big\{|y|, |x| + (1-w)|y|\big\}$ with weight $0 < w < 1$.

In this paper, for $m, n \in \mathbb{N}$, $m, n \geq 2$, we characterize $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}\big({}^m l_1^n\big)$, where $l_1^n = \mathbb{R}^n$ with the l_1-norm. As an application, we classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}\big({}^m l_1^n\big)$ for $n = 2, 3$ and $m = 2$.

2. The norming sets of $\mathcal{L}\big({}^m l_1^n\big)$.

Theorem A [10]. *Let* $n, m \geq 2$. *Let* $T \in \mathcal{L}\big({}^m l_1^n\big)$ *with*

$$T\Big(\big(x_1^{(1)}, \ldots, x_n^{(1)}\big), \ldots, \big(x_1^{(m)}, \ldots, x_n^{(m)}\big)\Big) = \sum_{1 \leq i_k \leq n, 1 \leq k \leq m} a_{i_1 \ldots i_m} x_{i_1}^{(1)} \ldots x_{i_m}^{(m)}$$

for some $a_{i_1 \ldots i_m} \in \mathbb{R}$. *Then*

$$\|T\| = \max\big\{|a_{i_1 \ldots i_m}| \colon 1 \leq i_k \leq n,\ 1 \leq k \leq m\big\}.$$

By simplicity we denote $T = (a_{i_1 \ldots i_m})_{1 \leq i_k \leq n, 1 \leq k \leq m}$. We call $a_{i_1 \ldots i_m}$'s the *coefficients* of T. Note that if $\|T\| = 1$, then $|a_{i_1 \ldots i_m}| \leq 1$ for all $1 \leq i_k \leq n$, $1 \leq k \leq m$.

Theorem B [10]. *Let* $n, m \geq 2$. *Let* $T \in \mathcal{L}\big({}^m l_1^n\big)$ *be the same as in Theorem A. Suppose that* $\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big) \in \mathrm{Norm}(T)$. *If* $|a_{i'_1 \ldots i'_m}| < \|T\|$ *for* $1 \leq i'_k \leq n$, $1 \leq k \leq m$, *then* $t_{i'_1}^{(1)} \ldots t_{i'_m}^{(m)} = 0$.

Theorem C. *Let* $n, m \geq 2$. *Let* $T = (a_{i_1 \ldots i_m})_{1 \leq i_k \leq n, 1 \leq k \leq m} \in \mathcal{L}\big({}^m l_1^n\big)$ *be the same as in Theorem A with* $\|T\| = 1$. *Let* $\delta_{i_1 \ldots i_m} = 1$ *if* $|a_{i_1 \ldots i_m}| = 1$ *and* $\delta_{i_1 \ldots i_m} = 0$ *if* $|a_{i_1 \ldots i_m}| < 1$. *We define*

$$T_\delta = (a_{i_1 \ldots i_m} \delta_{i_1 \ldots i_m})_{1 \leq i_k \leq n, 1 \leq k \leq m} \in \mathcal{L}\big({}^m l_1^n\big).$$

Then $\mathrm{Norm}(T) = \mathrm{Norm}(T_\delta)$.

Proof. $(\subseteq)$. Let $\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big) \in \mathrm{Norm}(T)$. Then

$$\begin{aligned} 1 &= \Big|T\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big)\Big| \\ &= \left| \sum_{|a_{i'_1 \ldots i'_m}| < 1} a_{i'_1 \ldots i'_m} t_{i'_1}^{(1)} \ldots t_{i'_m}^{(m)} + \sum_{|a_{i_1 \ldots i_m}| = 1} a_{i_1 \ldots i_m} t_{i_1}^{(1)} \ldots t_{i_m}^{(m)} \right| \\ &= \left| \sum_{|a_{i_1 \ldots i_m}| = 1} a_{i_1 \ldots i_m} t_{i_1}^{(1)} \ldots t_{i_m}^{(m)} \right| \quad \text{(by Theorem B)} \\ &= \left| \sum_{|a_{i'_1 \ldots i'_m}| < 1} (a_{i'_1 \ldots i'_m} \delta_{i'_1 \ldots i'_m}) t_{i'_1}^{(1)} \ldots t_{i'_m}^{(m)} + \sum_{|a_{i_1 \ldots i_m}| = 1} (a_{i_1 \ldots i_m} \delta_{i_1 \ldots i_m}) t_{i_1}^{(1)} \ldots t_{i_m}^{(m)} \right| \\ &= \Big|T_\delta\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big)\Big|. \end{aligned}$$

$(\supseteq)$. Let $\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big) \in \mathrm{Norm}(T_\delta)$. Write

$$
\begin{aligned}
&T\big((x_1^{(1)},\dots,x_n^{(1)}),\dots,(x_1^{(m)},\dots,x_n^{(m)})\big)\\
&\qquad= T_\delta\big((x_1^{(1)},\dots,x_n^{(1)}),\dots,(x_1^{(m)},\dots,x_n^{(m)})\big) + \sum_{|a_{i_1\dots i_m}|<1} a_{i_1\dots i_m} x_{i_1}^{(1)}\dots x_{i_m}^{(m)}.
\end{aligned}
$$

