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ON THE BEREZIN NUMBER OF OPERATORS ON THE REPRODUCING KERNEL OF HILBERT SPACE AND RELATED QUESTIONS

ПРО ЧИСЛО БЕРЕЗІНА ОПЕРАТОРІВ НА ВІДТВОРЮЮЧОМУ ЯДРІ ГІЛЬБЕРТОВОГО ПРОСТОРУ ТА ПОВ'ЯЗАНІ ПИТАННЯ

We obtain some new inequalities for the Berezin number of operators via the Cauchy – Schwarz-type inequalities. Some other related questions are also discussed.

За допомогою нерівностей типу Коші – Шварца отримано деякі нові нерівності для числа операторів в сенсі Березіна. Також обговорено деякі інші пов'язані питання.

1. Introduction. A reproducing kernel Hilbert space (shortly RKHS) $\mathcal{H} = \mathcal{H}(\Omega)$ is a Hilbert space of all complex valued functions on a nonempty set Ω, which has the property that for every $\lambda \in \Omega$ the linear functional (evaluation functional) $f \to f(\lambda)$ is bounded on $\mathcal{H}$.

Then by the classical Riesz representation theorem for each $\lambda \in \Omega$ there exists a unique function $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The function k_λ is called the reproducing kernel of the space $\mathcal{H}$. For any orthonormal basis $\{e_n(z)\}_{n\geq 0}$ of the space $\mathcal{H}(\Omega)$ (see [4, 19])

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z).$$

Let $\widehat{k}_\lambda = \dfrac{k_\lambda}{\|k_\lambda\|}$ denote the normalized reproducing kernel of the space $\mathcal{H}$ (note that by (ii), we surely have $k_\lambda \neq 0$). For a bounded linear operator A on the RKHS $\mathcal{H}$, its Berezin symbol $\widetilde{A}$ is defined by the formula (see [5])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathcal{H}}, \quad \lambda \in \Omega.$$

The Berezin symbol is a function that is bounded by norm of the operator.

Karaev in [13] defined the Berezin set and the Berezin number, respectively,

$$\operatorname{Ber}(A) := \operatorname{Range}\left(\widetilde{A}\right) = \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\} \quad \text{and} \quad \operatorname{ber}(A) := \sup\left\{\left|\widetilde{A}(\lambda)\right| : \lambda \in \Omega\right\}.$$

Recall that $W(A) := \{\langle Af, f\rangle : \|f\|_{\mathcal{H}} = 1\}$ is the numerical range of the operator A and

$$w(A) := \sup\{|\langle Af, f\rangle| : \|f\|_{\mathcal{H}} = 1\}$$

is the numerical radius of A (for more information, see [8]).

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Garayev and Yamancı [11] gave an upper bound for two Hilbert space operators as follows:

$$\operatorname{ber}^r(B^*A) \le \frac{1}{2}\operatorname{ber}\left(|A|^{2r} + |B|^{2r}\right). \tag{1.1}$$

They also proved that

$$\operatorname{ber}^{2r}(B^*A) \le \left[\alpha\operatorname{ber}\left(|A|^{\frac{2r}{\alpha}}\right) + (1-\alpha)\operatorname{ber}\left(|B|^{\frac{2r}{1-\alpha}}\right)\right].$$

Also, some important result related with Berezin number inequalities were obtained by authors in [6, 7, 9, 12, 17, 18, 20–24].

The Berezin number of an operator A holds the following properties:

(i) $\operatorname{ber}(A) \le \|A\|$,

(ii) $\operatorname{ber}(\alpha A) = |\alpha|\operatorname{ber}(A)$ for all $\alpha \in \mathbb{C}$,

(iii) $\operatorname{ber}(A+B) \le \operatorname{ber}(A) + \operatorname{ber}(B)$.

It is known that the Berezin number does not generally define a norm. But, if $\mathcal{H}$ is a RKHS of analytic functions (for instance, on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$), then $\operatorname{ber}(A)$ defines a norm on $\mathcal{B}(\mathcal{H}(\mathbb{D}))$, which follows from the following result (see, for instance, [25]):

Let $\mathcal{H} = \mathcal{H}(\mathbb{D})$ be a RKHS of analytic functions on $\mathbb{D}$ and $A \in \mathcal{B}(\mathcal{H})$ be an operator. Then the Berezin symbol $\widetilde{A}$ uniquely defines the operator A, i.e., $A = 0$ if and only if $\widetilde{A} = 0$.

