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MEASURE PSEUDOASYMPTOTICALLY BLOCH PERIODIC FUNCTIONS IN THE SENSE OF STEPANOV AND APPLICATIONS

МІРИ ПСЕВДОАСИМПТОТИЧНИХ БЛОХІВСЬКИХ ПЕРІОДИЧНИХ ФУНКЦІЙ У СЕНСІ СТЕПАНОВА ТА ЇХ ЗАСТОСУВАННЯ

We focus on the measures of Stepanov-like pseudoasymptotically Bloch τ-periodicity and its applications. First, we define a new notion of measures of Stepanov-like pseudoasymptotically Bloch periodic functions and diccuss some of its fundamental properties. Then the obtained results are applied to investigate the existence and uniqueness of the measure Stepanov-like pseudoasymptotically Bloch periodic mild solutions to semilinear delay differential equation in Banach spaces. Finally, an application is presented to illustrate the efficiency of the results.

Нашу увагу зосереджено на мірах степановської псевдоасимптотичної блохівської τ-періодичності та її застосуваннях. Спочатку визначено нове поняття міри степановських псевдоасимптотично періодичних функцій Блоха та встановлено деякі їхні фундаментальні властивості. Потім отримані результати застосовано до дослідження існування та єдиності мір псевдоасимптотично блохівських періодичних м'яких розв'язків типу Степанова напівлінійного диференціального рівняння із запізненням у банахових просторах. Насамкінець наведено застосування, яке ілюструє ефективність отриманих результатів.

1. Introduction. This work is based on two important principles. The first is the notion of the Bloch function which came back to the physicist Felix Bloch who worked on the conductivity of crystalline solids [4]. Recently, this type of function becomes the interest of several mathematicians, such is the example of Hasler and N'Guérékata [15], which focuses on the fundamental results of Bloch type τ-periodic functions. After, Y.-K. Chang and Y. Wei [6], develops the concept of the Bloch τ-periodic function in pseudo-S-asymptotically Bloch τ-periodicity, for more results on this type of function we can see [7, 14, 16, 21]. The second principle is the concept of Stepanov, in [22], Zhi Nan Xia introduces the notion of weighted pseudo-S-asymptotically periodicity in Stepanov sense. to better understand Stepanov-like functions, look at [8, 9, 11 – 13, 20].

In this paper, we combine the two principles to define the functions (μ_1, μ_2)-Stepanov-like pseudoasymptotically Bloch τ-periodic with the exponent p, defines, for given $\tau, \rho \in \mathbb{R}$, by

$$\lim_{r\to+\infty} \frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \|f(s+\tau) - e^{i\rho\tau} f(s)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t) = 0, \quad t \in \mathbb{R}.$$

We denote the set of all such functions by $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$, where $\mu_1,\ \mu_2$ two positive measures and Ω_r will define later. To see works that use the notion of measures, it is important to see [2, 3, 17 – 19].

We consider the nonlinear differential equation

$$u'(t) = Tu(t) + g(t, u(t)), \tag{1.1}$$

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where $g \in BS^p(\mathbb{R} \times X, X)$ and T is a linear (unbounded) operator generates an C_0-semigroup $\{S(s)\}_{s\geq 0} \subset BS^p(\mathbb{R}, X)$ on a Banach space X.

This paper is organized as follows. Section 2 is preliminaries composing some basic definitions, remarks and notations. Section 3 we treat some results on (μ_1, μ_2)-Stepanov-like pseudoasymptotically Bloch τ-periodic functions. Section 4 is concerned with the existence and uniqueness of (μ_1, μ_2)-Stepanov-like pseudoasymptotically Bloch τ-periodic solutions to (1.1). In Section 5, we give an application which explains the work. In the last section we added a conclusion.

2. Preliminaries. We consider the following notations:

$(X, \|.\|)$ is a Banach space,

$f_a(.) := f(. + a)$ with $a \in \mathbb{R}$,

$\Omega_r := [-r, r]$, $r > 0$,

$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$.

Definition 2.1 [19]. *For given $\tau, \rho \in \mathbb{R}$, a function $f \in \mathcal{C}_b(\mathbb{R}, X)$ is said to be Bloch τ-periodic if, for all $s \in \mathbb{R}$, $f(s + \tau) = e^{i\rho\tau} f(s)$.*

We denote by $BP_{\tau,\rho}(\mathbb{R}, X)$ the space of all Bloch τ-periodic functions from $\mathbb{R}$ to X.

***Remark* 2.1.** From Definition 2.1, we can see that f is τ-periodic if $\rho\tau = 0$, and f is τ-antiperiodic if $\rho\tau = \pi$.

In the continuation of this paper, we denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a_1, a_2]) < +\infty$ for all $a_1, a_2 \in \mathbb{R}$ $(a_1 \leq a_2)$.

Definition 2.2. *Let $\mu_1, \mu_2 \in \mathcal{M}$. The measures μ_1 and μ_2 are said to be equivalent $(\mu_1 \sim \mu_2)$ if there exist a constants $c_1, c_2 > 0$ and a bounded interval $\mathrm{J} \subset \mathbb{R}$ (eventually, $\mathrm{J} = \varnothing$) such that*

$$c_1\mu_2(A) \leq \mu_1(A) \leq c_2\mu_2(A)$$

for all $A \in \mathcal{B}$ satisfying $A \bigcap J = \varnothing$.

Let $\mu_1, \mu_2 \in \mathcal{M}$ and $r > 0$. Suppose that $\Theta_r = \dfrac{\mu_1(\Omega_r)}{\mu_2(\Omega_r)}$.

In this paper, we need the following hypotheses:

$(\mathbf{M1})$ Let $\mu_1, \mu_2 \in \mathcal{M}$ such that $\limsup_{r\to+\infty} \Theta_r < +\infty$.