Let $T_- \in \mathcal{L}(^m l_1^n)$ be such that

$$
\begin{aligned}
&T_-\big((x_1^{(1)},\dots,x_n^{(1)}),\dots,(x_1^{(m)},\dots,x_n^{(m)})\big)\\
&\qquad= T_\delta\big((x_1^{(1)},\dots,x_n^{(1)}),\dots,(x_1^{(m)},\dots,x_n^{(m)})\big) - \sum_{|a_{i_1\dots i_m}|<1} a_{i_1\dots i_m} x_{i_1}^{(1)}\dots x_{i_m}^{(m)}.
\end{aligned}
$$

By Theorem A, $\|T_-\| = 1$. It follows that

$$
\begin{aligned}
1 &\geq \big|T\big((t_1^{(1)},\dots,t_n^{(1)}),\dots,(t_1^{(m)},\dots,t_n^{(m)})\big)\big|\\
&= \left|T_\delta\big((t_1^{(1)},\dots,t_n^{(1)}),\dots,(t_1^{(m)},\dots,t_n^{(m)})\big) + \sum_{|a_{i_1\dots i_m}|<1} a_{i_1\dots i_m} t_{i_1}^{(1)}\dots t_{i_m}^{(m)}\right|,\\
1 &\geq \big|T_-\big((t_1^{(1)},\dots,t_n^{(1)}),\dots,(t_1^{(m)},\dots,t_n^{(m)})\big)\big|\\
&= \left|T_\delta\big((t_1^{(1)},\dots,t_n^{(1)}),\dots,(t_1^{(m)},\dots,t_n^{(m)})\big) - \sum_{|a_{i_1\dots i_m}|<1} a_{i_1\dots i_m} t_{i_1}^{(1)}\dots t_{i_m}^{(m)}\right|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
1 &\geq \big|T_\delta\big((t_1^{(1)},\dots,t_n^{(1)}),\dots,(t_1^{(m)},\dots,t_n^{(m)})\big)\big|\\
&\quad+ \left|\sum_{|a_{i_1\dots i_m}|<1} a_{i_1\dots i_m} t_{i_1}^{(1)}\dots t_{i_m}^{(m)}\right|\\
&= 1 + \left|\sum_{|a_{i_1\dots i_m}|<1} a_{i_1\dots i_m} t_{i_1}^{(1)}\dots t_{i_m}^{(m)}\right|.
\end{aligned}
$$

Thus,

$$
\left|\sum_{|a_{i_1\dots i_m}|<1} a_{i_1\dots i_m} t_{i_1}^{(1)}\dots t_{i_m}^{(m)}\right| = 0
$$

and so

$$
\left|T\big((t_1^{(1)},\dots,t_n^{(1)}),\dots,(t_1^{(m)},\dots,t_n^{(m)})\big)\right| = \left|T_\delta\big((t_1^{(1)},\dots,t_n^{(1)}),\dots,(t_1^{(m)},\dots,t_n^{(m)})\big)\right| = 1.
$$

Theorem C is proved.

The following shows that we can classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^n l_1^m)$ with $\|T\| = 1$ if we know $\mathrm{Norm}(S)$ for every $S = (b_{i_1\dots i_m})_{1\leq i_k\leq n, 1\leq k\leq m} \in \mathcal{L}(^n l_1^m)$ such that $\|S\| = 1 = |b_{i_1\dots i_m}|$ for every $1 \leq i_k \leq n$, $1 \leq k \leq m$.

Theorem D. *Let* $n, m \geq 2$. *Let* $T = (a_{i_1\ldots i_m})_{1\leq i_k\leq n, 1\leq k\leq m} \in \mathcal{L}\big({}^m l_1^n\big)$ *with* $\|T\| = 1$. *Let* $c_{i_1\ldots i_m} \in \mathbb{R}$ *by* $|c_{i_1\ldots i_m}| = 1$ *for every* $1 \leq i_k \leq n$, $1 \leq k \leq m$.

Define $S = (b_{i_1\ldots i_m})_{1\leq i_k\leq n, 1\leq k\leq m} \in \mathcal{L}({}^n l_1^m)$ *be such that* $b_{i_1\ldots i_m} = a_{i_1\ldots i_m}$ *if* $|a_{i_1\ldots i_m}| = 1$ *and* $b_{i_1\ldots i_m} = c_{i_1\ldots i_m}$ *if* $|a_{i_1\ldots i_m}| < 1$. *Then*

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{ \big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big) \in \mathrm{Norm}(S)\colon \\ t_{i_1'}^{(1)} = 0 \ \textit{or} \ \ldots, \ \textit{or} \ t_{i_m'}^{(m)} = 0 \ \textit{whenever} \ |a_{i_1'\ldots i_m'}| < 1 \\ \textit{for some} \ 1 \leq i_k' \leq n, \ 1 \leq k \leq m \Big\}.\end{aligned}$$

Proof. Let

$$\begin{aligned}\mathcal{F} = \Big\{ \big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big) \in \mathrm{Norm}(S) : t_{i_1'}^{(1)} \ldots t_{i_m'}^{(m)} = 0 \\ \text{if } |a_{i_1'\ldots i_m'}| < 1 \ \text{for some} \ 1 \leq i_k' \leq n, \ 1 \leq k \leq m \Big\}.\end{aligned}$$

We will show that $\mathrm{Norm}(T) = \mathcal{F}$.