The Berezin norm of the operator A on the RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ is defined by formula

$$\|A\|_{\mathrm{ber}} = \sup_{\lambda\in\Omega}\left\|A\widehat{k}_\lambda\right\|_{\mathcal{H}}.$$

Obviously, $\|A\|_{\mathrm{ber}}$ holds the properties (i)–(iii) with $\operatorname{ber}(A)$. Moreover, since the family $\{k_\lambda : \lambda \in \Omega\}$ is complete in $\mathcal{H}$, it is clear that $\|A\|_{\mathrm{ber}} = 0$ if and only if $A = 0$. So, all these properties mean that $\|A\|_{\mathrm{ber}}$ is a norm in $\mathcal{B}(\mathcal{H})$. Obviously, $\operatorname{ber}(A) \le \|A\|_{\mathrm{ber}}$ for any $A \in \mathcal{B}(\mathcal{H})$.

The Cauchy–Schwarz inequality in an inner product space is

$$|\langle x, y\rangle| \le \|x\|\|y\| \tag{1.2}$$

for all vectors x and y. In recent, Kittaneh and Moradi [15] gave the refinement of (1.2) as follows:

$$|\langle x, y\rangle|^2 \le |\langle x, y\rangle|\|x\|\,\|y\| + \frac{1}{2}\left(\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2\right) \le \|x\|^2\|y\|^2.$$

Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then the classical Schwarz inequality for positive operators has the form

$$|\langle Tx, y\rangle|^2 \le \langle Tx, x\rangle\langle Ty, y\rangle \tag{1.3}$$

for any vectors $x, y \in \mathcal{H}$.

A companion of Schwarz inequality (1.3) was introduced by Kato [14] in 1952, sometimes known as the mixed Schwarz inequality, which states that

$$|\langle Tx, y\rangle|^2 \le \Big\langle |T|^{2\alpha}x, x\Big\rangle \Big\langle |T^*|^{2(1-\alpha)}y, y\Big\rangle, \quad 0 \le \alpha \le 1,$$

for all operators $T \in \mathcal{B}(\mathcal{H})$ and any vectors $x, y \in \mathcal{H}$. In particular, we get

$$|\langle Tx, x\rangle| \le \sqrt{\langle |T|x, x\rangle \langle |T^*|x, x\rangle}.$$

Recently, Altwaijry et al. [3] have given some upper bounds for $\operatorname{ber}(S^*T)$ and $\operatorname{ber}(T)$ via the Cauchy – Schwarz-type inequalities. In this paper, using the Cauchy – Schwarz-type inequalities and well-known inequalities, we obtain some upper bounds for the Berezin number of operators which improve the results in [3].

2. Main results. In order to prove our results we need some well-known results as following.

The Power – Mean inequality means that

$$a^{\alpha} b^{(1-\alpha)} \le \alpha a + (1-\alpha) b \le (\alpha a^p + (1-\alpha) b^p)^{\frac{1}{p}} \tag{2.1}$$

for all $\alpha \in [0, 1]$, $a, b \ge 0$ and $p \ge 1$ [16].

Let $\mathcal{B}(\mathcal{H})^+$ be the cone of positive (semidefinite) operators, i.e.,

$$\mathcal{B}(\mathcal{H})^+ = \{A \in \mathcal{B}(\mathcal{H}) \colon \langle Ax, x\rangle \ge 0 \quad \forall x \in \mathcal{H}\}.$$

The following result is known as McCarthy inequality:

$$\langle Tx, x\rangle^p \le \langle T^p x, x\rangle, \quad p \ge 1, \tag{2.2}$$

for $T \in \mathcal{B}(\mathcal{H})^+$ and any vector $x \in \mathcal{H}$. Inequality (2.2) is reversed if $0 \le p \le 1$ [10].

Let $T, S \in B(H)$ be two positive operators. Then

$$\left\| f\left(\frac{T+S}{2}\right) \right\| \le \left\| \frac{f(T) + f(S)}{2} \right\|$$

for a nonnegative convex function f on $[0, \infty)$ (see [1]). Also,

$$\left\| \frac{T+S}{2} \right\|_{\text{ber}}^r \le \left\| \frac{T^r + S^r}{2} \right\|_{\text{ber}} \tag{2.3}$$

for $r \ge 1$ (see [3]). For all vectors x and y in an inner product space, one has

$$|\langle x, y\rangle|^2 \le (1-\beta)\langle x, y\rangle \|x\|\|y\| + \beta \|x\|^2 \|y\|^2 \le \|x\|^2 \|y\|^2 \tag{2.4}$$

for all $\beta \in [0, 1]$ (see [2]).