$(\mathbf{M2})$ For $\mu \in \mathcal{M}$, $\omega \in \mathbb{R}$, there exist $\gamma > 0$ and a bounded interval J such that $\mu(\{a + \omega;\ a \in \mathcal{A}\}) \leq \gamma\mu(\mathcal{A})$, when $\mathcal{A} \in \mathcal{B}$ satisfy $\mathcal{A} \bigcap \mathrm{J} = \varnothing$.

Lemma 2.1 [5]. *Hypothesis $(\mathbf{M2})$ implies, for all $\omega > 0$,*

$$\limsup_{r\to+\infty} \frac{\mu([-r - \omega, r + \omega])}{\mu([-r, r])} < +\infty.$$

3. (μ_1, μ_2)-Stepanov-like pseudoasymptotically Bloch τ-periodic functions.

Definition 3.1 [10, 20]. *The Bochner transformation $f^b(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, of a function $f : \mathbb{R} \to X$ is defined by*

$$f^b(t, s) := f(t + s).$$

Definition 3.2 [10, 20]. *Let $p > 1$. The space $BS^p(\mathbb{R}, X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to X$ such that $f^b \in L^\infty(\mathbb{R}, L^p([0, 1], X))$. This is a Banach space with the norm*

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int\limits_t^{t+1} \|f(\omega)\|^p d\omega \right)^{\frac{1}{p}}.$$

***Remark* 3.1.** Let $p > 1$. It is obvious to have the following results:

$$L^p(\mathbb{R}, X) \subset BS^p(\mathbb{R}, X),$$

$$BS^p(\mathbb{R}, X) \subset BS^q(\mathbb{R}, X) \quad \text{for} \quad p \geq q > 1.$$

***Example* 3.1.** There are some functions in $BS^p(\mathbb{R}, X)$, but are not bounded. For example, let $p > 1$ and Λ is the function defined by

$$\Lambda(t) = \begin{cases} k, & \text{if } k \leq t \leq k + \dfrac{1}{k^p}, \quad p \in \mathbb{N}^*, \\ 0, & \text{otherwise.} \end{cases}$$

Then Λ is not bounded; however $\Lambda \in BS^p(\mathbb{R}, X)$, in fact

$$\int\limits_t^{t+1} |\Lambda(s)|^p ds \leq \int\limits_{[t]}^{[t]+2} |\Lambda(s)|^p ds = \sum_{k=[t]}^{[t]+1} \int\limits_k^{k+\frac{1}{k^p}} |\Lambda(s)|^p ds = 2.$$

Definition 3.3. *Let $\mu_1, \mu_2 \in \mathcal{M}$. The function $f \in BS^p(\mathbb{R}, X)$ is said to be (μ_1, μ_2)-Stepanov-like pseudoasymptotically Bloch τ-periodic with the exponent p (or (μ_1, μ_2)-S^p-pseudoasymptotically Bloch τ-periodic), if, for given $\tau, \rho \in \mathbb{R}$,*

$$\lim_{r\to+\infty} \frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \left\|f(s + \tau) - e^{i\rho\tau} f(s)\right\|^p ds \right)^{\frac{1}{p}} d\mu_1(t) = 0, \quad t \in \mathbb{R}.$$

We denote the set of all such functions by $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$.

If $\rho\tau = 0$, we have

$$\lim_{r\to+\infty} \frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \|f(s + \tau) - f(s)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t) = 0, \quad t \in \mathbb{R}.$$

Then f is called (μ_1, μ_2)-Stepanov-like pseudoasymptotically τ-periodic with the exponent p (or (μ_1, μ_2)-S^p-pseudoasymptotically τ-periodic) denoted by $PSAP_\tau(\mathbb{R}, X, \mu_1, \mu_2, p)$.

If $\rho\tau = \pi$, we obtain

$$\lim_{r\to+\infty} \frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \|f(s + \tau) + f(s)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t) = 0, \quad t \in \mathbb{R}.$$

Then f is called (μ_1, μ_2)-Stepanov-like pseudoasymptotically τ-antiperiodic with the exponent p (or (μ_1, μ_2)-S^p-pseudoasymptotically τ-antiperiodic) denoted by $PSAAP_\tau(\mathbb{R}, X, \mu_1, \mu_2, p)$.

If $\mu_1 = \mu_2 = \mu$, we get

$$\lim_{r\to+\infty} \frac{1}{\mu(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \|f(s+\tau) - e^{i\rho\tau} f(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) = 0, \quad t \in \mathbb{R}.$$

Then f is called μ-Stepanov-like pseudoasymptotically Bloch τ-periodic with the exponent p (or μ-S^p-pseudoasymptotically Bloch τ-periodic) denoted by $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu, p)$.

Lemma 3.1. *Let* $\mu_1, \mu_2 \in \mathcal{M}$ *satisfy* $(\mathbf{M1})$ *and* $(\mathbf{M2})$, $f, g, h \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$. *Then we have the following results*:

(i) $g + h \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$, $cf \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$ *for each* $c \in \mathbb{R}$;

(ii) *the function* $f_a \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$ *for any* $a \in \mathbb{R}$;

(iii) *the space* $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$ *is a Banach space with the norm* $\|.\|_{S^p}$.

