

CROSSED MODULES WITH ACTION

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Following the idea of a group with action, in a higher dimension, we consider crossed modules with a given crossed-module action upon itself. A category of these objects \mathbf{XMod}^* is given by replacing the set of morphisms in the category \mathbf{Gr}^* of groups with action by the set of a modified form of φ -crossed homomorphisms. Moreover, the connections of \mathbf{XMod}^* with some other known categories, such as \mathbf{Gr}^* and \mathbf{XMod} , are given by describing some related functors.

За допомогою ідеї групи з дією у вищій вимірності розглянуто схрещені модулі із заданою схрещеною дією схрещеного модуля на себе. Категорія цих об'єктів \mathbf{XMod}^* задається шляхом заміни набору морфізмів у категорії \mathbf{Gr}^* груп з дією на набір φ -схрещених гомоморфізмів модифікованої форми. Крім того, зв'язки \mathbf{XMod}^* з деякими іншими відомими категоріями, такими як \mathbf{Gr}^* і \mathbf{XMod} , отримано шляхом опису деяких споріднених функторів.

1. Introduction. A group with action, as its name suggests, is an algebraic object consisting of a group and an action of this group on itself. The notion is defined by Datuashvili [4] for the solution of two problems stated by Loday [7, 8], which is related to the derivation of Lie–Leibniz algebras from a group like structure by describing the category \mathbf{Gr}^* of groups with action and some of its important subcategories related to the topic [4, 5].

The concept of action is defined not only in the category \mathbf{Gr} of groups, but also in many other algebraic categories, including the categories of crossed modules in these categories [2, 3, 9].

Since crossed modules are objects that generalize groups, and they have strong relationships with groups, similar to the fact that groups with action form an interesting category, it is possible to foresee that crossed modules with action would also constitute a category having similar characteristics with the category of groups with action.

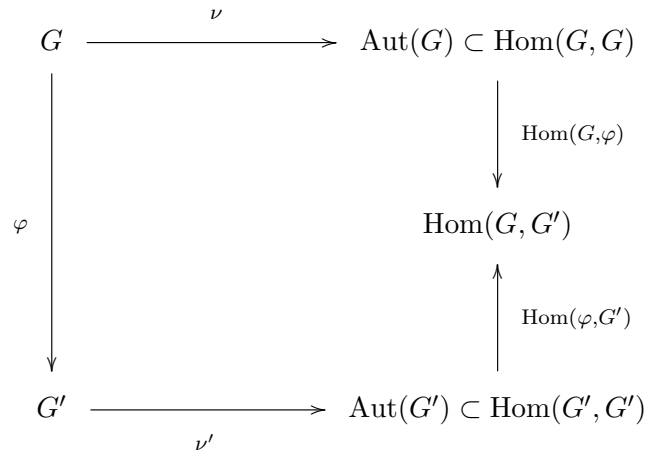
In this article, following this approach, which is inspired by the work [4], we define a category \mathbf{XMod}^* of crossed modules with action.

The point to be noted here is, we do not give the crossed modules of groups with action, but instead we define a new category of crossed modules with crossed module action, which seems independent from groups with action by definition. The action of a group with action on another one was defined and described recently in [6]. However, the category \mathbf{XMod}^* , which is introduced here, still has significant similarities and close relationships with the category \mathbf{Gr}^* . If it is possible to identify crossed modules in the category of groups with action properly, it seems certain that the new category, say $\mathbf{XMod}(\mathbf{Gr}^*)$, would have significant similarities and more or less common features with \mathbf{XMod}^* . We add that the category \mathbf{XMod}^* is interesting in a way, since it is known that groups with action do not form a modified category of interest [1], and this is also true for \mathbf{XMod}^* , as \mathbf{Gr}^* can be faithfully embedded into \mathbf{XMod}^* , via the functor \mathbf{In}^* introduced here.

Before describing the category of crossed modules with action, some required definitions are given in Section 2. The definition of a φ -derivation given at the end of this section, is slightly general than the definition of a φ -crossed homomorphism given in [9]. φ -Derivations are defined via crossed

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module morphisms, while φ -crossed homomorphisms also require the corresponding crossed module morphisms to be an endomorphism. The reason why they are needed, is that, they can explain one of the sets, corresponding the set $\text{Hom}(G, G')$ in [4], which is included in the diagram below, explaining the morphisms of groups with action.



In Section 3, crossed modules with action are defined, and the category \mathbf{XMod}^* is presented in various aspects. In this context, the functors \mathbf{Id}^* , \mathbf{In}^* , $\mathbf{\Gamma}^*$, $\mathbf{\Pi}_1^*$, $\mathbf{\Pi}_2^*$, \mathbf{XC} and \mathbf{XT} , which are the counterparts of the functors \mathbf{Id} , \mathbf{In} , $\mathbf{\Gamma}$, $\mathbf{\Pi}_1$, $\mathbf{\Pi}_2$, \mathbf{C} and \mathbf{T} , relating either the category of groups with action and the category of groups, or the category of crossed modules and the category of groups.

In the last section, a general discussion is made, and some arrows between the four categories are given on a diagram.

2. Preliminaries. Let $(G, +)$ be a group, which is not required to be Abelian, though the additive notation is used and given a left group action $\varepsilon : G \times G \rightarrow G$, of G on itself. Then we have

$$\begin{aligned}
 \varepsilon(g + h, k) &= \varepsilon(g, \varepsilon(h, k)), \\
 \varepsilon(0, g) &= g, \\
 \varepsilon(g, h + k) &= \varepsilon(g, h) + \varepsilon(g, k)
 \end{aligned}$$

for all $g, h, k \in G$. (G, ε) is called a group with action. Equivalently, it is also possible to take ε as a right action, as in [4].

Throughout this study, given a group with action (G, ε) , we use the notation ${}^g h$ for $\varepsilon(g, h)$. More details about the following series of definitions can be found in [9].

Continuing to adopt the additive notation, a crossed module is a triple (T, G, ∂) , where T and G are groups, $\partial : T \rightarrow G$ is a homomorphism, G acts on T from the left, and the following conditions are satisfied, where $g \triangleright t$ is used to denote the action of a $g \in G$ on a $t \in T$, to avoid confusion with actions of groups with action:

- (CM1) $\partial(g \triangleright t) = g + \partial(t) - g$,
- (CM2) $\partial(s \triangleright t) = s + t - s$.