The following result is important for proving the next results.

Lemma 1. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS on Ω and $T \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$\begin{aligned}\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^{2p} &\le \beta \widetilde{|T|^{2p\alpha}}(\lambda) \widetilde{|T^*|^{2p(1-\alpha)}}(\mu) \\ &\quad + (1-\beta)\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^p \sqrt{\widetilde{|T|^{2p\alpha}}(\lambda) \widetilde{|T^*|^{2p(1-\alpha)}}(\mu)} \\ &\le \widetilde{|T|^{2p\alpha}}(\lambda) \widetilde{|T^*|^{2p(1-\alpha)}}(\mu)\end{aligned} \tag{2.5}$$

for all λ, $\mu \in \Omega$, $0 \le \alpha \le 1$ and $p \ge 1$.

Proof. Utilizing inequality (2.2), we can reach that

$$\begin{aligned}
&\beta\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)+(1-\beta)\left|\left\langle T\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right|^p\sqrt{\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)}\\
&\quad\geq\beta\left[\widetilde{|T|^{2\alpha}}(\lambda)\right]^p\left[\widetilde{|T^*|^{2(1-\alpha)}}(\mu)\right]^p\\
&\qquad+(1-\beta)\left|\left\langle T\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right|^p\sqrt{\left[\widetilde{|T|^{2\alpha}}(\lambda)\right]^p\left[\widetilde{|T^*|^{2(1-\alpha)}}(\mu)\right]^p}\\
&\quad\geq\beta\left|\left\langle T\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right|^{2p}+(1-\beta)\left|\left\langle T\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right|^p\left|\left\langle T\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right|^p=\left|\left\langle T\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right|^{2p}
\end{aligned}\tag{2.6}$$

for all $\beta\in[0,1]$ and $p\geq 1$.

Also, we get

$$\begin{aligned}
&\beta\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)\\
&\qquad+(1-\beta)\left|\left\langle T\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right|^p\sqrt{\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)}\\
&\quad\leq\beta\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)\\
&\qquad+(1-\beta)\sqrt{\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)}\sqrt{\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)}\\
&\quad=\beta\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)+(1-\beta)\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)\\
&\quad=\beta\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu).
\end{aligned}\tag{2.7}$$

Combining (2.6) and (2.7), we reach that

$$\begin{aligned}
\left|\left\langle T\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right|^{2p}&\leq\beta\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)\\
&\qquad+(1-\beta)\left|\left\langle T\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right|^p\sqrt{\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)}\\
&\leq\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\mu)
\end{aligned}$$

for any $p\geq 1$, which proves the (2.5).

The lemma is proved.

Now we give a upper bound for a product of two reproducing kernel Hilbert space operators.

Theorem 1. *Let* $T, S \in \mathcal{B}(\mathcal{H})$. *Then*

$$\mathrm{ber}^{2r}(S^*T) \le \frac{1-\beta}{2}\,\mathrm{ber}^r(S^*T)\Big\||T|^{2r}+|S|^{2r}\Big\|_{\mathrm{ber}} + \frac{\beta}{2}\Big\||T|^{4r}+|S|^{4r}\Big\|_{\mathrm{ber}}$$

for all $\beta \in [0,1]$ *and* $r \ge 1$.

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Setting $x = T\widehat{k}_\lambda$ and $y = S\widehat{k}_\lambda$ in the first inequality in (2.4), we have

$$\Big|\Big\langle T\widehat{k}_\lambda, S\widehat{k}_\lambda\Big\rangle\Big|^2 = \Big|\Big\langle S^*T\widehat{k}_\lambda, \widehat{k}_\lambda\Big\rangle\Big|^2 \le (1-\beta)\Big|\Big\langle T\widehat{k}_\lambda, S\widehat{k}_\lambda\Big\rangle\Big|\Big\|T\widehat{k}_\lambda\Big\|\Big\|S\widehat{k}_\lambda\Big\| + \beta\Big\|T\widehat{k}_\lambda\Big\|^2\Big\|S\widehat{k}_\lambda\Big\|^2$$

$$= (1-\beta)\Big|\widetilde{S^*T}(\lambda)\Big|\Big(\widetilde{|T|^2}(\lambda)\Big)^{\frac{1}{2}}\Big(\widetilde{|S|^2}(\lambda)\Big)^{\frac{1}{2}} + \beta\widetilde{|T|^2}(\lambda)\widetilde{|S|^2}(\lambda).$$