Proof. (i) Hence,

$$\frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \|(g+h)(s+\tau) - e^{i\rho\tau}(g+h)(s)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$\leq \frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \|g(s+\tau) - e^{i\rho\tau} g(s)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$+ \frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \|h(s+\tau) - e^{i\rho\tau} h(s)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

and

$$\frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \|(cf)(s+\tau) - e^{i\rho\tau}(cf)(s)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$\leq |c| \frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \int\limits_t^{t+1} \left\|f(s+\tau) - e^{i\rho\tau} f(s)\right\|^p ds)^{\frac{1}{p}} d\mu_1(t).$$

Then

$$g + h, cf \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p).$$

(ii) For each $a \in \mathbb{R}$, hence,

$$\frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \|f(s+a+\tau) - e^{i\rho\tau} f(s+a)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$\leq \frac{\gamma}{\mu_2(\Omega_r)} \int\limits_{-r+a}^{r+a} \left(\int\limits_t^{t+1} \left\| f(s+\tau) - e^{i\rho\tau} f(s) \right\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$\leq \frac{\gamma}{\mu_2(\Omega_r)} \int\limits_{-r-|a|}^{r+|a|} \left(\int\limits_t^{t+1} \left\| f(s+\tau) - e^{i\rho\tau} f(s) \right\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$= \gamma \frac{\mu_2([-r-|a|, r+|a|])}{\mu_2([-r,r])}$$

$$\times \left(\frac{1}{\mu_2([-r-|a|, r+|a|])} \int\limits_{-r-|a|}^{r+|a|} \left(\int\limits_t^{t+1} \left\| f(s+\tau) - e^{i\rho\tau} f(s) \right\|^p ds \right)^{\frac{1}{p}} d\mu_1(t) \right).$$

For sufficiently large r, by using Lemma 2.1, we obtain that $f_a \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$.

(iii) Let $\{f_n\}_n \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$ converge to f as $r \to \infty$. Then, for any $\epsilon > 0$, we can choose suitable constants $N > 0$ and r_ϵ such that, for $n > N$ and $r > r_\epsilon$, we have

$$\frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \| f_n(s+\tau) - e^{i\rho\tau} f_n(s) \|^p ds \right)^{\frac{1}{p}} d\mu_1(t) \leq \frac{\epsilon}{3}$$

and

$$\| f_n - f \|_{S^p} \leq \frac{\epsilon}{3\Theta_r}.$$

Then we get

$$\frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \left\| f(s+\tau) - e^{i\rho\tau} f(s) \right\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$= \frac{1}{\mu_2(\Omega_r)} \left(\int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \| f(s+\tau) - f_n(s+\tau) + f_n(s+\tau) \right. \right.$$

$$\left. \left. - e^{ip\tau} f_n(s) + e^{i\rho\tau} f_n(s) - e^{i\rho\tau} f(s) \|^p ds \right)^{\frac{1}{p}} d\mu_1(t) \right)$$

$$\leq \frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \| f(s+\tau) - f_n(s+\tau) \|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$+ \frac{1}{\mu_2(\Omega_r)} \int\limits_{\Omega_r} \left(\int\limits_t^{t+1} \| e^{i\rho\tau} f_n(s) - e^{i\rho\tau} f(s) \|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$+\frac{1}{\mu_2(\Omega_r)}\int\limits_{\Omega_r}\left(\int\limits_t^{t+1}\|f_n(s+\tau)-e^{i\rho\tau}f_n(s)\|^p ds\right)^{\frac{1}{p}}d\mu_1(t)$$

$$\leq 2\Theta_r\|f_n-f\|_{S^p}+\left(\frac{1}{\mu_2(\Omega_r)}\int\limits_{\Omega_r}\left(\int\limits_t^{t+1}\|f_n(s+\tau)-e^{i\rho\tau}f_n(s)\|^p ds\right)^{\frac{1}{p}}d\mu_1(t)\right)$$

$$\leq 2\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.$$

Which gives that the space $PSABP_{\tau,\rho}(\mathbb{R},X,\mu_1,\mu_2,p)$ is a closed subspace of $BS^p(\mathbb{R},X)$, it is, therefore, a Banach space equipped with norm $\|.\|_{S^p}$.

Theorem 3.1. *Let $\mu_1,\mu_2\in\mathcal{M}$ satisfies $(\mathbf{M1})$ and* J *be a bounded interval* (*eventually*, $\mathrm{J}=\varnothing$) *and $f\in BS^p(\mathbb{R},X)$. Then the following assertions are equivalent*:

(i) $f\in PSABP_{\tau,\rho}(\mathbb{R},X,\mu_1,\mu_2,p)$;

(ii) $\lim_{r\to\infty}\dfrac{1}{\mu_2(\Omega_r\backslash\mathrm{J})}\displaystyle\int_{\Omega_r\backslash\mathrm{J}}\left(\int_t^{t+1}\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|^p ds\right)^{\frac{1}{p}}d\mu_1(t)=0$;

(iii) *for any* $\epsilon>0$,

$$\lim_{r\to\infty}\frac{\mu_1\left(s\in\Omega_r\backslash\mathrm{J},\left(\int_t^{t+1}\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|^p ds\right)^{\frac{1}{p}}>\epsilon\right)}{\mu_2(\Omega_r\backslash\mathrm{J})}=0.$$

Proof. (i) $\Leftrightarrow$ (ii) Denote by

$$A_1=\mu_2(\mathrm{J}),\qquad A_2=\int\limits_{\mathrm{J}}\left(\int\limits_t^{t+1}\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|^p ds\right)^{\frac{1}{p}}d\mu_1(t),\qquad A_3=\mu_1(\mathrm{J}).$$