($\subseteq$). Let $\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big) \in \mathrm{Norm}(T)$. Then

$$\begin{aligned}&\Big|S\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big)\Big| \\ &= \left| \sum_{|a_{i_1'\ldots i_m'}|<1} c_{i_1'\ldots i_m'} t_{i_1'}^{(1)} \ldots t_{i_m'}^{(m)} + \sum_{|a_{i_1\ldots i_m}|=1} a_{i_1\ldots i_m} t_{i_1}^{(1)} \ldots t_{i_m}^{(m)} \right| \\ &= \left| \sum_{|a_{i_1\ldots i_m}|=1} a_{i_1\ldots i_m} t_{i_1}^{(1)} \ldots t_{i_m}^{(m)} \right| \quad \Big(\text{by Theorem B}, \ t_{i_1'}^{(1)} \ldots t_{i_m'}^{(m)} = 0\Big) \\ &= \left| \sum_{|a_{i_1'\ldots i_m'}|<1} a_{i_1'\ldots i_m'} t_{i_1'}^{(1)} \ldots t_{i_m'}^{(m)} + \sum_{|a_{i_1\ldots i_m}|=1} a_{i_1\ldots i_m} \delta_{i_1\ldots i_m} t_{i_1}^{(1)} \ldots t_{i_m}^{(m)} \right| \\ &= \Big|T\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big)\Big| = 1,\end{aligned}$$

so $\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big) \in \mathrm{Norm}(S)$ satisfying $t_{i_1'}^{(1)} \ldots t_{i_m'}^{(m)} = 0$ if $|a_{i_1'\ldots i_m'}| < 1$. Thus, $\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big) \in \mathcal{F}$.

($\supseteq$). Let $\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big) \in \mathcal{F}$. It follows that

$$\begin{aligned}1 &= \Big|S\big(\big(t_1^{(1)}, \ldots, t_n^{(1)}\big), \ldots, \big(t_1^{(m)}, \ldots, t_n^{(m)}\big)\big)\Big| \\ &= \left| \sum_{|a_{i_1'\ldots i_m'}|<1} c_{i_1'\ldots i_m'} t_{i_1'}^{(1)} \ldots t_{i_m'}^{(m)} + \sum_{|a_{i_1\ldots i_m}|=1} a_{i_1\ldots i_m} t_{i_1}^{(1)} \ldots t_{i_m}^{(m)} \right|\end{aligned}$$

$$= \left| \sum_{|a_{i_1\ldots i_m}|=1} a_{i_1\ldots i_m} t_{i_1}^{(1)} \ldots t_{i_m}^{(m)} \right| \quad \Big(\text{since } t_{i_1'}^{(1)} \ldots t_{i_m'}^{(m)} = 0\Big)$$

$$= \big|T_\delta\big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\big)\big| \quad \text{(by Theorem C)}$$

$$= \Big|T\big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\big)\Big|,$$

which implies that $\big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\big) \in \mathrm{Norm}(T)$.

Theorem D is proved.

Theorem E. *Let* $n, m \geq 2$. *Let* $T = (a_{i_1\ldots i_m})_{1\leq i_k\leq n, 1\leq k\leq m} \in \mathcal{L}\big({}^m l_1^n\big)$ *with* $\|T\| = 1$. *Then*

$$\mathrm{Norm}(T) = \Big\{\big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\big) \in (S_{l_1^n})^m : \\ \Big(\big(t_1^{(1)},\ldots,t_n^{(1)},0\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)},0\big)\Big) \in \mathrm{Norm}(T')\Big\},$$

where $T' = (b_{i_1\ldots i_m})_{1\leq i_k\leq n+1, 1\leq k\leq m} \in \mathcal{L}\big({}^m l_1^{n+1}\big)$ *is such that* $b_{i_1\ldots i_m} = a_{i_1\ldots i_m}$ *if* $1 \leq i_k \leq n$ *for every* $1 \leq k \leq m$ *and* $b_{i_1\ldots i_m} = 1$ *otherwise.*

Proof. By Theorem A, $\|T'\| = 1$. Let

$$\mathcal{M} = \Big\{\big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\big) \in (S_{l_1^n})^m : \\ \big(\big(t_1^{(1)},\ldots,t_n^{(1)},0\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)},0\big)\big) \in \mathrm{Norm}(T')\Big\}.$$

$(\subseteq)$. Let $\big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\big) \in \mathrm{Norm}(T)$. Obviously,

$$\Big\|\big(\big(t_1^{(1)},\ldots,t_n^{(1)},0\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)},0\big)\big)\Big\|_{l_1^{n+1}} = 1.$$

Then

$$\Big|T'\big(\big(t_1^{(1)},\ldots,t_n^{(1)},0\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)},0\big)\big)\Big|$$

$$= \left| \sum_{1\leq i_k\leq n+1, 1\leq k\leq m} b_{i_1\ldots i_m} t_{i_1}^{(1)} \ldots t_{i_m}^{(m)} \right|$$

$$= \left| \sum_{1\leq i_k\leq n, 1\leq k\leq m} a_{i_1\ldots i_m} t_{i_1}^{(1)} \ldots t_{i_m}^{(m)} \right|$$

$$= \Big|T\big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\big)\Big| = \|T'\| = 1,$$

thus, $\big(\big(t_1^{(1)},\ldots,t_n^{(1)},0\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)},0\big)\big) \in \mathcal{M}$.