A crossed module morphism $\varphi : (\varphi_1, \varphi_2) : (T, G, \partial) \rightarrow (T', G', \partial')$ is a pair of group homomorphisms $\varphi_1 : T \rightarrow T'$, $\varphi_2 : G \rightarrow G'$ such that $\partial' \varphi_1 = \varphi_2 \partial$ and $\varphi_1(g \triangleright t) = \varphi_2(g) \triangleright \varphi_1(t)$ for all

$t \in T, g \in G$. The group of all crossed module automorphisms of (T, G, ∂) , that is, all invertible crossed module morphisms $(\varphi_1, \varphi_2) : (T, G, \partial) \rightarrow (T, G, \partial)$ with the pairwise composition of pairs of group automorphisms, is denoted with $\text{Aut}(T, G, \partial)$.

A derivation on a crossed module (T, G, ∂) is a function $\chi : G \rightarrow T$, satisfying the identity

$$\chi(g + h) = \chi(g) + (g \triangleright \chi(h))$$

for all $g, h \in G$. Each crossed module derivation χ on (T, G, ∂) defines a crossed module morphism $(\theta_\chi, \sigma_\chi) : (T, G, \partial) \rightarrow (T, G, \partial)$, defined as $\theta_\chi(t) = \chi\partial(t) + t$ and $\sigma_\chi(g) = \partial\sigma(g) + g$. In particular, if $(\theta_\chi, \sigma_\chi)$ is a crossed module automorphism, then χ is called a regular derivation.

The set of all derivations on (T, G, ∂) is denoted with $\text{Der}(G, T)$, and it is a monoid with the operation \diamond defined by

$$(\chi \diamond \psi)(g) = \chi\sigma_\psi(g) + \psi(g)$$

or, equivalently,

$$(\chi \diamond \psi)(g) = \theta_\chi\psi(g) + \chi(g).$$

The set of all regular derivations is denoted with $\text{D}(G, T)$ and it is a group under the operation \diamond , called the Whitehead group. The inverse of a derivation χ in $\text{D}(G, T)$ is denoted by $\chi^\#$ and given with

$$\chi^\#(g) = -\chi\sigma_\chi^{-1}(g) = -\theta_\chi^{-1}\chi(g).$$

Actor of a crossed module (T, G, ∂) is defined as the crossed module

$$\mathcal{A}(T, G, \partial) := (\text{D}(G, T), \text{Aut}(T, G, \partial), \Delta),$$

where $\Delta : \text{D}(G, T) \rightarrow \text{Aut}(T, G, \partial)$ is given by $\Delta(\chi) = (\theta_\chi, \sigma_\chi)$, and $\text{Aut}(T, G, \partial)$ acts on $\text{D}(G, T)$ by $(\varphi_1, \varphi_2) \triangleright \chi = \varphi_1\chi\varphi_2$.

An action of a crossed module (T, G, ∂) on a crossed module (T', G', ∂') is a crossed module morphism

$$\alpha : (T, G, \partial) \rightarrow \mathcal{A}(T', G', \partial').$$

To a crossed module action $\alpha = (\alpha_1, \alpha_2)$, we associate the mappings

$$\psi_t : G' \rightarrow T',$$

$$\theta_g : T' \rightarrow T',$$

$$\sigma_g : G' \rightarrow G'$$

for each $t \in T$ and $g \in G$ such that

$$\alpha_1(t) = \psi_t,$$

$$\alpha_2(g) = (\theta_g, \sigma_g).$$

Moreover, for the crossed module automorphism $(\theta_{\psi_t}, \sigma_{\psi_t})$ corresponding to the derivation ψ_t , we have $\theta_{\psi_t} = \theta_{\partial(t)}$ and $\sigma_{\psi_t} = \sigma_{\partial(t)}$.

Given two crossed modules (T, G, ∂) and (T', G', ∂') , we denote the set of all crossed module morphisms $\varphi : (T, G, \partial) \rightarrow (T', G', \partial')$ by $\text{Hom}((T, G, \partial), (T', G', \partial'))$. We use the notation

$$\text{End}(T, G, \partial) := \text{Hom}((T, G, \partial), (T, G, \partial)).$$

For any crossed module morphism $\varphi = (\varphi_1, \varphi_2) \in \text{Hom}((T, G, \partial), (T', G', \partial'))$, the set of all functions $\xi: G \rightarrow T'$ such that

$$\xi(g + h) = \xi(g) + (\varphi_2(g) \triangleright \xi(h))$$

is denoted by $\text{XHom}_\varphi(G, T')$, and the elements of this set are called φ -derivations.

3. Crossed modules with an action on itself. Below, we present a crossed module analog of the concept of a group with action defined in [4].

Definition 3.1. A crossed module with action is a quadruple (T, G, ∂, α) , where (T, G, ∂) is a crossed module, and $\alpha: (T, G, \partial) \rightarrow \mathcal{A}(T, G, \partial)$ is a crossed module action.

For clarity, given a crossed module with action (T, G, ∂, α) , we denote the action of G on T , originating from the crossed module structure, by $\triangleright: G \times T \rightarrow T$. Thus, we write $g \triangleright t$ to denote the result of the action of $g \in G$ on $t \in T$. On the other hand, for the group actions inherited by the crossed module action α , we use the notations ${}^{g_1}g_2 := \sigma_{g_1}(g_2)$, ${}^g t := \theta_g(t)$, and ${}^{t_1}t_2 := \theta_{\partial(t_1)}(t_2)$.

If (T, G, ∂, α) is a crossed module with action, then both the groups (T, θ) and (G, σ) are groups with action, where $\theta: T \times T \rightarrow T$, $\theta(t, _) = \theta_{\partial(t)}$, and $\sigma: G \times G \rightarrow G$, $\sigma(g, _) = \sigma_g$. In this case, note that the boundary homomorphism ∂ becomes a morphism also in the category of groups with actions, since $(\theta_{\partial(t_1)}, \sigma_{\partial(t_1)})$ is an automorphism of (T, G, ∂) , so that the diagram

$$\begin{array}{ccc} T & \xrightarrow{\partial} & G \\ \theta_{\partial(t_1)} \downarrow & & \downarrow \sigma_{\partial(t_2)} \\ T & \xrightarrow{\partial} & G \end{array}$$

commutes and

$$\partial(\theta(t_1, t_2)) = \partial(\theta_{\partial(t_1)}(t_2)) = \sigma_{\partial(t_1)}(\partial(t_2)) = \sigma(\partial(t_1), \partial(t_2)) = \sigma(\partial, \partial)(t_1, t_2),$$