From power mean inequality (2.1), we obtain

$$\Big|\widetilde{S^*T}(\lambda)\Big|^2 \le \left((1-\beta)\Big|\widetilde{S^*T}(\lambda)\Big|^r\Big(\widetilde{|T|^2}(\lambda)\Big)^{\frac{r}{2}}\Big(\widetilde{|S|^2}(\lambda)\Big)^{\frac{r}{2}} + \beta\Big(\widetilde{|T|^2}(\lambda)\Big)^r\Big(\widetilde{|S|^2}(\lambda)\Big)^r\right)^{\frac{1}{r}},$$

which implies that

$$\Big|\widetilde{S^*T}(\lambda)\Big|^{2r} \le (1-\beta)\Big|\widetilde{S^*T}(\lambda)\Big|^r\Big(\widetilde{|T|^2}(\lambda)\Big)^{\frac{r}{2}}\Big(\widetilde{|S|^2}(\lambda)\Big)^{\frac{r}{2}} + \beta\Big(\widetilde{|T|^2}(\lambda)\Big)^r\Big(\widetilde{|S|^2}(\lambda)\Big)^r$$

$$\le (1-\beta)\Big|\widetilde{S^*T}(\lambda)\Big|^r\Big(\widetilde{|T|^{2r}}(\lambda)\Big)^{\frac{1}{2}}\Big(\widetilde{|S|^{2r}}(\lambda)\Big)^{\frac{1}{2}} + \beta\widetilde{|T|^{2r}}(\lambda)\widetilde{|S|^{2r}}(\lambda) \text{ (by (2.2))}$$

$$\le \frac{1}{2}(1-\beta)\Big|\widetilde{S^*T}(\lambda)\Big|^r\Big(\widetilde{|T|^{2r}}(\lambda) + \widetilde{|S|^{2r}}(\lambda)\Big) + \frac{1}{2}\beta\Big(\widetilde{|T|^{4r}}(\lambda) + \widetilde{|S|^{4r}}(\lambda)\Big) \quad \text{(by (2.1))}$$

for all $\lambda \in \Omega$. Taking the supremum over all $\lambda \in \Omega$, we have the desired inequality.

The theorem is proved.

For Theorem 1, we can give the following example.

***Example* 1.** If $\Omega = \{1,2\}$, then $\mathcal{H}(\Omega) = \mathbb{R}^2$ and $\mathcal{B}(\mathcal{H}) = \mathcal{M}_2(\mathbb{R})$. Now, by taking $\widehat{k}_1 = (1,0), \widehat{k}_1 = (0,1)$, then, for matrices $S = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.33 \end{bmatrix}$, $T = \begin{bmatrix} 0.1 & 0.04 \\ 0.04 & 0.2 \end{bmatrix}$, $r = 2$ and $\beta = 1/2$, we have $S^*T = \begin{bmatrix} 0.058 & 0.06 \\ 0.033 & 0.074 \end{bmatrix}$. Therefore,

$$\mathrm{ber}^4(S^*T) = 0.000002 < \frac{1-1/2}{2}\mathrm{ber}^2(S^*T)\Big\||T|^4+|S|^{2r}\Big\|_{\mathrm{ber}} + \frac{1/2}{2}\Big\||T|^8+|S|^8\Big\|_{\mathrm{ber}}$$

$$= \frac{1-1/2}{2}(0.00088) + \frac{1/2}{2}(0.00033) = 0.000119.$$

Theorem 2. *Let $T, S \in \mathcal{B}(\mathcal{H})$. Then*

$$\operatorname{ber}^{2r}(S^*T) \leq \frac{1}{4}\beta\left\| |T|^{2r} + |S|^{2r} \right\|_{\operatorname{ber}}^2 + \frac{1-\beta}{2}\operatorname{ber}^r(T)\left\| |T|^{2r} + |S|^{2r} \right\|_{\operatorname{ber}}$$

$$\leq \frac{1}{2}\beta\left\| |T|^{4r} + |S|^{4r} \right\|_{\operatorname{ber}} + \frac{1-\beta}{2}\operatorname{ber}^r(T)\left\| |T|^{2r} + |S|^{2r} \right\|_{\operatorname{ber}} \tag{2.8}$$

for $r \geq 1$ and $\beta \in [0, 1]$.