$(\supseteq)$. Let $\big(\big(t_1^{(1)},\ldots,t_n^{(1)},0\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)},0\big)\big) \in \mathcal{M}$. Obviously,

$$\Big\|\big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\big)\Big\|_{l_1^n} = 1.$$

Then

$$\begin{aligned}
&\left|T\big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\big)\right| \\
&\qquad= \left|\sum_{1\le i_k\le n, 1\le k\le m} a_{i_1\ldots i_m} t_{i_1}^{(1)}\ldots t_{i_m}^{(m)}\right| \\
&\qquad= \left|\sum_{1\le i_k\le n, 1\le k\le m} b_{i_1\ldots i_m} t_{i_1}^{(1)}\ldots t_{i_m}^{(m)}\right| \\
&\qquad= \left|T'\big(\big(t_1^{(1)},\ldots,t_n^{(1)},0\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)},0\big)\big)\right| = \|T'\| = 1,
\end{aligned}$$

thus, $\big(\big(t_1^{(1)},\ldots,t_n^{(1)}\big),\ldots,\big(t_1^{(m)},\ldots,t_n^{(m)}\big)\big) \in \mathrm{Norm}(T)$.

Theorem E is proved.

3. Applications. In this section, as an application of Theorems D and E, we classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}\big({}^m l_1^n\big)$ for $n = 2, 3$ and $m = 2$.

Theorem F [10]. *Let* $(x, y) \in S_{l_1^2}$.

(1) *If* $|x \pm y| = 1$, *then* $|x| = 1$ *or* $|y| = 1$.

(2) *Let* $a, b \in \mathbb{R}$ *such that* $0 < |x| < 1$, $|a| \le 1$ *and* $|b| \le 1$. *If* $1 = |xa + yb|$, *then* $|a| = |b| = 1$ *and* $\mathrm{sign}(xy) = \mathrm{sign}(ab)$.

Let $T((x_1, y_1, z_1), (x_2, y_2, z_2)) = ax_1x_2 + by_1y_2 + cz_1z_2 + dx_1y_2 + ex_2y_1 + fx_1z_2 + gx_2z_1 + hy_1z_2 + iy_2z_1 \in \mathcal{L}\big({}^2 l_1^3\big)$ for some $a, b, c, d, e, f, g, h, i \in \mathbb{R}$. For simplicity we denote $T = (a, b, c, d, e, f, g, h, i)$. By Theorem A,

$$\|T\| = \max\big\{|a|, |b|, |c|, |d|, |e|, |f|, |g|, |h|, |i|\big\}.$$

Theorem G. *Let* $T((x_1, y_1, z_1), (x_2, y_2, z_2)) = ax_1x_2 + by_1y_2 + cz_1z_2 + dx_1y_2 + ex_2y_1 + fx_1z_2 + gx_2z_1 + hy_1z_2 + iy_2z_1 \in \mathcal{L}\big({}^2 l_1^3\big)$ *for some* $a, b, c, d, e, f, g, h, i \in \mathbb{R}$. *Then there are* $a^*, b^*, c^*, d^*, e^*, f^* \in \{\pm a, \pm b, \pm c, \pm d, \pm e, \pm f, \pm g, \pm h, \pm i\}$ *such that* $a^* \ge b^* \ge c^* \ge 0$, $|d^*| \ge |e^*|$, $f^* \ge 0, h^* \ge 0$ *and* $\|T\| = \|(a^*, b^*, c^*, d^*, e^*, f^*, g^*, h^*, i^*)\|$.

Proof. If $|b| > |a|$, we consider $T_1((x_1, y_1, z_1), (x_2, y_2, z_2)) = T((y_1, x_1, z_1), (y_2, x_2, z_2))$. Note that $\|T_1\| = \|T\|$. Hence, we may assume that $|a| \ge |b|$. If $|c| > |a|$, we consider $T_2((x_1, y_1, z_1), (x_2, y_2, z_2)) = T((z_1, x_1, y_1), (z_2, x_2, y_2))$. Note that $\|T_2\| = \|T\|$. Hence, we may assume that $|a| \ge \max\{|b|, |c|\}$. If $|c| > |b|$, we consider $T_3((x_1, y_1, z_1), (x_2, y_2, z_2)) = T((x_1, z_1, y_1), (x_2, z_2, y_2))$. Note that $\|T_3\| = \|T\|$. Hence, we may assume that $|a| \ge |b| \ge |c|$. If $a < 0$, $T_4 = -T \in \mathcal{L}_s\big({}^2 l_1^3\big)$. Note that $\|T_4\| = \|T\|$. Thus, we may assume that $a \ge |b| \ge |c| \ge 0$. If $b < 0$, we let $T_5((x_1, y_1, z_1), (x_2, y_2, z_2)) = T((x_1, -y_1, z_1), (x_2, y_2, z_2))$. Note that $\|T_5\| = \|T\|$. Thus we may assume that $b \ge 0$. If $c < 0$, we let $T_6((x_1, y_1, z_1), (x_2, y_2, z_2)) = T((x_1, y_1, -z_1), (x_2, y_2, z_2))$. Note that $\|T_6\| = \|T\|$. Thus, we may assume that $c \ge 0$. Hence, we assume that $a \ge b \ge c \ge 0$. If $d < 0$, we let $T_7((x_1, y_1, z_1), (x_2, y_2, z_2)) = T((-x_1, y_1, z_1), (-x_2, y_2, z_2))$. Note that $\|T_7\| = \|T\|$. Thus we may assume that $d \ge 0$. If $e < 0$, we let $T_8((x_1, y_1, z_1), (x_2, y_2, z_2)) = T((x_1, y_1, -z_1), (x_2, y_2, -z_2))$. Note that $\|T_8\| = \|T\|$. Thus, we may assume that $e \ge 0$.

Theorem G is proved.