Since the interval J is bounded and $f\in BS^p(\mathbb{R},X)$, then A_1, A_2 and A_3 are finite. For $r>0$ such that $\mathrm{J}\subset\Omega_r$ and $\mu_2(\Omega_r\backslash\mathrm{J})>0$, we have

$$\frac{1}{\mu_2(\Omega_r\backslash\mathrm{J})}\int\limits_{\Omega_r\backslash\mathrm{J}}\left(\int\limits_t^{t+1}\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|^p ds\right)^{\frac{1}{p}}d\mu_1(t)$$

$$=\frac{1}{\mu_2(\Omega_r)-A_1}\left(\int\limits_{\Omega_r}\left(\int\limits_t^{t+1}\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|^p ds\right)^{\frac{1}{p}}d\mu_1(t)-A_2\right)$$

$$=\frac{\mu_2(\Omega_r)}{\mu_2(\Omega_r)-A_1}\left(\frac{1}{\mu_2(\Omega_r)}\int\limits_{\Omega_r}\left(\int\limits_t^{t+1}\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|^p ds\right)^{\frac{1}{p}}d\mu_1(t)-\frac{A_2}{\mu_2(\Omega_r)}\right).$$

Since $\mu_2(\mathbb{R})=\infty$, we deduce that (ii) $\Leftrightarrow$ (i).

(iii) $\Rightarrow$ (ii) Denote by Φ_r^ϵ and Ψ_r^ϵ the following sets:

$$\Phi_r^\epsilon = \left\{ s \in \Omega_r \backslash \mathrm{J}, \left(\int_t^{t+1} \|f(s+\tau) - e^{i\rho\tau} f(s)\|^p ds \right)^{\frac{1}{p}} > \epsilon \right\},$$

$$\Psi_r^\epsilon = \left\{ s \in \Omega_r \backslash \mathrm{J}, \left(\int_t^{t+1} \big\|f(s+\tau) - e^{i\rho\tau} f(s)\big\|^p ds \right)^{\frac{1}{p}} \leq \epsilon \right\}.$$

Assume that (iii) holds, that is,

$$\lim_{r\to\infty} \frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r \backslash \mathrm{J})} = 0.$$

Since

$$\int_{\Omega_r \backslash \mathrm{J}} \left(\int_t^{t+1} \big\|f(s+\tau) - e^{i\rho\tau} f(s)\big\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$= \int_{\Phi_r^\epsilon} \left(\int_t^{t+1} \big\|f(s+\tau) - e^{i\rho\tau} f(s)\big\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$+ \int_{\Psi_r^\epsilon} \left(\int_t^{t+1} \big\|f(s+\tau) - e^{i\rho\tau} f(s)\big\|^p ds \right)^{\frac{1}{p}} d\mu_1(t),$$

then

$$\frac{1}{\mu_2(\Omega_r \backslash \mathrm{J})} \int_{\Omega_r \backslash \mathrm{J}} \left(\int_t^{t+1} \big\|f(s+\tau) - e^{i\rho\tau} f(s)\big\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$\leq 2\|f\|_{S^p} \frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r \backslash \mathrm{J})} + \frac{\mu_1(\Psi_r^\epsilon)}{\mu_2(\Omega_r \backslash \mathrm{J})}\epsilon$$

$$\leq 2\|f\|_{S^p} \frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r \backslash \mathrm{J})} + \frac{\mu_1(\Omega_r \backslash \mathrm{J})}{\mu_2(\Omega_r \backslash \mathrm{J})}\epsilon$$

$$= 2\|f\|_{S^p} \frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r \backslash \mathrm{J})} + \frac{\mu_1(\Omega_r) - A_3}{\mu_2(\Omega_r) - A_1}\epsilon$$

$$= 2\|f\|_{S^p} \frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r \backslash \mathrm{J})} + \frac{\mu_1(\Omega_r)\left(1 - \dfrac{A_3}{\mu_1(\Omega_r)}\right)}{\mu_2(\Omega_r)\left(1 - \dfrac{A_1}{\mu_2(\Omega_r)}\right)}\epsilon$$

$$\leq 2\|f\|_{S^p}\frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r\backslash \mathrm{J})}+\Theta_r\frac{1-\dfrac{A_3}{\mu_1(\Omega_r)}}{1-\dfrac{A_1}{\mu_2(\Omega_r)}}\epsilon.$$

For sufficiently large r, since $\mu_1(\mathbb{R})=\mu_2(\mathbb{R})=\infty$, we have, for all $\epsilon>0$,

$$\frac{1}{\mu_2(\Omega_r\backslash \mathrm{J})}\int\limits_{\Omega_r\backslash \mathrm{J}}\left(\int\limits_t^{t+1}\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|^p ds\right)^{\frac{1}{p}} d\mu_1(t)\leq\epsilon.$$

Then

$$\lim_{r\to\infty}\frac{1}{\mu_2(\Omega_r\backslash \mathrm{J})}\int\limits_{\Omega_r\backslash \mathrm{J}}\left(\int\limits_t^{t+1}\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|^p ds\right)^{\frac{1}{p}} d\mu_1(t)=0.$$

Consequently, (ii) holds.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds. Then

$$\frac{1}{\mu_2(\Omega_r\backslash \mathrm{J})}\int\limits_{\Omega_r\backslash \mathrm{J}}\left(\int\limits_t^{t+1}\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|^p ds\right)^{\frac{1}{p}} d\mu_1(t)$$

$$\geq\frac{1}{\mu_2(\Omega_r\backslash \mathrm{J})}\int\limits_{\Phi_r^\epsilon}\left(\int\limits_t^{t+1}\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|^p ds\right)^{\frac{1}{p}} d\mu_1(t)$$

$$\geq\epsilon\frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r\backslash \mathrm{J})}.$$

For sufficiently large r, we obtain (iii).

Theorem 3.1 is proved.

***Corollary* 3.1.** *A continuous function $f\in BS^p(\mathbb{R},X)$ satisfying*

$$\lim_{s\to\infty}f(s)=0.$$

Then $f\in PSABP_{\tau,\rho}(\mathbb{R},X,\mu_1,\mu_2,p)$ for all $\mu_1,\mu_2\in\mathcal{M}$ satisfies $(\mathbf{M1})$.