which makes the diagram

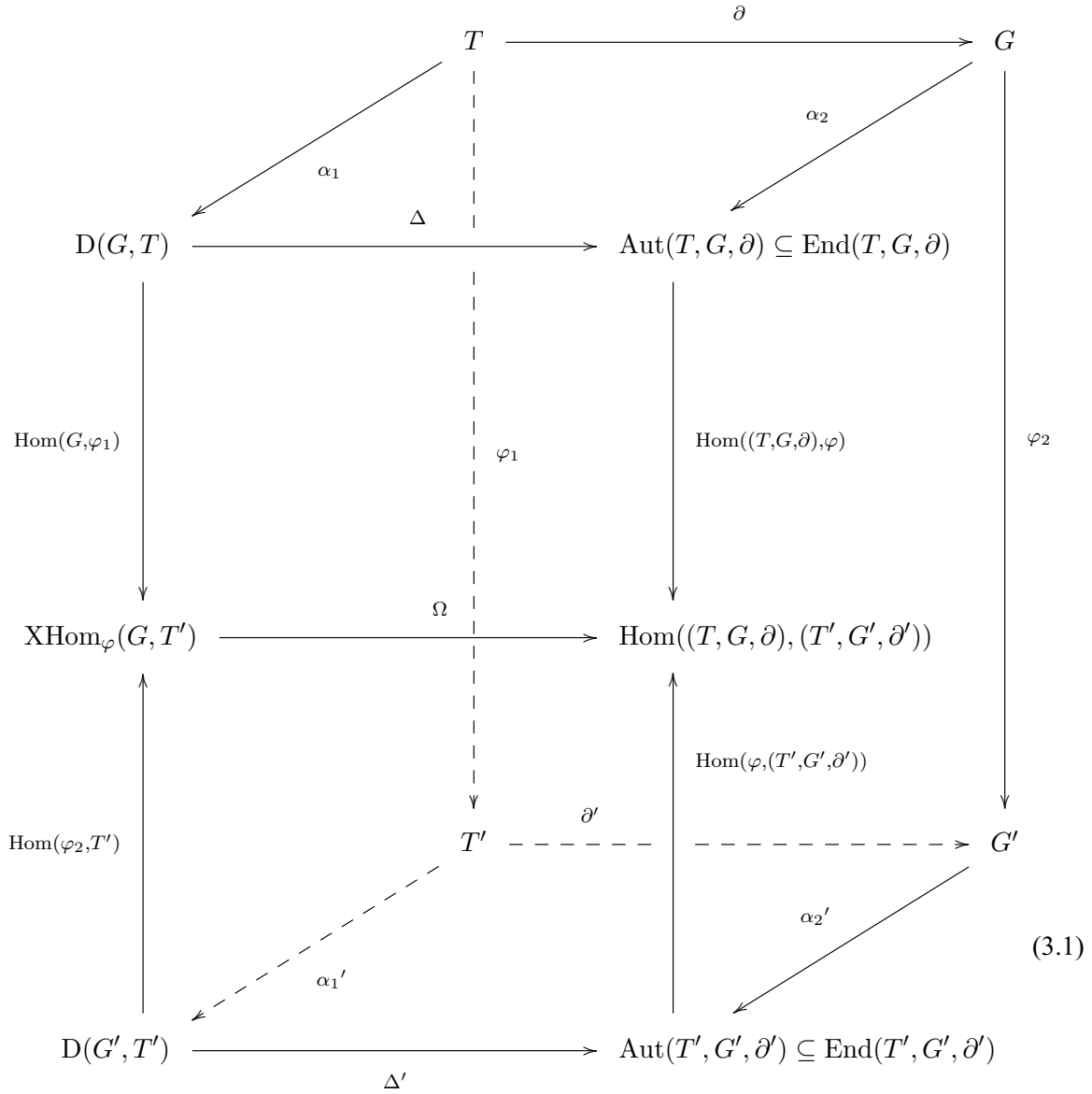
$$\begin{array}{ccc} T \times T & \xrightarrow{\theta} & T \\ (\partial, \partial) \downarrow & & \downarrow \partial \\ G \times G & \xrightarrow{\sigma} & G \end{array}$$

commute.

A morphism of crossed modules with action

$$\varphi : (T, G, \partial, \alpha) \longrightarrow (T', G', \partial', \alpha')$$

is a crossed module morphism such that the diagram



commutes. Here, $\varphi = (\varphi_1, \varphi_2)$ and

$$\begin{aligned} \text{Hom}((T, G, \partial), \varphi) &= (\text{Hom}(T, \varphi_1), \text{Hom}(G, \varphi_2)), \\ \text{Hom}(\varphi, (T', G', \partial')) &= (\text{Hom}(\varphi_1, T'), \text{Hom}(\varphi_2, G')), \end{aligned}$$

and also

$$\Omega : \text{XHom}_\varphi(G, T') \longrightarrow \text{Hom}((T, G, \partial), (T', G', \partial'))$$

is defined by

$$\psi \longmapsto \Omega(\psi) = (\beta_\psi, \gamma_\psi),$$

where $\beta_\psi : T \longrightarrow T'$ and $\gamma_\psi : G \longrightarrow G'$ are defined by

$$t \longmapsto \beta_\psi(t) = \psi\partial(t) + \varphi_1(t),$$

$$g \longmapsto \gamma_\psi(g) = \partial'\psi(g) + \varphi_2(g),$$

respectively. Note that $\Omega(\psi) : (T, G, \partial) \longrightarrow (T', G', \partial')$ is indeed a crossed module morphism since

$$\partial'\beta_\psi(t) = \partial'(\psi\partial(t) + \varphi_1(t)) = \partial'\psi\partial(t) + \partial'\varphi_1(t) = \partial'\psi\partial(t) + \varphi_2\partial(t) = \gamma_\psi\partial(t)$$

makes the diagram

$$\begin{array}{ccc} T & \xrightarrow{\partial} & G \\ \beta_\psi \downarrow & & \downarrow \gamma_\psi \\ T' & \xrightarrow{\partial'} & G' \end{array}$$

commute, and

$$\begin{aligned} \beta_\psi(g \triangleright t) &= \psi\partial(g \triangleright t) + \varphi_1(g \triangleright t) = \psi\partial(g \triangleright t) + (\varphi_2(g) \triangleright \varphi_1(t)) \\ &= \psi(g + \partial(t) - g) + (\varphi_2(g) \triangleright \varphi_1(t)) \\ &= \psi(g) + (\varphi_2(g) \triangleright \psi(\partial t - g)) + (\varphi_2(g) \triangleright \varphi_1(t)) \\ &= \psi(g) + (\varphi_2(g) \triangleright (\psi\partial(t) + (\varphi_2\partial(t) \triangleright \psi(-g)))) + (\varphi_2(g) \triangleright \varphi_1(t)) \\ &= \psi(g) + (\varphi_2(g) \triangleright (\psi\partial(t) + (\partial'\varphi_1(t) \triangleright \psi(-g)) + \varphi_1(t))) \\ &= \psi(g) + (\varphi_2(g) \triangleright (\psi\partial(t) + \varphi_1(t) + \psi(-g))) \\ &= \psi(g) + (\varphi_2(g) \triangleright (\psi\partial(t) + \varphi_1(t))) - \psi(g) \\ &= (\partial'\psi(g) + \varphi_2(g)) \triangleright (\psi\partial(t) + \varphi_1(t)) = \gamma_\psi(g) \triangleright \beta_\psi(t). \end{aligned}$$

An isomorphism of crossed modules with action is defined as an invertible morphism. An endomorphism of $(T \xrightarrow{\partial} G, \alpha)$ is a morphism $(T \xrightarrow{\partial} G, \alpha) \longrightarrow (T \xrightarrow{\partial} G, \alpha)$, and an automorphism of a crossed module with action is an endomorphism that is also an isomorphism.