Proof. Using inequality (1.1), we have, for all $\beta \in [0, 1]$,

$$\operatorname{ber}^{2r}(S^*T) = \beta \operatorname{ber}^{2r}(S^*T) + (1-\beta)\operatorname{ber}^{2r}(S^*T)$$

$$= \beta\operatorname{ber}^{2r}(S^*T) + (1-\beta)\operatorname{ber}^r(S^*T)\operatorname{ber}^r(S^*T)$$

$$\leq \frac{1}{4}\beta\left\| |T|^{2r} + |S|^{2r} \right\|_{\operatorname{ber}}^2 + \frac{(1-\beta)}{2}\operatorname{ber}^r(S^*T)\left\| |T|^{2r} + |S|^{2r} \right\|_{\operatorname{ber}},$$

which gives the first inequality in (2.8). Also, from (2.3) we obtain

$$\operatorname{ber}^{2r}(S^*T) \leq \frac{\beta}{4}\left\| |T|^{2r} + |S|^{2r} \right\|_{\operatorname{ber}}^2 + \frac{1-\beta}{2}\operatorname{ber}^r(S^*T)\left\| |T|^{2r} + |S|^{2r} \right\|_{\operatorname{ber}}$$

$$= \frac{\beta}{4}\left\| \frac{2|T|^{2r} + 2|S|^{2r}}{2} \right\|_{\operatorname{ber}}^2 + \frac{1-\beta}{2}\operatorname{ber}^r(S^*T)\left\| |T|^{2r} + |S|^{2r} \right\|_{\operatorname{ber}}$$

$$\leq \frac{\beta}{4}\left\| \frac{\left(2|T|^{2r}\right)^2 + \left(2|S|^{2r}\right)^2}{2} \right\|_{\operatorname{ber}} + \frac{1-\beta}{2}\operatorname{ber}^r(S^*T)\left\| |T|^{2r} + |S|^{2r} \right\|_{\operatorname{ber}}$$

$$= \frac{\beta}{2}\left\| |T|^{4r} + |S|^{4r} \right\|_{\operatorname{ber}} + \frac{1-\beta}{2}\operatorname{ber}^r(S^*T)\left\| |T|^{2r} + |S|^{2r} \right\|_{\operatorname{ber}},$$

which gives the second inequality in (2.8).

The theorem is proved.

Theorem 3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\operatorname{ber}^{2p}(T) \leq \beta\left\| \alpha|T|^{2p} + (1-\alpha)|T^*|^{2p} \right\|_{\operatorname{ber}} + \frac{1-\beta}{2}\operatorname{ber}^p(T)\left\| |T|^{2p\alpha} + |T^*|^{2p(1-\alpha)} \right\|_{\operatorname{ber}}$$

for all $p \geq 1$ and $0 \leq \alpha,\ \beta \leq 1$.

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Putting the $\widehat{k}_\mu = \widehat{k}_\lambda$ in (2.5), we have

$$\left|\widetilde{T}(\lambda)\right|^{2p} \leq \beta\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\lambda)$$

$$+ (1-\beta)\left|\widetilde{T}(\lambda)\right|^p\sqrt{\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\lambda)}$$

$$\le \beta\left(\widetilde{|T|^{2p}}(\lambda)\right)^{\alpha}\left(\widetilde{|T^*|^{2p}}(\lambda)\right)^{(1-\alpha)} \quad (\text{by}(2.2))$$

$$+\,\frac{1}{2}\,(1-\beta)\left|\widetilde{T}(\lambda)\right|^{p}\left(\widetilde{|T|^{2p\alpha}}(\lambda)+\widetilde{|T^*|^{2p(1-\alpha)}}(\lambda)\right) \text{ (by (2.1))}$$

$$\le \beta\left[\alpha\widetilde{|T|^{2p}}(\lambda)+(1-\alpha)\left(\widetilde{|T^*|^{2p}}(\lambda)\right)\right]$$

$$+\,\frac{1-\beta}{2}\left|\widetilde{T}(\lambda)\right|^{p}\left\langle\left(|T|^{2p\alpha}+|T^*|^{2p(1-\alpha)}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle$$

$$=\beta\left\langle\alpha\left(|T|^{2p}+(1-\alpha)|T^*|^{2p}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle$$

$$+\,\frac{1}{2}\,(1-\beta)\left|\widetilde{T}(\lambda)\right|^{p}\left\langle\left(|T|^{2p\alpha}+|T^*|^{2p(1-\alpha)}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle.$$

for all $\lambda\in\Omega$. Taking the supremum over all $\lambda\in\Omega$, we get the required result.