Proof. We have $\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|_{S^p}\leq 2\|f\|_{S^p}$.

By hypothesis $(\mathbf{M1})$,

$$\lim_{t\to\infty}\left(\int\limits_t^{t+1}\left\|f(s+\tau)-e^{i\rho\tau}f(s)\right\|^p ds\right)^{\frac{1}{p}}=0$$

for all $\epsilon>0$, there exists $\upsilon>0$ such that

$$s \geq \upsilon \Rightarrow \left(\int\limits_t^{t+1} \left\| f(s+\tau) - e^{i\rho\tau} f(s) \right\|^p ds \right)^{\frac{1}{p}} \leq \epsilon.$$

Then, for all $r > \upsilon$,

$$\left\{ s \in \Omega_r \backslash (-\upsilon, \upsilon); \left(\int\limits_t^{t+1} \left\| f(s+\tau) - e^{i\rho\tau} f(s) \right\|^p ds \right)^{\frac{1}{p}} > \epsilon \right\} = \varnothing.$$

We conclude by using Theorem 3.1.

Proposition 3.1. *Assume that* $(\mathbf{M1})$ *holds. If* $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$ *and* $\mu_1 \sim \nu_1,\ \mu_2 \sim \nu_2$, *then* $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p) = PSABP_{\tau,\rho}(\mathbb{R}, X, \nu_1, \nu_2, p)$.

Proof. Since $\mu_1 \sim \nu_1,\ \mu_2 \sim \nu_2$, then, for all $\mathcal{A} \in \mathcal{B}$ satisfying $\mathcal{A} \bigcap \Omega_r = \varnothing$, by Definition 2.2, there exists $\alpha_1, \alpha_2 > 0,\ \beta_1, \beta_2 > 0$ such that $\alpha_1 \nu_1(\mathcal{A}) \leq \mu_1(\mathcal{A}) \leq \beta_1 \nu_1(\mathcal{A})$, and $\alpha_2 \nu_2(\mathcal{A}) \leq \mu_2(\mathcal{A}) \leq \beta_2 \nu_2(\mathcal{A})$.

For sufficiently large r, we have

$$\frac{\alpha_1}{\beta_2} \frac{\nu_1\left(\left\{ s \in \Omega_r \backslash \mathrm{J}; \left(\int_t^{t+1} \left\| f(s+\tau) - e^{i\rho\tau} f(s) \right\|^p ds \right)^{\frac{1}{p}} > \epsilon \right\}\right)}{\nu_2(\Omega_r \backslash \mathrm{J})}$$

$$\leq \frac{\mu_1\left(\left\{ s \in \Omega_r \backslash \mathrm{J}; \left(\int_t^{t+1} \left\| f(s+\tau) - e^{i\rho\tau} f(s) \right\|^p ds \right)^{\frac{1}{p}} > \epsilon \right\}\right)}{\mu_2(\Omega_r \backslash \mathrm{J})}$$

$$\leq \frac{\beta_1}{\alpha_2} \frac{\nu_1\left(\left\{ s \in \Omega_r \backslash \mathrm{J}; \left(\int_t^{t+1} \left\| f(s+\tau) - e^{i\rho\tau} f(s) \right\|^p ds \right)^{\frac{1}{p}} > \epsilon \right\}\right)}{\nu_2(\Omega_r \backslash \mathrm{J})}.$$

Hence, $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p) = PSABP_{\tau,\rho}(\mathbb{R}, X, \nu_1, \nu_2, p)$ by Theorem 3.1.

Proposition 3.1 is proved.

We use the following assumptions which will be applied in the rest of this paper:

(A1) let $g \in BS^p(\mathbb{R} \times X, X)$ for all $(s, x) \in \mathbb{R} \times X,\ g(s+\tau, x) = e^{i\rho\tau} g(s, e^{-i\rho\tau} x)$;

(A2) there exists a constant $L_g > 0$ such that

$$\left\| g(s, x_1) - g(s, x_2) \right\| \leq L_g \|x_1 - x_2\|;$$

(A3) let $g \in BS^p(\mathbb{R} \times X, X)$ with $p > 1$ such that, for all $x_1, x_2 \in X,\ s \in \mathbb{R}$, we have

$$\left\| g(s, x_1) - g(s, x_2) \right\| \leq L_g(t) \|x_1 - x_2\|,$$

where $L_g \in BS^m(\mathbb{R}, \mathbb{R}_+)$ with $m \geq \max\left\{p, \dfrac{p}{p-1}\right\}$.

Theorem 3.2. *Let* $\mu_1, \mu_2 \in \mathcal{M}$ *and* $g \in BS^p(\mathbb{R} \times X, X)$ *satisfy* (A1) *and* (A2). *Then, for each* $\varphi \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p),\ g(., \varphi(.)) \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$.

Proof. Let $\varphi \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2),\ s \in \mathbb{R}$. Then we obtain

$$\lim_{r\to+\infty} \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \left(\int_t^{t+1} \|\varphi(s+\tau) - e^{i\rho\tau}\varphi(s)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t) = 0.$$

On the other hand,

$$\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \left(\int_t^{t+1} \|g(s+\tau, \varphi(s+\tau)) - e^{i\rho\tau} g(s, \varphi(s))\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$= \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \left(\int_t^{t+1} \|e^{i\rho\tau} g(s, e^{-i\rho\tau}\varphi(s+\tau)) - e^{i\rho\tau} g(s, \varphi(s))\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$\le \frac{L_g}{\mu_2(\Omega_r)} \int_{\Omega_r} \left(\int_t^{t+1} \|e^{-i\rho\tau}\varphi(s+\tau) - \varphi(s)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)$$

$$= L_g\big(\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \left(\int_t^{t+1} \|\varphi(s+\tau) - e^{i\rho\tau}\varphi(s)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t)\big).$$

Thus,

$$\lim_{r\to+\infty} \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \left(\int_t^{t+1} \|g(s+\tau, \varphi(s+\tau)) - e^{i\rho\tau} g(s, \varphi(s))\|^p ds \right)^{\frac{1}{p}} d\mu_1(t) = 0,$$

i.e.,

$$g(., \varphi(.)) \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p).$$

Theorem 3.2 is proved.