Let $\mathcal{W} \subseteq S_{l_1^3} \times S_{l_1^3}$. We denote

$$\mathrm{Sym}(\mathcal{W}) := \big\{(X,Y),(Y,X) : (X,Y) \in \mathcal{W}\big\}.$$

To classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^2 l_1^3)$, by Theorem D, it suffices to classify $\mathrm{Norm}(S)$ for every $S \in \mathcal{L}(^2 l_1^3)$ such that the absolute value of every coefficient of S is equal to $\|S\|$. By Theorem G, if $T = (a,b,c,d,e,f,g,h,i) \in \mathcal{L}(^2 l_1^3)$, then we may assume that $a \ge b \ge c \ge 0,\ |d| \ge |e|,\ f \ge 0,$ $h \ge 0$.

Theorem H. *Let* $T((x_1,y_1,z_1),(x_2,y_2,z_2)) = x_1x_2 + y_1y_2 + z_1z_2 + dx_1y_2 + ex_2y_1 + x_1z_2 + gx_2z_1 + y_1z_2 + iy_2z_1 \in \mathcal{L}(^2 l_1^3)$ *such that* $\|T\| = 1$ *with* $1 = a = b = c = |d| = |e| = f = |g| = h = |i|$. *We consider* 16 *cases*:

Case 1: $1 = a = b = c = d = e = f = g = h = i.$

$$\mathrm{Norm}(T) = \Big\{\big(\pm(r,s,t),\ \pm(r',s',t')\big) : \\ |r+s+t| = |r|+|s|+|t| = |r'+s'+t'| = |r'|+|s'|+|t'| = 1\Big\}.$$

Case 2: $1 = a = b = c = -d = -e = f = g = h = i.$

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\big(\pm(1,0,0),\pm(r,-s,t)\big),\ \big(\pm(0,1,0),\pm(r,-s,-t)\big), \\ \big(\pm(0,0,1),\pm(r,s,t)\big),\ \big(\pm(0,u,1-u),\pm(0,v,1-v)\big), \\ \big(\pm(u,0,1-u),\pm(v,0,1-v)\big),\ \big(\pm(u,-(1-u),0),\pm(v,-(1-v),0)\big) : \\ 0 \le u,\ v \le 1,\ |r+s+t| = |r|+|s|+|t| = 1\Big\}\Big).$$

Case 3: $1 = a = b = c = d = e = f = -g = h = -i.$

$$\mathrm{Norm}(T) = \Big\{\big(\pm(u,1-u,0),\pm(r,s,t)\big),\ \big(\pm(0,0,1),\pm(r,s,-t)\big), \\ \big(\pm(0,u,-(1-u)),\pm(v,1-v,0)\big),\ \big(\pm(u,0,-(1-u)),\pm(v,1-v,0)\big), \\ \big(\pm(u,0,1-u),\pm(0,0,1)\big),\ \big(\pm(0,u,1-u),\pm(0,0,1)\big) : 0 \le u,v \le 1, \\ |r+s+t| = |r|+|s|+|t| = 1\Big\}.$$

Case 4: $1 = a = b = c = d = e = f = -g = h = i.$

$$\mathrm{Norm}(T) = \Big\{\big(\pm(u,1-u,0),\pm(r,s,t)\big),\ \big(\pm(r,s,t),\pm(0,u,1-u)\big), \\ \big(\pm(0,0,1),\pm(r,-s,-t)\big),\ \big(\pm(0,u,-(1-u)),\pm(1,0,0)\big), \\ \big(\pm(u,0,-(1-u)),\pm(1,0,0)\big) : 0 \le u \le 1, \\ |r+s+t| = |r|+|s|+|t| = 1\Big\}.$$

Case 5: $1 = a = b = c = d = e = f = g = h = -i.$

$$\mathrm{Norm}(T) = \Big\{\big(\pm(u,1-u,0),\pm(r,s,t)\big),\ \big(\pm(0,0,1),\pm(r,-s,t)\big), \\ \big(\pm(r,s,t),\pm(u,0,1-u)\big),\ \big(\pm(r,s,-t),\pm(0,1,0)\big) : \\ 0 \le u \le 1,\ |r+s+t| = |r|+|s|+|t| = 1\Big\}.$$

Case 6: $1 = a = b = c = -d = e = f = g = h = i.$

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\big(\pm(0,u,1-u),\pm(r,s,t)\big),\ \big(\pm(r,s,t),\pm(u,0,1-u)\big),\\ &\big(\pm(r,-s,-t),\pm(0,1,0)\big),\ \big(\pm(1,0,0),\pm(r,-s,t)\big):\\ &0 \le u \le 1,\ |r+s+t| = |r|+|s|+|t| = 1\Big\}.\end{aligned}$$

Case 7: $1 = a = b = c = -d = e = f = -g = h = i.$

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\big(\pm(1,0,0),\pm(r,-s,t)\big),\ \big(\pm(r,s,-t),\pm(1,0,0)\big),\\ &\big(\pm(0,1,0),\pm(r,s,t)\big),\ \big(\pm(r,-s,-t),\pm(0,1,0)\big),\\ &\big(\pm(0,0,1),\pm(r,s,t)\big),\ \big(\pm(r,-s,-t),\pm(0,0,1)\big),\\ &\big(\pm(0,u,1-u),\pm(0,v,1-v)\big),\ \big(\pm(u,0,-(1-u)),\pm(v,-(1-v),0)\big),\\ &\big(\pm(u,1-u,0),\pm(v,0,1-v)\big):0 \le u,\ v \le 1,\ |r+s+t| = |r|+|s|+|t| = 1\Big\}.\end{aligned}$$