Note that in the diagram of the morphisms of crossed module with action, the triple

$$(\text{XHom}_\varphi(G, T'), \text{Hom}((T, G, \partial), (T', G', \partial')), \Omega),$$

in particular, does not determine a crossed module. In fact, these sets do not even have to merit a group structure in general, like the role of the set $\text{Hom}(G, G')$ in the diagram for the morphisms of groups with action. So the arrows coming in and out here are ordinary functions that make the diagram commuting. Since this diagram is complex enough, we can break it down as suggested in the following proposition for an easier understanding of it.

Proposition 3.1. *The commutativity of the diagram (3.1), is equivalent to the commutativity of the diagrams*

$$\begin{array}{ccc}
 G \times G & \xrightarrow{\sigma} & G \\
 \downarrow (\varphi_2, \varphi_2) & & \downarrow \varphi_2 \\
 G' \times G' & \xrightarrow{\sigma'} & G'
 \end{array}
 \qquad
 \begin{array}{ccc}
 G \times T & \xrightarrow{\tilde{\theta}} & T \\
 \downarrow (\varphi_2, \varphi_1) & & \downarrow \varphi_1 \\
 G' \times T' & \xrightarrow{\tilde{\theta}'} & T'
 \end{array}$$

$$\begin{array}{ccc}
 T \times G & \xrightarrow{\tilde{\alpha}_1} & T \\
 \downarrow (\varphi_1, \varphi_2) & & \downarrow \varphi_1 \\
 T' \times G' & \xrightarrow{\tilde{\alpha}'_1} & T'
 \end{array}$$

where $\tilde{\theta}(g, t) = \theta_g(t)$, $\tilde{\theta}'(g', t') = \theta'_{g'}(t')$, $\tilde{\alpha}_1(t, g) = \alpha_1(t)(g) = \psi_t(g)$ and $\tilde{\alpha}'_1(t', g') = \alpha'_1(t')(g') = \psi'_{t'}(g')$.

Proof. The squares

$$\begin{array}{ccc}
 T & \xrightarrow{\partial} & G \\
 \downarrow \alpha_1 & & \downarrow \alpha_2 \\
 D(G, T) & \xrightarrow{\Delta} & \text{Aut}(T, G, \partial)
 \end{array}
 \qquad
 \begin{array}{ccc}
 T' & \xrightarrow{\partial'} & G' \\
 \downarrow \alpha'_1 & & \downarrow \alpha'_2 \\
 D(G', T') & \xrightarrow{\Delta'} & \text{Aut}(T', G', \partial')
 \end{array}$$

already commute since α and α' are crossed module morphisms, and also does

$$\begin{array}{ccc}
 T & \xrightarrow{\partial} & G \\
 \varphi_1 \downarrow & & \downarrow \varphi_2 \\
 T' & \xrightarrow{\partial'} & G'
 \end{array}$$

since φ is a crossed module morphism. Furthermore,

$$\begin{aligned}
 \text{Hom}((T, G, \partial), \varphi)\Delta(\psi) &= \text{Hom}((T, G, \partial), \varphi)(\psi\partial(_) + (_), \partial\psi(_) + (_)) \\
 &= (\varphi_1\psi\partial(_) + \varphi_1(_), \varphi_2\partial\psi(_) + \varphi_2(_)) \\
 &= (\varphi_1\psi\partial(_) + \varphi_1(_), \partial'\varphi_1\psi(_) + \varphi_2(_)) \\
 &= \Omega(\varphi_1\psi) = \Omega(\text{Hom}(G, \varphi_1)(\psi))
 \end{aligned}$$

gives

$$\begin{array}{ccc}
 D(G, T) & \xrightarrow{\Delta} & \text{Aut}(T, G, \partial) \subseteq \text{End}(T, G, \partial) \\
 \text{Hom}(G, \varphi_1) \downarrow & & \downarrow \text{Hom}((T, G, \partial), \varphi) \\
 \text{XHom}_\varphi(G, T') & \xrightarrow{\Omega} & \text{Hom}((T, G, \partial), (T', G', \partial'))
 \end{array}$$

and

$$\begin{aligned}
 \text{Hom}(\varphi, (T', G', \partial'))\Delta'(\psi') &= \text{Hom}(\varphi, (T', G', \partial'))(\psi'\partial'(_) + (_), \partial'\psi'(_) + (_)) \\
 &= (\psi'\partial'\varphi_1(_) + \varphi_1(_), \partial'\psi'\varphi_2(_) + \varphi_2(_)) \\
 &= (\psi'\varphi_2\partial(_) + \varphi_1(_), \partial'\psi'\varphi_2(_) + \varphi_2(_)) \\
 &= \Omega(\psi'\varphi_2) = \Omega(\text{Hom}(\varphi_2, T')(\psi'))
 \end{aligned}$$

gives

$$\begin{array}{ccc}
 \text{XHom}_\varphi(G, T') & \xrightarrow{\Omega} & \text{Hom}((T, G, \partial), (T', G', \partial')) \\
 \uparrow \text{Hom}(\varphi_2, T') & & \uparrow \text{Hom}(\varphi, (T', G', \partial')) \\
 D(G', T') & \xrightarrow{\Delta'} & \text{Aut}(T', G', \partial') \subseteq \text{End}(T', G', \partial')
 \end{array}$$