Theorem 3.3. *Let $\mu_1, \mu_2 \in \mathcal{M}$ and $g \in BS^p(\mathbb{R} \times X, X)$ satisfy* (A1) *and* (A3). *If $\varphi \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$, then there exists $q \in [1, p)$ such that $g(., \varphi(.)) \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, q)$.*

Proof. Let $\varphi \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2),\ s \in \mathbb{R}$. Then we have

$$\lim_{r\to+\infty} \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \left(\int_t^{t+1} \|\varphi(s+\tau) - e^{i\rho\tau}\varphi(s)\|^p ds \right)^{\frac{1}{p}} d\mu_1(t) = 0.$$

Since $m \ge \dfrac{p}{p-1}$, there exists $q \in [1, p)$ such that $m = \dfrac{pq}{p-q}$. We pose $p_1 = \dfrac{p}{p-q} > 1$ and $q_1 = \dfrac{p}{q} > 1$, then $\dfrac{1}{p_1} + \dfrac{1}{q_1} = 1$.

On the other hand,

$$\frac{1}{\mu_2(\Omega_r)}\int\limits_{\Omega_r}\left(\int\limits_t^{t+1}\|g(s+\tau,\varphi(s+\tau))-e^{i\rho\tau}g(s,\varphi(s))\|^q ds\right)^{\frac{1}{q}} d\mu_1(t)$$

$$=\frac{1}{\mu_2(\Omega_r)}\int\limits_{\Omega_r}\left(\int\limits_t^{t+1}\|e^{i\rho\tau}g(s,e^{-i\rho\tau}\varphi(s+\tau))-e^{i\rho\tau}g(s,\varphi(s))\|^q ds\right)^{\frac{1}{q}} d\mu_1(t)$$

$$\le\frac{1}{\mu_2(\Omega_r)}\int\limits_{\Omega_r}\left(\int\limits_t^{t+1}(L_g(s)\|e^{-i\rho\tau}\varphi(s+\tau)-\varphi(s)\|)^q ds\right)^{\frac{1}{q}} d\mu_1(t)$$

$$\le\frac{1}{\mu_2(\Omega_r)}\int\limits_{\Omega_r}\left(\left(\int\limits_t^{t+1}(L_g(s)^q)^{p_1}ds\right)^{\frac{1}{p_1}}\left(\int\limits_t^{t+1}\|\varphi(s+\tau)-e^{i\rho\tau}\varphi(s)\|^{qq_1}ds\right)^{\frac{1}{q_1}}\right)^{\frac{1}{q}} d\mu_1(t))$$

$$\le\frac{1}{\mu_2(\Omega_r)}\int\limits_{\Omega_r}\left(\int\limits_t^{t+1}(L_g(s)^{qp_1}ds)^{\frac{1}{qp_1}}\left(\int\limits_t^{t+1}\|\varphi(s+\tau)-e^{i\rho\tau}\varphi(s)\|^{qq_1}ds\right)^{\frac{1}{qq_1}} d\mu_1(t)\right)$$

$$\le\frac{1}{\mu_2(\Omega_r)}\int\limits_{\Omega_r}\left(\int\limits_t^{t+1}(L_g(s)^{m}ds)^{\frac{1}{m}}\left(\int\limits_t^{t+1}\|\varphi(s+\tau)-e^{i\rho\tau}\varphi(s)\|^{p}ds\right)^{\frac{1}{p}} d\mu_1(t)\right)$$

$$\le\|L_g\|_{S^m}\left(\frac{1}{\mu_2(\Omega_r)}\int\limits_t^{t+1}\|\varphi(s+\tau)-e^{i\rho\tau}\varphi(s)\|^{p}ds\right)^{\frac{1}{p}} d\mu_1(t).$$

Then $\varphi\in PSABP_{\tau,\rho}(\mathbb{R},X,\mu_1,\mu_2,p)$ and $L_g\in BS^m(\mathbb{R},X)$. Thus,

$$\lim_{r\to+\infty}\frac{1}{\mu_2(\Omega_r)}\int\limits_{\Omega_r}\left(\int\limits_t^{t+1}\|g(s+\tau,\varphi(s+\tau))-e^{i\rho\tau}g(s,\varphi(s))\|^q ds\right)^{\frac{1}{q}} d\mu_1(t)=0,$$

i.e.,

$$g(.,\varphi(.))\in PSABP_{\tau,\rho}(\mathbb{R},X,\mu_1,\mu_2,q).$$

Theorem 3.3 is proved.

4. Existence of (μ_1,μ_2)-Stepanov-like pseudoasymptotically Bloch τ-periodic solutions. In this section, we will tackle the problem of existence and uniqueness of a (μ_1,μ_2)-S^p-pseudoasymptotically Bloch τ-periodic solution of the equation

$$u'(t)=Tu(t)+g(t,u(t)),$$

where $g \in BS^p(\mathbb{R} \times X, X)$ and T is a linear (unbounded) operator generates an C_0-semigroup $\{S(s)\}_{s\geq 0} \subset BS^p(\mathbb{R}, X)$ on a Banach space X.