Case 8: $1 = a = b = c = -d = e = f = g = h = -i.$

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\big(\pm(u,0,1-u),\pm(r,-s,t)\big),\ \big(\pm(0,u,1-u),\pm(v,0,1-v)\big),\\ &\big(\pm(r,s,t),\pm(u,0,1-u)\big),\ \big(\pm(r,-s,t),\pm(0,1,0)\big):\\ &0 \le u,\ v \le 1,\ |r+s+t| = |r|+|s|+|t| = 1\Big\}.\end{aligned}$$

Case 9: $1 = a = b = c = -d = e = f = -g = h = -i.$

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\big(\pm(1,0,0),\pm(r,-s,t)\big),\ \big(\pm(0,1,0),\pm(r,s,t)\big),\\ &\big(\pm(0,0,1),\pm(r,s,-t)\big),\ \big(\pm(r,s,-t),\pm(1,0,0)\big),\\ &\big(\pm(r,-s,t),\pm(0,1,0)\big),\ \big(\pm(r,s,t),\pm(0,0,1)\big),\\ &\big(\pm(0,u,-(1-u)),\pm(v,1-v,0)\big),\\ &\big(\pm(u,0,1-u),\pm(0,v,-(1-v))\big),\\ &\big(\pm(u,1-u,0),\pm(v,0,1-v)\big):0 \le u,v \le 1,\\ &|r+s+t| = |r|+|s|+|t| = 1\Big\}.\end{aligned}$$

Case 10: $1 = a = b = c = d = -e = f = g = h = i.$

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\big(\pm(0,1,0),\pm(r,-s,-t)\big),\ \big(\pm(u,0,1-u),\pm(r,s,t)\big),\\ &\big(\pm(r,s,t),\pm(0,u,1-u)\big),\ \big(\pm(r,-s,t),\pm(1,0,0)\big),\\ &\big(\pm(0,u,1-u),\pm(0,v,1-v)\big):0 \le u,v \le 1,\\ &|r+s+t| = |r|+|s|+|t| = 1\Big\}.\end{aligned}$$

Case 11: $1 = a = b = c = d = -e = f = -g = h = i$.

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{ & \big(\pm(1,0,0), \pm(r,s,t)\big),\ \big(\pm(0,u,1-u), \pm(r,-s,-t)\big),\\ & \big(\pm(r,-s,-t), \pm(1,0,0)\big),\ \big(\pm(r,s,t), \pm(0,u,1-u)\big),\\ & \big(\pm(u,1-u,0), \pm(0,v,1-v)\big) : 0 \le u,v \le 1,\\ & |r+s+t| = |r|+|s|+|t| = 1\Big\}.\end{aligned}$$

Case 12: $1 = a = b = c = d = -e = f = g = h = -i$.

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{ & \big(\pm(1,0,0), \pm(r,s,t)\big),\ \big(\pm(0,1,0), \pm(r,-s,-t)\big),\\ & \big(\pm(0,0,1), \pm(r,-s,t)\big),\ \big(\pm(r,-s,t), \pm(1,0,0)\big),\\ & \big(\pm(r,s,-t), \pm(0,1,0)\big),\ \big(\pm(r,s,t), \pm(0,0,1)\big),\\ & \big(\pm(0,u,-(1-u)), \pm(v,-(1-v),0)\big),\\ & \big(\pm(u,0,1-u), \pm(v,0,1-v)\big),\\ & \big(\pm(u,1-u,0), \pm(0,v,1-v)\big) : 0 \le u,v \le 1,\\ & |r+s+t| = |r|+|s|+|t| = 1\Big\}.\end{aligned}$$

Case 13: $1 = a = b = c = d = -e = f = -g = h = -i$.

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{ & \big(\pm(1,0,0), \pm(r,s,t)\big),\ \big(\pm(0,1,0), \pm(r,-s,-t)\big),\\ & \big(\pm(0,0,1), \pm(r,s,-t)\big),\ \big(\pm(r,-s,-t), \pm(1,0,0)\big),\\ & \big(\pm(r,s,-t), \pm(0,1,0)\big),\ \big(\pm(r,s,t), \pm(0,0,1)\big),\\ & \big(\pm(0,u,1-u), \pm(v,0,-(1-v))\big),\\ & \big(\pm(u,0,-(1-u)), \pm(v,1-v,0)\big),\\ & \big(\pm(u,1-u,0), \pm(0,v,1-v)\big) : 0 \le u,v \le 1,\\ & |r+s+t| = |r|+|s|+|t| = 1\Big\}.\end{aligned}$$

Case 14: $1 = a = b = c = -d = -e = f = -g = h = -i$.