The commutativity of the diagrams

$$\begin{array}{ccc}
 G \times T & \xrightarrow{\tilde{\theta}} & T \\
 \downarrow (\varphi_2, \varphi_1) & & \downarrow \varphi_1 \\
 G' \times T' & \xrightarrow{\tilde{\theta}'} & T'
 \end{array}
 \qquad
 \begin{array}{ccc}
 G \times G & \xrightarrow{\sigma} & G \\
 \downarrow (\varphi_2, \varphi_2) & & \downarrow \varphi_2 \\
 G' \times G' & \xrightarrow{\sigma'} & G'
 \end{array}$$

is equivalent to the commutativity of

$$\begin{array}{ccc}
 G & \xrightarrow{\alpha_2} & \text{Aut}(T, G, \partial) \subseteq \text{End}(T, G, \partial) \\
 \downarrow \varphi_2 & & \downarrow \text{Hom}((T, G, \partial), \varphi) \\
 & & \text{Hom}((T, G, \partial), (T', G', \partial')) \\
 & & \uparrow \text{Hom}(\varphi, (T', G', \partial')) \\
 G' & \xrightarrow{\alpha'_2} & \text{Aut}(T', G', \partial') \subseteq \text{End}(T', G', \partial')
 \end{array}$$

since

$$\begin{aligned}
 (\text{Hom}((T, G, \partial), \varphi)\alpha_2(t, h)) &= (\text{Hom}((T, G, \partial), \varphi)(\theta_g, \sigma_g))(t, h) = (\varphi_1\theta_g(t), \varphi_2\sigma_g(h)) \\
 &= (\varphi_1\tilde{\theta}(g, t), \varphi_2\sigma(g, h))
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\tilde{\theta}'(\varphi_2, \varphi_1)(g, t), \sigma'(\varphi_2, \varphi_2)(g, h) \right) \\
 &= \left(\tilde{\theta}'(\varphi_2(g), \varphi_1(t)), \sigma'(\varphi_2(g), \varphi_2(h)) \right) \\
 &= \left(\tilde{\theta}'_{\varphi_2(g)}(\varphi_1(t)), \sigma'_{\varphi_2(g)}(\varphi_2(h)) \right) \\
 &= \left(\text{Hom}(\varphi, (T', G', \partial')) \left(\tilde{\theta}'_{\varphi_2(g)}, \sigma'_{\varphi_2(g)} \right) \right) (t, h) \\
 &= \left(\text{Hom}(\varphi, (T', G', \partial')) \alpha'_2(\varphi_2(g)) \right) (t, h).
 \end{aligned}$$

The equivalence of commutativities of the diagrams

$$\begin{array}{ccc}
 T \times G & \xrightarrow{\tilde{\alpha}_1} & T \\
 \downarrow (\varphi_1, \varphi_2) & & \downarrow \varphi_1 \\
 T' \times G' & \xrightarrow{\tilde{\alpha}'_1} & T'
 \end{array}$$

and

$$\begin{array}{ccc}
 T & \xrightarrow{\alpha_1} & D(G, T) \\
 \downarrow \varphi_1 & & \downarrow \text{Hom}(G, \varphi_1) \\
 & & \text{XHom}_{\varphi}(G, T') \\
 & & \uparrow \text{Hom}(\varphi_2, T') \\
 T' & \xrightarrow{\alpha'_1} & D(G', T')
 \end{array}$$

follows from

$$\begin{aligned}
 (\text{Hom}(G, \varphi_1)(\alpha_1(t)))(g) &= (\text{Hom}((G, \varphi_1)(\psi_t)))(g) = \varphi_1 \psi_t(g) = \varphi_1 \tilde{\alpha}_1(t, g) = \tilde{\alpha}'_1(\varphi_1, \varphi_2)(t, g) \\
 &= \tilde{\alpha}'_1(\varphi_1(t), \varphi_2(g)) = \psi'_{\varphi_1(t)}(\varphi_2(g)) = \text{Hom}(\varphi_2, T') \psi'_{\varphi_1(t)}(g) \\
 &= (\text{Hom}(\varphi_2, T') \alpha'_1(\varphi_1(t)))(g)
 \end{aligned}$$

as desired.

Proposition 3.1 is proved.

Having also explained their morphisms, we are now in a position to define the category of crossed modules with action.

Remark 3.1. The category of crossed modules with action and their morphisms is denoted by \mathbf{XMod}^* .

It is useful to note that, in addition to the three diagrams that explain the morphisms of crossed modules with action, given in Proposition 3.1, and, in particular, to the first diagram containing only the group G , we also have the following supplementary diagram, from the group T .

Proposition 3.2. *If $(T \xrightarrow{\partial} G, \alpha) \longrightarrow (T' \xrightarrow{\partial'} G', \alpha')$ is a morphism of crossed modules with action, then the diagram*

$$\begin{array}{ccc}
 T \times T & \xrightarrow{\theta} & T \\
 \downarrow (\varphi_1, \varphi_1) & & \downarrow \varphi_1 \\
 T' \times T' & \xrightarrow{\theta'} & T'
 \end{array}$$

commutes.

Proof. It follows from the diagram

$$\begin{array}{ccc}
 G \times T & \xrightarrow{\tilde{\theta}} & T \\
 \downarrow (\varphi_2, \varphi_1) & & \downarrow \varphi_1 \\
 G' \times T' & \xrightarrow{\tilde{\theta}'} & T'
 \end{array}$$

that

$$\begin{aligned}
 \varphi_1 \theta(s, t) &= \varphi_1 \theta_{\partial(s)}(t) = \varphi_1 \tilde{\theta}(\partial(s), t) = \tilde{\theta}(\varphi_2, \varphi_1)(\partial(s), t) = \tilde{\theta}(\varphi_2 \partial(s), \varphi_1(t)) \\
 &= \theta'_{\varphi_2 \partial(s)} \varphi_1(t) = \theta'_{\partial' \varphi_1(s)} \varphi_1(t) = \theta'(\varphi_1(s), \varphi_1(t)) = \theta'(\varphi_1, \varphi_1)(s, t),
 \end{aligned}$$

so that $\varphi_1 \theta = \theta'(\varphi_1, \varphi_1)$.

Proposition 3.2 is proved.

From the previous two propositions, we immediately have the following corollaries.

Corollary 3.1. Any morphism $\varphi: (T \xrightarrow{\partial} G, \alpha) \rightarrow (T' \xrightarrow{\partial'} G', \alpha')$ of crossed modules with action, gives rise to two morphism of groups with action

$$\varphi_1: (T, \theta) \rightarrow (T', \theta'),$$

$$\varphi_2: (G, \sigma) \rightarrow (G', \sigma').$$

Corollary 3.2. The correspondences

$$\Pi_1^* \left((T \xrightarrow{\partial} G, \alpha) \xrightarrow{\varphi} (T' \xrightarrow{\partial'} G', \alpha') \right) = (T, \theta) \xrightarrow{\varphi_1} (T', \theta'),$$

$$\Pi_2^* \left((T \xrightarrow{\partial} G, \alpha) \xrightarrow{\varphi} (T' \xrightarrow{\partial'} G', \alpha') \right) = (G, \sigma) \xrightarrow{\varphi_2} (G', \sigma')$$

define the functors $\Pi_1^*, \Pi_2^*: \mathbf{XMod}^* \rightarrow \mathbf{Gr}^*$.