The theorem is proved.

Theorem 4. *Let $T\in\mathcal{B}(\mathcal{H})$. Then*

$$\operatorname{ber}^{2p}(T)\le\beta\left\|\alpha|T|^{2p}+(1-\alpha)|T^*|^{2p}\right\|_{\text{ber}}$$

$$+\,(1-\beta)\operatorname{ber}^{p}(T)\sqrt{\left\|\alpha|T|^{2p\alpha}+(1-\alpha)|T^*|^{2p}\right\|_{\text{ber}}}$$

$$\le\left\|\alpha|T|^{2p\alpha}+(1-\alpha)|T^*|^{2p}\right\|_{\text{ber}} \tag{2.9}$$

for all $p\ge 1$ and $0\le\alpha,\ \beta\le 1$.

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Placing the $\widehat{k}_\mu=\widehat{k}_\lambda$ in (2.5), it follows that

$$\left|\widetilde{T}(\lambda)\right|^{2p}\le\beta\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\lambda)$$

$$+\,(1-\beta)\left|\widetilde{T}(\lambda)\right|^{p}\sqrt{\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\lambda)}$$

$$\le\beta\left(\widetilde{|T|^{2p}}(\lambda)\right)^{\alpha}\left(\widetilde{|T^*|^{2p}}(\lambda)\right)^{(1-\alpha)} \quad (\text{by (2.2)})$$

$$+\,(1-\beta)\left|\widetilde{T}(\lambda)\right|^{p}\sqrt{\left(\widetilde{|T|^{2p}}(\lambda)\right)^{\alpha}\left(\widetilde{|T^*|^{2p}}(\lambda)\right)^{(1-\alpha)}}$$

$$\le\beta\left[\alpha\widetilde{|T|^{2p}}(\lambda)+(1-\alpha)\left(\widetilde{|T^*|^{2p}}(\lambda)\right)\right]$$

$$+ (1-\beta)\left|\widetilde{T}(\lambda)\right|^{p}\sqrt{\left[\alpha\widetilde{|T|^{2p}}(\lambda) + (1-\alpha)\left(\widetilde{|T^{*}|^{2p}}(\lambda)\right)\right]} \quad \text{(by (2.1))}.$$

Taking the supremum over all $\lambda \in \Omega$, we get the first in (2.9). By using the above inequality, we obtain

$$\operatorname{ber}^{2p}(T) \le \left\|\alpha|T|^{2p} + (1-\alpha)|T^{*}|^{2p}\right\|_{\operatorname{ber}} + (1-\beta)\operatorname{ber}^{p}(T)\sqrt{\left\|\alpha|T|^{2p} + (1-\alpha)|T^{*}|^{2p}\right\|_{\operatorname{ber}}}$$

$$\le \left\|\alpha|T|^{2p} + (1-\alpha)|T^{*}|^{2p}\right\|_{\operatorname{ber}},$$

which gives the desired consequence.

The theorem is proved.

Theorem 5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\operatorname{ber}^{p}(T) \le \frac{1}{2}\beta\left\||T|^{2pa} + |T^{*}|^{2p(1-\alpha)}\right\|_{\operatorname{ber}} + \frac{1}{\sqrt{2}}(1-\beta)\operatorname{ber}^{\frac{p}{2}}(T)\left\||T|^{2p\alpha} + |T^{*}|^{2p(1-\alpha)}\right\|_{\operatorname{ber}}^{\frac{1}{2}}$$

for all $p \ge 1$ and $0 \le \alpha,\ \beta \le 1$.