We assume that g and S verify the following hypothesis:

(H) The C_0-semigroup $\{S(s)\}_{s\geq 0}$ is exponentially stable, i.e, there exist constants $\delta > 0$, $M > 0$ such that, for all $s \geq 0$,

$$\|S(s)\| \leq Me^{-\delta s}. \tag{4.1}$$

We can now tackle the existence and uniqueness problem of the (μ_1, μ_2)-S^p-pseudoasymptotically Bloch τ-periodic mild solution of the evolution equation

$$u'(t) = Tu(t) + g(t). \tag{4.2}$$

Equation (4.2) admits the mild solution given by

$$u(s) = \int_{-\infty}^{s} S(s-\xi)g(\xi)d\xi, \quad s \in \mathbb{R}. \tag{4.3}$$

Lemma 4.1. *If $g \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$, then the mild solution u(s) given by* (4.3) *belongs to $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$.*

To clearly show Lemma 4.1, it is enough to apply Lemma 3.4 and Theorem 3.6 from paper [1].

Definition 4.1. *We say that the function $u\colon \mathbb{R} \to X$ is a mild solution to equation* (1.1) *if the function $\xi \mapsto S(s-\xi)g(\xi, u(\xi))$ is integrable on $(-\infty, s]$ for all $s \in \mathbb{R}$ and*

$$u(s) = \int_{-\infty}^{s} S(s-\xi)g(\xi, u(\xi))d\xi, \quad s \in \mathbb{R}.$$

Theorem 4.1. *Let $\mu_1, \mu_2 \in \mathcal{M}$ satisfy* (**M1**) *and* (**M2**). *Assume that* (H) *holds. Then we have*:

(i) *If $g \in BS^p(\mathbb{R} \times X, X)$ satisfies* (A1) *and* (A2), *then equation* (1.1) *admits a unique mild solution $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$ whenever*

$$\frac{ML_g}{\delta} < 1.$$

(ii) *If $g \in BS^p(\mathbb{R} \times X, X)$ satisfies* (A1) *and* (A3), *then system* (1.1) *admits a unique mild solution $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$ whenever*

$$\|L_g\|_{S^m} < \frac{1-e^{-\delta}}{M}\left(\frac{\delta m_0}{1-e^{-\delta m_0}}\right)^{\frac{1}{m_0}},$$

where $m_0 > 1$ and $\dfrac{1}{m} + \dfrac{1}{m_0} = 1$.

Proof. We define the operator $\mathcal{F}\colon PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p) \to PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$ by

$$(\mathcal{F}u)(s) := \int_{-\infty}^{s} S(s-\xi)g(\xi, u(\xi))d\xi, \quad s \in \mathbb{R},$$

where $\{S(s)\}_{s\geq 0}$ verifies the relation (4.1).

For each $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$, using Theorems 3.2 and 3.3, the function $\xi \mapsto g(\xi, u(\xi))$ belongs to $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$. From Lemma 4.1 we have that $\mathcal{F}u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$, which give $\mathcal{F}$ is well defined.

(i) For $u_1, u_2 \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$ and $s \in \mathbb{R}$, we get

$$\begin{aligned}\|(\mathcal{F}u_1)(s) - (\mathcal{F}u_2)(s)\| &\leq \int_{-\infty}^{s} \big\|S(s-\xi)\big[g(\xi, u_1(\xi)) - g(\xi, u_2(\xi))\big]\big\| d\xi \\ &\leq \int_{-\infty}^{s} L_g \|S(s-\xi)\| \|u_1(\xi) - u_2(\xi)\| d\xi \\ &\leq L_g \|u_1 - u_2\|_\infty \int_0^\infty \|S(\xi)\| d\xi \leq \frac{L_g M}{\delta} \|u_1 - u_2\|_\infty.\end{aligned}$$

Therefore,

$$\|\mathcal{F}u_1 - \mathcal{F}u_2\|_\infty \leq \frac{ML_g}{\delta} \|u_1 - u_2\|_\infty.$$

It means that $\mathcal{F}$ is contractive for the assumption $\dfrac{ML_g}{\delta} < 1$.

So there is a unique $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$ such that $\mathcal{F}(u) = u$ via the Banach fixed point theorem.

(ii) For $u_1, u_2 \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$ and $s \in \mathbb{R}$, we obtain

$$\begin{aligned}\big\|(\mathcal{F}u_1)(s) - (\mathcal{F}u_2)(s)\big\| &\leq \int_{-\infty}^{s} \|S(s-\xi)[g(\xi, u_1(\xi)) - g(\xi, u_2(\xi))]\| d\xi \\ &\leq M \int_{-\infty}^{s} e^{-\delta(s-\xi)} L_g(\xi) \|u_1(\xi) - u_2(\xi)\| d\xi \\ &\leq M \|u_1 - u_2\|_\infty \sum_{n=1}^{\infty} \int_{s-n}^{s-n+1} e^{-\delta(s-\xi)} L_g(\xi) d\xi \\ &\leq M \|u_1 - u_2\|_\infty \sum_{n=1}^{\infty} \left(\int_{s-n}^{s-n+1} e^{-\delta m_0 (s-\xi)} d\xi \right)^{\frac{1}{m_0}} \|L_g\|_{S^m} \\ &\leq \frac{M}{1 - e^{-\delta}} \left(\frac{1 - e^{-m_0 \delta}}{m_0 \delta} \right)^{\frac{1}{m_0}} \|L_g\|_{S^m} \|u_1 - u_2\|_\infty.\end{aligned}$$

Therefore,

$$\|\mathcal{F}u_1 - \mathcal{F}u_2\|_\infty \le \frac{M}{1-e^{-\delta}}\left(\frac{1-e^{-m_0\delta}}{m_0\delta}\right)^{\frac{1}{m_0}} \|L_g\|_{S^m}\|u_1-u_2\|_\infty.$$

As well as the first result (i), the system (1.1) admits a unique mild solution $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2, p)$ for $\|L_g\|_{S^m} < \dfrac{1-e^{-\delta}}{M}\Big(\dfrac{\delta m_0}{1-e^{-\delta m_0}}\Big)^{\frac{1}{m_0}}$.