In addition, to the functors Π_1^* and Π_2^* mentioned above, another interesting functor throughout semidirect product from the category \mathbf{XMod}^* to the category \mathbf{Gr}^* is given below.

Theorem 3.1. The correspondence

$$\Gamma^* \left((T \xrightarrow{\partial} G, \alpha) \xrightarrow{\varphi} (T' \xrightarrow{\partial'} G', \alpha') \right) = (T \rtimes G, \Gamma^*(\alpha)) \xrightarrow{(\varphi_1, \varphi_2)} (T' \rtimes G', \Gamma^*(\alpha'))$$

defines a functor $\Gamma^*: \mathbf{XMod}^* \rightarrow \mathbf{Gr}^*$, where $\Gamma^*(\alpha): (T \rtimes G) \times (T \rtimes G) \rightarrow (T \rtimes G)$ is given by

$$[\Gamma^*(\alpha)]((s, g), (t, h)) = ({}^s({}^g t) + \psi_s({}^g h), {}^g h),$$

and $T \rtimes G$ stands for the semidirect product of G acting on T , which is equal to $T \times G$ as a set, given with an operation $(s, g)(t, h) = (s({}^g t), {}^g h)$ for $(s, g), (t, h) \in T \times G$.

Proof. First, we show that $(T \rtimes G, \Gamma^*(\alpha))$ is a group with action. In the following, we write $({}^{s,g})(t, h)$ instead of $[\Gamma^*(\alpha)]((s, g), (t, h))$:

$$({}^{0,0})(t, g) = ({}^0({}^0 t) + \psi_0({}^0 g), {}^0 g) = (t, g),$$

since $\alpha_1(0) = \psi_0$ must be the unit element of the group $D(G, T)$, which is known to be the constant function with the value 0_T . Also,

$$({}^{t,g})(0, 0) = ({}^t({}^g 0) + \psi_t({}^g 0), {}^g 0) = (0, 0).$$

Moreover,

$$\begin{aligned} ({}^{r,g})((s, h) + (t, k)) &= ({}^{r,g})(s + (h \triangleright t), h + k) = \left({}^r \left({}^g (s + (h \triangleright t)) \right) + \psi_r({}^g (h + k)), {}^g (h + k) \right) \\ &= \left({}^r({}^g s) + {}^r({}^g (h \triangleright t)) + \psi_r({}^g h + {}^g k), {}^g h + {}^g k \right) \\ &= \left({}^r({}^g s) + {}^r({}^g (h \triangleright t)) + \psi_r({}^g h) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k \right) \\ &= \left({}^r({}^g s) + \theta_r \theta_g (h \triangleright t) + \psi_r({}^g h) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k \right) \\ &= \left({}^r({}^g s) + \psi_r \partial \theta_g (h \triangleright t) + \theta_g (h \triangleright t) + \psi_r({}^g h) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k \right) \end{aligned}$$

$$\begin{aligned}
&= ({}^r(g_s) + \psi_r \sigma_g \partial(h \triangleright t) + \theta_g(h \triangleright t) + \psi_r({}^g h) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r \sigma_g(h + \partial t - h) + \theta_g(h \triangleright t) + \psi_r({}^g h) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r \sigma_g(h + \partial t) + (\sigma_g(h + \partial t) \triangleright \psi_r \sigma_g(-h)) + \theta_g(h \triangleright t) + \psi_r({}^g h) \\
&\quad + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r \sigma_g(h + \partial t) + (\sigma_g(h) \triangleright (\sigma_g \partial t \triangleright \psi_r \sigma_g(-h))) + (\sigma_g(t) \triangleright \theta_g(t)) \\
&\quad + (\sigma_g(h) \triangleright ((-\sigma_g(h)) \triangleright \psi_r({}^g h))) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r \sigma_g(h + \partial t) + \sigma_g(h) \triangleright ((\partial \theta_g(t) \triangleright \psi_r \sigma_g(-h)) + \theta_g(t) \\
&\quad + ((-{}^g h) \triangleright \psi_r({}^g h))) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r \sigma_g(h + \partial t) + \sigma_g(h) \triangleright (\theta_g(t) + \psi_r \sigma_g(-h) - \theta_g(t) + \theta_g(t) \\
&\quad + ({}^g(-h) \triangleright \psi_r({}^g h))) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r \sigma_g(h + \partial t) + \sigma_g(h) \triangleright (\theta_g(t) + \psi_r({}^g(-h)) + ({}^g(-h) \triangleright \psi_r({}^g h))) \\
&\quad + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r \sigma_g(h + \partial t) + \sigma_g(h) \triangleright (\theta_g(t) + \psi_r({}^g(-h) + {}^g h)) \\
&\quad + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r \sigma_g(h + \partial t) + (\sigma_g(h) \triangleright \theta_g(t)) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r(\sigma_g(h) + \sigma_g \partial(t)) + (\sigma_g(h) \triangleright \theta_g(t)) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r \sigma_g(h) + (\sigma_g(h) \triangleright \psi_r \sigma_g \partial(t)) \\
&\quad + (\sigma_g(h) \triangleright \theta_g(t)) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r \sigma_g(h) + (\sigma_g(h) \triangleright (\psi_r \partial \theta_g(t) + \theta_g(t))) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r \sigma_g(h) + (\sigma_g(h) \triangleright (\theta_r \theta_g(t))) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r({}^g h) + ({}^g h \triangleright ({}^r({}^g t))) + ({}^g h \triangleright \psi_r({}^g k)), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r({}^g h) + ({}^g h \triangleright ({}^r({}^g t) + \psi_r \kappa({}^g k))), {}^g h + {}^g k) \\
&= ({}^r(g_s) + \psi_r({}^g h), {}^g h) + ({}^r({}^g t) + \psi_r({}^g k), {}^g k) = ({}^{r,g})(s, h) + ({}^{r,g})(t, k)
\end{aligned}$$