Proof. By using the similar techniques in (2.6) and (2.7), we observe that

$$\left|\left\langle T\widehat{k}_{\lambda}, \widehat{k}_{\mu}\right\rangle\right|^{p} \le \beta\left(\widetilde{|T|^{2p\alpha}}(\lambda)\right)^{1/2}\left(\widetilde{|T^{*}|^{2p(1-\alpha)}}(\mu)\right)^{1/2}$$

$$+ (1-\beta)\left|\left\langle T\widehat{k}_{\lambda}, \widehat{k}_{\mu}\right\rangle\right|^{\frac{p}{2}}\sqrt{\left(\widetilde{|T|^{2p\alpha}}(\lambda)\right)^{1/2}\left(\widetilde{|T^{*}|^{2p(1-\alpha)}}(\mu)\right)^{1/2}}$$

$$\le \left(\widetilde{|T|^{2p\alpha}}(\lambda)\right)^{1/2}\left(\widetilde{|T^{*}|^{2p(1-\alpha)}}(\mu)\right)^{1/2} \tag{2.10}$$

for all $p \ge 1$ and $0 \le \alpha,\ \beta \le 1$.

Putting the $\widehat{k}_{\mu} = \widehat{k}_{\lambda}$ in (2.10), we have

$$\left|\widetilde{T}(\lambda)\right|^{p} \le \beta\left(\widetilde{|T|^{2p\alpha}}(\lambda)\right)^{1/2}\left(\widetilde{|T^{*}|^{2p(1-\alpha)}}(\lambda)\right)^{1/2}$$

$$+ (1-\beta)\left|\widetilde{T}(\lambda)\right|^{\frac{p}{2}}\sqrt{\left(\widetilde{|T|^{2p\alpha}}(\lambda)\right)^{1/2}\left(\widetilde{|T^{*}|^{2p(1-\alpha)}}(\lambda)\right)^{1/2}}$$

$$\le \frac{1}{2}\beta\left(\widetilde{|T|^{2p\alpha}}(\lambda) + \widetilde{|T^{*}|^{2p(1-\alpha)}}(\lambda)\right)$$

$$+ \frac{1}{\sqrt{2}}(1-\beta)\left|\widetilde{T}(\lambda)\right|^{\frac{p}{2}}\sqrt{\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^{*}|^{2p(1-\alpha)}}(\lambda)} \quad \text{(by (2.1))}.$$

Taking the supremum over all $\lambda \in \Omega$, we get the required consequence.

The theorem is proved.

Theorem 6. *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then*

$$\mathrm{ber}^{2p}(T) \leq \frac{1}{2}\beta\left\| |T|^{4p\alpha} + |T^*|^{4p(1-\alpha)} \right\|_{\mathrm{ber}} + \frac{1}{2}(1-\beta)\mathrm{ber}^p(T)\left\| |T|^{2p\alpha} + |T^*|^{2p(1-\alpha)} \right\|_{\mathrm{ber}}$$

for all $p \geq 1$ *and* $0 \leq \alpha,\ \beta \leq 1$.

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Putting the $\widehat{k}_\mu = \widehat{k}_\lambda$ in (2.5), we have

$$\begin{aligned}
\left|\widetilde{T}(\lambda)\right|^{2p} &\leq \beta\left(\widetilde{|T|^{2p\alpha}}(\lambda) + \widetilde{|T^*|^{2p(1-\alpha)}}(\lambda)\right) \\
&\quad + (1-\beta)\left|\widetilde{T}(\lambda)\right|^{p}\sqrt{\widetilde{|T|^{2p\alpha}}(\lambda)\widetilde{|T^*|^{2p(1-\alpha)}}(\lambda)} \\
&\leq \frac{1}{2}\beta\left[\left(\widetilde{|T|^{2p\alpha}}(\lambda)\right)^2 + \left(\widetilde{|T^*|^{2p(1-\alpha)}}(\lambda)\right)^2\right] \quad \text{(by (2.1))} \\
&\quad + \frac{1}{2}(1-\beta)\left|\widetilde{T}(\lambda)\right|^{p}\left(\widetilde{|T|^{2p\alpha}}(\lambda) + \widetilde{|T^*|^{2p(1-\alpha)}}(\lambda)\right) \quad \text{(by (2.1))} \\
&\leq \frac{1}{2}\beta\left(\widetilde{|T|^{4p\alpha}}(\lambda) + \widetilde{|T^*|^{4p(1-\alpha)}}(\lambda)\right) \quad \text{(by (2.2))} \\
&\quad + \frac{1}{2}(1-\beta)\left|\widetilde{T}(\lambda)\right|^{p}\left(\widetilde{|T|^{2p\alpha}}(\lambda) + \widetilde{|T^*|^{2p(1-\alpha)}}(\lambda)\right)
\end{aligned}$$

for all $\lambda \in \Omega$. Taking the supremum over all $\lambda \in \Omega$, we get the required consequence.

The theorem is proved.

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