Theorem 4.1 is proved.

If $\mu_1 = \mu_2 = \mu$, we have the following result.

***Corollary* 4.1.** *Let $\mu \in \mathcal{M}$ satisfies* $(\mathbf{M2})$. *Assume that* (H) *holds. Then we have*:

(i) *If $g \in BS^p(\mathbb{R}\times X, X)$ satisfies* (A1) *and* (A2), *then equation* (1.1) *admits a unique mild solution $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu, p)$ whenever*

$$\frac{ML_g}{\delta} < 1.$$

(ii) *If $g \in BS^p(\mathbb{R}\times X, X)$ satisfies* (A1) *and* (A3), *then system* (1.1) *admits a unique mild solution $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu, p)$ whenever*

$$\|L_g\|_{S^m} < \frac{1-e^{-\delta}}{M}\left(\frac{\delta m_0}{1-e^{-\delta m_0}}\right)^{\frac{1}{m_0}},$$

where $m_0 > 1$ and $\dfrac{1}{m} + \dfrac{1}{m_0} = 1$.

5. Application. Let μ be the measure defined by the following weight: $\rho(s) = e^{\sin(s)}$ for all $s \in \mathbb{R}$. Then, for all $z > 0$, we get

$$\frac{2z}{e} \le \mu([-z,z]) = \int_{-z}^{z} e^{\sin s} ds \le 2ez.$$

Therefore, $\mu \in \mathcal{M}$ satisfies $(\mathbf{M2})$. Indeed, for all $a \in A$ and $\tau \in \mathbb{R}$, we obtain $2+\sin(a) \ge \sin(\tau+a)$, which implies that $\mu(\tau + A) \le e^2\mu(A)$.

We consider the problem

$$\begin{aligned} &\frac{\partial u}{\partial s}(s,x) = \frac{\partial^2 u}{\partial x^2}(s,x) + g(s,u(s)), \\ &u(0,s) = u(\pi,s) = 0, \end{aligned} \tag{5.1}$$

with $x \in [0,\pi],\ s \in \mathbb{R}$.

Let $X := L^2([0,\pi]),\ T := \dfrac{d^2}{dx^2}$ with domain $D(T) = \big\{h \in H^2([0,\pi]),\ h(0) = h(\pi) = 0\big\}$. Then we can write problem (5.1) in the abstract form (1.1).

It is well-known that T is the infinitesimal generator of C_0-semigroup $\{S(s)\}_{s>0}$ on X such that $\|S(s)\| \le e^{-s}$.

We take $g(s,\varphi)(t) := \eta(s)\varphi(t)$. Assume that $\eta(s)$ is a continuous τ-periodic function, i.e., $\eta(s+\tau) = \eta(s)$. Then we have

$$\begin{aligned} g(s+\tau,\varphi)(t) &= \eta(s+\tau)e^{i\rho\tau}e^{-i\rho\tau}\varphi(t) \\ &= e^{i\rho\tau}\eta(s)e^{-i\rho\tau}\varphi(t) = e^{i\rho\tau}g(s, e^{-i\rho\tau}\varphi)(t), \end{aligned}$$

and (A1) is verified.

We also get

$$\|g(s,\varphi_1) - g(s,\varphi_2)\|_{L^2([0,\pi])} \leq |\eta(s)| \|\varphi_1(t) - \varphi_2(t)\|_{L^2([0,\pi])}.$$

If η is a bounded function, we apply Corollary 4.1(i) with $L_g = \|\eta\|_\infty$, $\delta = 1$, and $M = 1$. Then equation (5.1) has a unique μ-Stepanov-like pseudoasymptotically Bloch τ-periodic mild solution on $\mathbb{R}$ with $\|\eta\|_\infty < 1$.

If $\eta \in BS^m(\mathbb{R}, X)$ is a not bounded function, we apply Corollary 4.1(ii) with $L_g = \|\eta\|_{S^m}$, $\delta = 1$, and $M = 1$. We can take as an Example 3.1 with $\eta(s) = \dfrac{(1-e^{-1})\Big(\dfrac{m_0}{1-e^{-m_0}}\Big)^{\frac{1}{m_0}} \Lambda(s)}{3}$.

Hence, $\|\eta\|_{S^m} < (1-e^{-1})\Big(\dfrac{m_0}{1-e^{-m_0}}\Big)^{\frac{1}{m_0}}$, then system (5.1) admits a unique mild solution $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu, p)$.

6. Conclusion. In nature there is a small number of purely periodic phenomena, which gives us the idea in this paper to define and consider the new space for Stepanov pseudoasymptotic Bloch periodic functions with measures. Moreover, after the study of some important properties and characterizations of this new space and with some useful assumptions, we succeeded in showing the existence and the uniqueness of a measures Stepanov pseudoasymptotically Bloch periodic solution of the following evolution problem: $u'(t) = Tu(t) + g(t, u(t))$, where $g \in BS^p(\mathbb{R} \times X, X)$ and T is a linear operator generates an C_0-semigroup $\{S(s)\}_{s \geq 0} \subset BS^p(\mathbb{R}, X)$ on a Banach space X. Finally, we finalized the work with a physical model to justify the main result of our paper.

Conflict of interest. The authors declare that they have no potential conflict of interest in relation to the study in this paper.

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