and

$$\begin{aligned}
{}^{(r,g)}\left({}^{(s,h)}(t, k)\right) &= {}^{(r,g)}\left(s\left(h_t\right) + \psi_s\left(h_k\right), h_k\right) \\
&= {}^r\left(g\left(s\left(h_t\right)\right)\psi_s\left(h_k\right)\right) + \psi_r\left(g\left(h_k\right)\right), g\left(h_k\right) \\
&= \left(\theta_r\theta_g\left(\theta_s\theta_h(t) + \psi_s\left(h_k\right)\right) + \psi_r\left(\sigma_g\sigma_h(k)\right), g\left(h_k\right)\right) \\
&= \left(\theta_r\theta_g\theta_s\theta_h(t) + \theta_r\theta_g\psi_s\sigma_h(k) + \psi_r\sigma_{g+h}(k), g\left(h_k\right)\right) \\
&= \left(\theta_r\theta_g\theta_s\theta_h(t) + \theta_r\theta_g\psi_s\sigma_{-g}\sigma_g\sigma_h(k) + \psi_r\sigma_{g+h}(k), g\left(h_k\right)\right) \\
&= \left(\theta_{\partial r}\theta_g\theta_{\partial s}\theta_{-g}\theta_g\theta_h(t) + \theta_r\psi_{g\triangleright s}\sigma_{g+h}(k) + \psi_r\sigma_{g+h}(k), g\left(h_k\right)\right) \\
&= \left(\theta_{\partial r+(g\triangleright s)}\theta_g\theta_h(t) + (\psi_r \diamond \psi_{g\triangleright s})\sigma_{g+h}(k), g\left(h_k\right)\right) \\
&= \left(\theta_{r+(g\triangleright s)}\theta_{g+h}(t) + \psi_{r+(g\triangleright s)}\sigma_{g+h}(k), g\left(h_k\right)\right) \\
&= \left({}^{r+(g\triangleright s)}\left(g+h_t\right) + \psi_{r+(g\triangleright s)}\left(g+h_k\right), g+h_t\right) \\
&= {}^{(r+(g\triangleright s), g+h)}(t, k) = {}^{((r,g)+(s,h))}(t, k).
\end{aligned}$$

Given a morphism of crossed modules with action $\varphi: (T \xrightarrow{\partial} G, \alpha) \rightarrow (T' \xrightarrow{\partial'} G', \alpha')$. It is obvious that $\varphi = (\varphi_1, \varphi_2): T \times G \rightarrow T' \times G'$ is a group homomorphism. It remains to show that φ respects the additional actions $\mathbf{\Gamma}^*(\alpha)$ and $\mathbf{\Gamma}^*(\alpha')$:

$$\begin{aligned}
\varphi\mathbf{\Gamma}^*(\alpha)((s, g), (t, h)) &= \varphi\left(s(g_t) + \psi_s(g_h), g_h\right) = \left(\varphi_1\left(s(g_t) + \psi_s(g_h)\right), \varphi_2(g_h)\right) \\
&= \left(\varphi_1^{(s)}\left(\varphi_2^{(g)}\varphi_1(t)\right) + \varphi_1\psi_s(g_h), \varphi_2(g_h)\right) \\
&= \left(\varphi_1^{(s)}\left(\varphi_2^{(g)}\varphi_1(t)\right) + \varphi_1\tilde{\alpha}_1(s, g_h), \varphi_2(g_h)\right) \\
&= \left(\varphi_1^{(s)}\left(\varphi_2^{(g)}\varphi_1(t)\right) + \tilde{\alpha}'_1(\varphi_1, \varphi_2)(s, g_h), \varphi_2(g_h)\right) \\
&= \left(\varphi_1^{(s)}\left(\varphi_2^{(g)}\varphi_1(t)\right) + \tilde{\alpha}'_1(\varphi_1(s), \varphi_2(g_h)), \varphi_2(g_h)\right) \\
&= \left(\varphi_1^{(s)}\left(\varphi_2^{(g)}\varphi_1(t)\right) + \psi'_{\varphi_1(s)}\varphi_2(g_h), \varphi_2(g_h)\right) \\
&= \left(\varphi_1^{(s)}\left(\varphi_2^{(g)}\varphi_1(t)\right) + \psi'_{\varphi_1(s)}\left(\varphi_2^{(g)}\varphi_2(h)\right), \varphi_2^{(g)}\varphi_2(h)\right) \\
&= \mathbf{\Gamma}^*(\alpha')((\varphi_1(s), \varphi_2(g)), \varphi_1(t), \varphi_2(h))
\end{aligned}$$

$$= \mathbf{\Gamma}^*(\alpha')(\varphi, \varphi)((s, g), (t, h))$$

yields commutativity of the diagram

$$\begin{array}{ccc}
 (T \rtimes G) \times (T \rtimes G) & \xrightarrow{\mathbf{\Gamma}^*(\alpha)} & T \rtimes G \\
 \downarrow (\varphi, \varphi) & & \downarrow \varphi \\
 (T' \rtimes G') \times (T' \rtimes G') & \xrightarrow{\mathbf{\Gamma}^*(\alpha')} & T' \rtimes G'
 \end{array}$$

completing the proof.

In the sequel, we present some more functors related to the category \mathbf{XMod}^* .

We can consider any given crossed module (T, G, ∂) , as a crossed module with action, considering the “trivial” action

$$\alpha : (T, G, \partial) \longrightarrow (D(G, T), \text{Aut}(T, G, \partial), \Delta),$$

where

$$\alpha_1(t) = 0_{D(G, T)},$$

$$\alpha_2(g) = (\text{id}_T, \text{id}_G).$$

We denote the functor corresponding to this consideration by

$$\mathbf{XT} : \mathbf{XMod} \longrightarrow \mathbf{XMod}^*.$$

A crossed module (T, G, ∂) , can also be considered as a crossed module with action utilizing the crossed module “conjugation”, that is, the action

$$\alpha : (T, G, \partial) \longrightarrow (D(G, T), \text{Aut}(T, G, \partial), \Delta),$$

given through canonical morphism of (T, G, ∂) , more precisely,

$$\alpha_1(t) = \chi_t,$$

$$\alpha_2(g) = (\triangleright_g, c_g),$$

where

$$\chi_t(g) = t - (g \triangleright t),$$

$$\triangleright_g(t) = g \triangleright t,$$

$$c_g(h) = g + h - g.$$

It is easy to verify that a morphism

$$\begin{array}{ccc}
 T & \xrightarrow{\partial} & G \\
 \varphi_1 \downarrow & & \downarrow \varphi_2 \\
 T' & \xrightarrow{\partial'} & G'
 \end{array}$$

of crossed modules also preserves these actions. Then the corresponding functor is denoted by

$$\mathbf{XC} : \mathbf{XMod} \longrightarrow \mathbf{XMod}^*.$$

Proposition 3.3. *Consider the functors*

$$\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{\Gamma} : \mathbf{XMod} \longrightarrow \mathbf{Gr},$$

$$\mathbf{T}, \mathbf{C} : \mathbf{Gr} \longrightarrow \mathbf{Gr}^*,$$

given by

$$\mathbf{\Pi}_1 \left((T \xrightarrow{\partial} G) \xrightarrow{\varphi} (T' \xrightarrow{\partial'} G') \right) = T \xrightarrow{\varphi_1} T',$$