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{ & \big(\pm(1,0,0), \pm(r,-s,t)\big),\ \big(\pm(0,1,0), \pm(r,-s,-t)\big),\\ & \big(\pm(0,0,1), \pm(r,s,-t)\big),\ \big(\pm(r,-s,-t), \pm(1,0,0)\big),\\ & \big(\pm(r,-s,t), \pm(0,1,0)\big),\ \big(\pm(r,s,t), \pm(0,0,1)\big),\\ & \big(\pm(0,u,1-u), \pm(v,0,-(1-v))\big),\\ & \big(\pm(u,0,1-u), \pm(0,v,-(1-v))\big),\end{aligned}$$

$$\big(\pm(u,-(1-u),0),\pm(v,-(1-v),0)\big) : 0\le u,v\le 1,$$
$$|r+s+t|=|r|+|s|+|t|=1\Big\}.$$

Case 15: $1=a=b=c=-d=-e=f=-g=h=i.$

$$\mathrm{Norm}(T)=\Big\{\big(\pm(1,0,0),\pm(r,-s,t)\big),\ \big(\pm(0,u,1-u),\pm(r,-s,-t)\big),$$
$$\big(\pm(r,s,t),\pm(0,0,1)\big),\ \big(\pm(r,-s,-t),\pm(u,-(1-u),0)\big):$$
$$0\le u\le 1,\ |r+s+t|=|r|+|s|+|t|=1\Big\}.$$

Case 16: $1=a=b=c=-d=-e=f=g=h=-i.$

$$\mathrm{Norm}(T)=\Big\{\big(\pm(u,0,1-u),\pm(r,-s,t)\big),\ \big(\pm(r,-s,t),\pm(u,-(1-u),0)\big),$$
$$\big(\pm(r,s,t),\pm(0,0,1)\big),\ \big(\pm(0,1,0),\pm(r,-s,-t)\big):$$
$$0\le u\le 1,\ |r+s+t|=|r|+|s|+|t|=1\Big\}.$$

Proof. We only prove Case 12 because the proofs of the other cases are analogous.

Case 12: $1=a=b=c=d=-e=f=g=h=-i.$

Let $\big((x_1,y_1,z_1),(x_2,y_2,z_2)\big)\in\mathrm{Norm}(T)$. Without loss of generality we may assume that $x_j\ge 0$ for $j=1,2$. It follows that

$$\begin{aligned}1&=|T((x_1,y_1,z_1),(x_2,y_2,z_2))|\\&=\big|x_1(x_2+dy_2+z_2)+y_1(ex_2+y_2+z_2)+z_1(gx_2+iy_2+z_2)\big|\\&=|x_1|\,|x_2+y_2+z_2|+|y_1|\,|-x_2+y_2+z_2|+|z_1|\,|x_2-y_2+z_2|.\end{aligned}$$

Suppose that $x_1=0$. If $y_1=0$, then $(x_1,y_1,z_1)=(0,0,\pm1)$, $(x_2,y_2,z_2)=(r,-s,t)$ for $|r+s+t|=|r|+|s|+|t|=1$.

If $|y_1|=1$, then $(x_1,y_1,z_1)=(0,\pm1,0)$, $(x_2,y_2,z_2)=(r,-s,-t)$ for $|r+s+t|=|r|+|s|+|t|=1$.

Let $0<|y_1|<1$. Then $0<|z_1|<1$ and

$$1=|y_1|\,|-x_2+y_2+z_2|+|z_1|\,|x_2-y_2+z_2|.$$

By Theorem F(2),

$$1=|-x_2+y_2+z_2|=|x_2-y_2+z_2|.$$

By Theorem F(1), $|z_2|=1$ or $|x_2-y_2|=1$. Note that if $|z_2|=1$, then $|y_1+z_1|=1$, so $(x_1,y_1,z_1)=\pm(0,u,1-u)$, $(x_2,y_2,z_2)=(0,0,\pm1)$ for $0\le u\le 1$. Note that if $|x_2-y_2|=1$, then $|y_1-z_1|=1$, so $(x_1,y_1,z_1)=\pm(0,u,-(1-u))$, $(x_2,y_2,z_2)=(v,-(1-v),0)$ for $0\le u,v\le 1$.

Suppose that $x_1=1$. Then $(x_1,y_1,z_1)=(1,0,0)$, $(x_2,y_2,z_2)=(r,s,t)$ for $|r+s+t|=|r|+|s|+|t|=1$.

Suppose that $0<x_1<1$. Note that if $y_1=0$, then $0<|z_1|<1$ and

$$1 = |x_1|\,|x_2 + y_2 + z_2| + |z_1|\,|x_2 - y_2 + z_2|.$$

By Theorem F(2), $|x_2 + y_2 + z_2| = 1 = |x_2 - y_2 + z_2|$. By Theorem F(1), $|y_2| = 1$ or $|x_2 + z_2| = 1$. If $|y_2| = 1$, then $|x_1 - z_1| = 1$, so $(x_1, y_1, z_1) = (u, 0, -(1-u))$, $(x_2, y_2, z_2) = (0, \pm 1, 0)$ for $0 \leq u \leq 1$. If $|x_2 + z_2| = 1$, then $|x_1 + z_1| = 1$, so $(x_1, y_1, z_1) = (u, 0, 1-u)$, $(x_2, y_2, z_2) = (v, 0, 1-v)$ for $0 \leq u, v \leq 1$.

Let $0 < |y_1| < 1$. Note that if $z_1 = 0$, then

$$1 = |x_1|\,|x_2 + y_2 + z_2| + |y_1|\,| - x_2 + y_2 + z_2|.$$

By Theorem F(2), $|x_2 + y_2 + z_2| = 1 = |-x_2 + y_2 + z_2|$. By Theorem F(1), $|x_2| = 1$ or $|y_2 + z_2| = 1$. If $|x_2| = 1$, then $|x_1 - y_1| = 1$, so $(x_1, y_1, z_1) = (u, -(1-u), 0)$, $(x_2, y_2, z_2) = (1, 0, 0)$ for $0 \leq u \leq 1$. If $|x_2 + y_2| = 1$, then $|y_1 + z_1| = 1$, so $(x_1, y_1, z_1) = (u, 1-u, 0)$, $(x_2, y_2, z_2) = \pm(0, v, 1-v)$ for $0 \leq u,\ v \leq 1$.