$$\mathbf{\Pi}_2 \left((T \xrightarrow{\partial} G) \xrightarrow{\varphi} (T' \xrightarrow{\partial'} G') \right) = G \xrightarrow{\varphi_2} G',$$

$$\mathbf{\Gamma} \left((T \xrightarrow{\partial} G) \xrightarrow{\varphi} (T' \xrightarrow{\partial'} G') \right) = T \times G \xrightarrow{(\varphi_1, \varphi_2)} T' \times G',$$

and

$$\mathbf{T} \left(G \xrightarrow{f} G' \right) = (G, \varepsilon) \xrightarrow{f} (G', \varepsilon'), \quad \varepsilon(g, h) = h, \quad \varepsilon'(g', h') = h',$$

$$\mathbf{C} \left(G \xrightarrow{f} G' \right) = (G, \varepsilon) \xrightarrow{f} (G', \varepsilon'), \quad \varepsilon(g, h) = g + h - g, \quad \varepsilon'(g', h') = g' + h' - g'.$$

Then one has six commutative diagrams of functors with all possible compositions

$$\begin{array}{ccc}
 \mathbf{XMod} & \xrightarrow{\mathbf{XU}} & \mathbf{XMod}^* \\
 \mathbf{V} \downarrow & & \downarrow \mathbf{V}^* \\
 \mathbf{Gr} & \xrightarrow{\mathbf{U}} & \mathbf{Gr}^*
 \end{array}$$

where $\mathbf{U} = \mathbf{T}$ or \mathbf{C} , and $\mathbf{V} = \mathbf{\Pi}_1, \mathbf{\Pi}_2$, or $\mathbf{\Gamma}$.

Proof. Straightforward calculations.

From the category of groups to the category of crossed modules, there are two well-known functors, $\mathbf{In}, \mathbf{Id} : \mathbf{Gr} \rightarrow \mathbf{XMod}$ given by

$$\mathbf{In}(G \xrightarrow{f} H) = (0, G, \text{inc}) \xrightarrow{(0,f)} (0, H, \text{inc})$$

and

$$\mathbf{Id}(G \xrightarrow{f} H) = (G, G, \text{id}) \xrightarrow{(f,f)} (H, H, \text{id}).$$

Now, if (G, ε) is a group with action, then

$$\mathcal{A}(0, G, \text{inc}) = (D(G, 0), \text{Aut}(0, G, \text{inc}), \Delta) \cong (0, \text{Aut } G, \text{inc})$$

and $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1(0) = 0$, $\alpha_2(g) = (0, \varepsilon(g))$ is an action of the crossed module $\mathbf{In}(G)$ on itself. On the other hand, note that

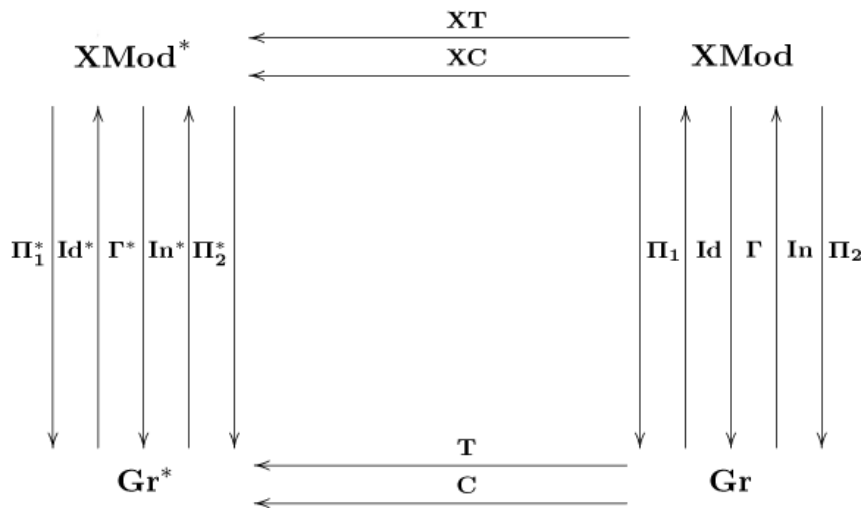
$$\mathcal{A}(G, G, \text{id}) = (D(G, G), \text{Aut}(G, G, \text{inc}), \Delta) \cong \text{Aut } G \times \text{Aut } G \times \text{Aut } G,$$

and, this time, $\alpha = (\alpha_1, \alpha_2)$, where $(\alpha_1(g))(h) = (\varepsilon(g))(h) - h$, $\alpha_2(g) = (\varepsilon(g), \varepsilon(g))$, is in $\mathcal{A}(G, G, \text{id})$. As can be verified easily, these give two corresponding functors

$$\mathbf{In}^*, \mathbf{Id}^* : \mathbf{Gr}^* \rightarrow \mathbf{XMod}^*$$

from the category of groups with action to the category of crossed modules with action.

4. Conclusion. In relation to the category \mathbf{XMod}^* , here we defined the functors $\mathbf{Id}^*, \mathbf{In}^*, \Gamma^*, \Pi_1^*, \Pi_2^*, \mathbf{XC}$, and \mathbf{XT} . With the addition of some previously known ones to these, we form the diagram of functors



with equal compositions $\mathbf{T}\Pi_1 = \Pi_1^*(\mathbf{XT})$, $\mathbf{C}\Pi_1 = \Pi_1^*(\mathbf{XC})$, $(\mathbf{XT})(\mathbf{Id}) = (\mathbf{Id}^*)\mathbf{T}$, $(\mathbf{XC})(\mathbf{Id}) = (\mathbf{Id}^*)\mathbf{C}$, $\mathbf{T}\Gamma = \Gamma^*(\mathbf{XT})$, $\mathbf{C}\Gamma = \Gamma^*(\mathbf{XC})$, $(\mathbf{XT})(\mathbf{In}) = (\mathbf{In}^*)\mathbf{T}$, $(\mathbf{XC})(\mathbf{In}) = (\mathbf{In}^*)\mathbf{C}$, $\mathbf{T}\Pi_2 = \Pi_2^*(\mathbf{XT})$, $\mathbf{C}\Pi_2 = \Pi_2^*(\mathbf{XC})$.

Crossed modules with action, which are presented here, form a generalization of groups with action, which are quite interesting objects, that bridge a gap in algebra. They are especially known for yielding Lie–Leibniz algebras through a generalization of Witt’s theorem. Also, crossed modules with action can similarly be expected to offer interesting categorical connections.

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