Note that if $0 < |z_1| < 1$, then, by Theorem F(2),

$$|x_2 + y_2 + z_2| = 1 = | - x_2 + y_2 + z_2| = |x_2 - y_2 + z_2|.$$

By Theorem F(1), $x_2 = 1$, $|y_2| = 1$ or $|z_2| = 1$. If $x_2 = 1$, then $|x_1 - y_1 + z_1| = 1$ and so $(x_1, y_1, z_1) = (r, -s, t)$, $(x_2, y_2, z_2) = (1, 0, 0)$ for $|r + s + t| = |r| + |s| + |t| = 1$. If $|y_2| = 1$, then $|x_1 + y_1 - z_1| = 1$ and so $(x_1, y_1, z_1) = (r, s, -t)$, $(x_2, y_2, z_2) = (0, \pm 1, 0)$ for $|r + s + t| = |r| + |s| + |t| = 1$. If $|z_2| = 1$, then $|x_1 + y_1 + z_1| = 1$ and so $(x_1, y_1, z_1) = (r, s, t)$, $(x_2, y_2, z_2) = (0, 0, \pm 1)$ for $|r + s + t| = |r| + |s| + |t| = 1$. Therefore,

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{ & \big(\pm(1,0,0), \pm(r,s,t)\big),\ \big(\pm(0,1,0), \pm(r,-s,-t)\big),\\ & \big(\pm(0,0,1), \pm(r,-s,t)\big),\ \big(\pm(r,-s,t), \pm(1,0,0)\big),\\ & \big(\pm(r,s,-t), \pm(0,1,0)\big),\ \big(\pm(r,s,t), \pm(0,0,1)\big),\\ & \big(\pm(0,u,-(1-u)), \pm(v,-(1-v),0)\big),\ \big(\pm(u,0,1-u), \pm(v,0,1-v)\big),\\ & \big(\pm(u,1-u,0), \pm(0,v,1-v)\big) : 0 \leq u, v \leq 1,\\ & |r+s+t| = |r| + |s| + |t| = 1 \Big\}.\end{aligned}$$

Theorem H is proved.

Kim classified in [10] $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^2 l_1^2)$ or $\mathcal{L}_s(^2 l_1^3)$. In the following we classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^2 l_1^3)$.

Theorem I. *Let* $T((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sum_{1 \leq i,j \leq 3} a_{ij} x_i y_j \in \mathcal{L}(^2 l_1^3)$ *such that* $\|T\| = 1$. *Let* $M = \{(i,j) : |a_{ij}| < 1,\ i,j = 1,2,3\}$. *Define* $S((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sum_{1 \leq i,j \leq 3} b_{ij} x_i y_j \in \mathcal{L}(^2 l_1^3)$ *by* $b_{ij} = a_{ij}$ *if* $(i,j) \notin M$ *and* $b_{ij} = 1$ *if* $(i,j) \in M$. (*Note that by Theorem H*, $\mathrm{Norm}(S)$ *is known.*)

Then

$$\mathrm{Norm}(T) = \Big\{ \big((x_1, y_1, z_1), (x_2, y_2, z_2)\big) \in \mathrm{Norm}(S) : x_i = 0 \ \textit{or}\ y_j = 0 \ \textit{for every}\ (i,j) \in M \Big\}.$$

Proof. This follows from Theorems D and H.

Remarks. **(a)** Let $T((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sum_{1\le i,j\le 3} a_{ij}x_iy_j \in \mathcal{L}(^2 l_1^3)$ such that $a_{33} = \frac{1}{2}$, $a_{12} = -1$ and $a_{ij} = 1$ for every $(i, j) \notin \{(1, 2), (3, 3)\}$.

Let $M = \{(3, 3)\}$. Define $S((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sum_{1\le i,j\le 3} b_{ij}x_iy_j \in \mathcal{L}(^2 l_1^3)$ by $b_{ij} = a_{ij}$ if $(i, j) \notin M$ and $b_{33} = 1$.

By Theorem I,

$$\begin{aligned}\operatorname{Norm}(T) &= \big\{\big((x_1, y_1, z_1), (x_2, y_2, z_2)\big) \in \operatorname{Norm}(S) : x_3 = 0 \text{ or } y_3 = 0\big\}\\ &= \Big\{\big(\pm(0, 1, 0), \pm(r, s, t)\big),\ \big(\pm(u, 1 - u, 0), \pm(v, 0, 1 - v)\big),\\ &\qquad \big(\pm(u, -(1 - u), 0), \pm(0, 1, 0)\big),\ \big(\pm(1, 0, 0), \pm(r, -s, t)\big),\\ &\qquad \big(\pm(0, u, 1 - u), \pm(v, 1 - v, 0)\big),\ \big(\pm(r, s, t), \pm(1, 0, 0)\big),\\ &\qquad \big(\pm(r, -s, -t), \pm(0, 1, 0)\big) : 0 \le u, v \le 1, |r + s + t| = |r| + |s| + |t| = 1\Big\}.\end{aligned}$$

(b) By Theorems E and I, we can classify $\operatorname{Norm}(T)$ for every $T \in \mathcal{L}(^2 l_1^2)$ which was a result in [10].

The author states that there is no conflict of interest.

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